Uniqueness and non-uniqueness of deriving transformations

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Differential categories I

Diff cats are a categorical axiomatization of differentiation based on symmetric monoidal categories.

—— 1st part of the def: "the linear maps" ——

A diff cat is first an *additive symmetric monoidal category* $(\mathsf{C}, \otimes, I)$, that is, a smc enriched over $(\mathsf{cMon}, \otimes, \mathbb{N})$.

Concretely, it means that each hom-set is a (additive) commutative monoid, and both composition and the tensor product of morphisms preserve $+$ and 0 in each argument.

Intuition: $(\mathsf{C}, \otimes, I)$ must be seen as a *category of linear maps*.

Running example: $(\mathbb{R}\text{-}\mathsf{Vec}, \otimes, \mathbb{R})$.

Differential categories II

—— 2nd part of the def: "the smooth maps" ——

We then ask for a monad (T, μ, η) on C and natural transformations $m_A \colon TA \otimes TA \to TA$ and $u_A \colon I \to TA$. This is called an algebra modality.

Intuition: TA must be seen as an object of:

- *formal* smooth functions in elements of A
- or *actual* smooth functions on the dual A^* of A

We can compose and multiply them, and there are units for these operations.

Running examples:

- Symm. alg. $S =$ free comm. $\mathbb{R}$-alg. monad on $(\mathbb{R}\text{-Vec}, \otimes, \mathbb{R})$. If $(e_i)_{i \in I}$ is a basis of A, then $SA \simeq k[x_i,\ i \in I]$.
- Free C^∞-ring S^∞ on $(\mathbb{R}\text{-Vec}, \otimes, \mathbb{R})$. We have $S^\infty(\mathbb{R}^n) = C^\infty(\mathbb{R}^n)$.

Differential categories III

—— 3rd part of the def: "differentiation of smooth maps" ——

Finally, we ask for a *deriving transformation*. This is a natural transformation $\mathrm{d}_A \colon TA \to TA \otimes A$ which must satisfy the usual rules of calculus: product rule, chain rule, symmetry of partial derivatives etc.

Running examples:

- Symmetric algebra S on $(\mathbb{R}\text{-Vec}, \otimes, \mathbb{R})$. If $(e_i)_{i \in I}$ is a basis of $A \in \mathbb{R}\text{-Vec}$, then $SA \simeq k[x_i,\ i \in I]$ and $\mathrm{d}(f) = \sum_{i \in I} \frac{\partial f}{\partial x_i} \otimes e_i$.
- Free C^∞-ring S^∞ on $(\mathbb{R}\text{-Vec}, \otimes, \mathbb{R})$. We have $S^\infty(\mathbb{R}^n) = C^\infty(\mathbb{R}^n)$ and $\mathrm{d}_{\mathbb{R}^n}(f) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \otimes e_i$.

Seely isomorphisms

Let (T, μ, η, m, u) be an algebra modality on an a.s.m.c. with finite coproducts.

We say it has the *binary Seely iso* (Seely 2) if the nat. trans.

$$\chi_{\otimes} := \left(TA \otimes TB \xrightarrow{T\iota_1 \otimes T\iota_2} T(A \oplus B) \otimes T(A \oplus B) \xrightarrow{m} T(A \oplus B) \right)$$

is an isomorphism.

We say it has the *nullary Seely iso* (Seely 0) if the morphism

$$\chi_I := u_0 \colon I \to T0$$

is an isomorphism.

Running examples:

- The symmetric algebra on $(\mathbb{R}\text{-Vec}, \otimes, \mathbb{R})$ has Seely 0 and 2.
- The free C^∞-ring on $(\mathbb{R}\text{-Vec}, \otimes, \mathbb{R})$ has only Seely 0.

Lemay's uniqueness theorem

Theorem

If an algebra modality on an a.s.m.c. with finite coproducts has both Seely isomorphisms, then it has at most one der. trans.

Example

The symmetric algebra on $(\mathbb{R}\text{-Vec}, \otimes, \mathbb{R})$ *has both Seely iso thus its deriving transformation is unique.*

After proving this, JS (Lemay) didn't know about the general case: does every algebra modality on an a.s.m.c. admit at most one deriving transformation??

The answer to this question is in fact: **No!!** We will see examples.

A refined uniqueness theorem

Before the counterexamples, we will refine our knowledge on uniqueness of deriving transformations.

First, Seely 0 is not necessary in JS' theorem!

Theorem

If an algebra modality on an a.s.m.c. with finite coproducts has the ***binary*** *Seely isomorphism, then it has at most one deriving transformation.*

AI disclosure: The above theorem has been proved by GPT Astra.

Example

The free commutative ring algebra modality S' *on* $(\mathsf{cMon}, \otimes_{\mathbb{N}}, \mathbb{N})$ *defined by* $S'A := US(\mathbb{Z} \otimes_{\mathbb{N}} A)$ *where* S *is the symmetric algebra on* $(\mathsf{Ab}, \otimes_{\mathbb{Z}}, \mathbb{Z})$ *and* $U \colon \mathsf{Ab} \to \mathsf{cMon}$ *has Seely 2 but not Seely* 0*. Moreover,* S' *has a deriving transformation, which is thus unique.*

Uniqueness without Seely 2

The refined theorem gives a necessary but not sufficient condition:

- The free C^∞-ring algebra modality S^∞ on $(\mathbb{R}\text{-Vec}, \otimes, \mathbb{R})$ has Seely 0 but not Seely 2. But it admits a unique der. trans.
- The free C^∞-ring algebra modality T^∞ on $(\mathbb{Q}\text{-Vec}, \otimes, \mathbb{Q})$ has neither Seely 0 nor Seely 2. But it admits a unique der. trans.

T^∞ is defined by $T^\infty A := US^\infty(\mathbb{R} \otimes_{\mathbb{Q}} A)$ where
$U : \mathbb{R}\text{-Vec} \to \mathbb{Q}\text{-Vec}$ is the forgetful functor.

Counterexamples to uniqueness I

Theorem

If $\mathbb{T}$ *is an algebraic theory containing the theory of commutative rigs then the functor* $U\colon \mathsf{Alg}_{\mathbb{T}} \to \mathsf{cMon}$ *forgetting the operations other than* $+$ *and* 0 *admits a left adjoint and the induced free* $\mathbb{T}$*-algebra monad* T *on* cMon *is an algebra modality on* $(\mathsf{cMon}, \otimes, \mathbb{N})$*.*

Example

- *We obtain an algebra modality* S^f *on* $(\mathsf{cMon}, \otimes, \mathbb{N})$ *from the algebraic theory of commutative rigs* R *with an endofunction* $f\colon R \to R$*.*
- *We obtain an algebra modality* T^f *on* $(\mathsf{cMon}, \otimes, \mathbb{N})$ *from the algebraic theory of commutative rigs* R *with an endofunction* $f\colon R \to R$ *preserving* 0*,* $+$*, and* 1*.*

Counterexamples to uniqueness II

Write $\mathrm{Der}(T)$ for the set of all deriving transformations on an algebra modality T.

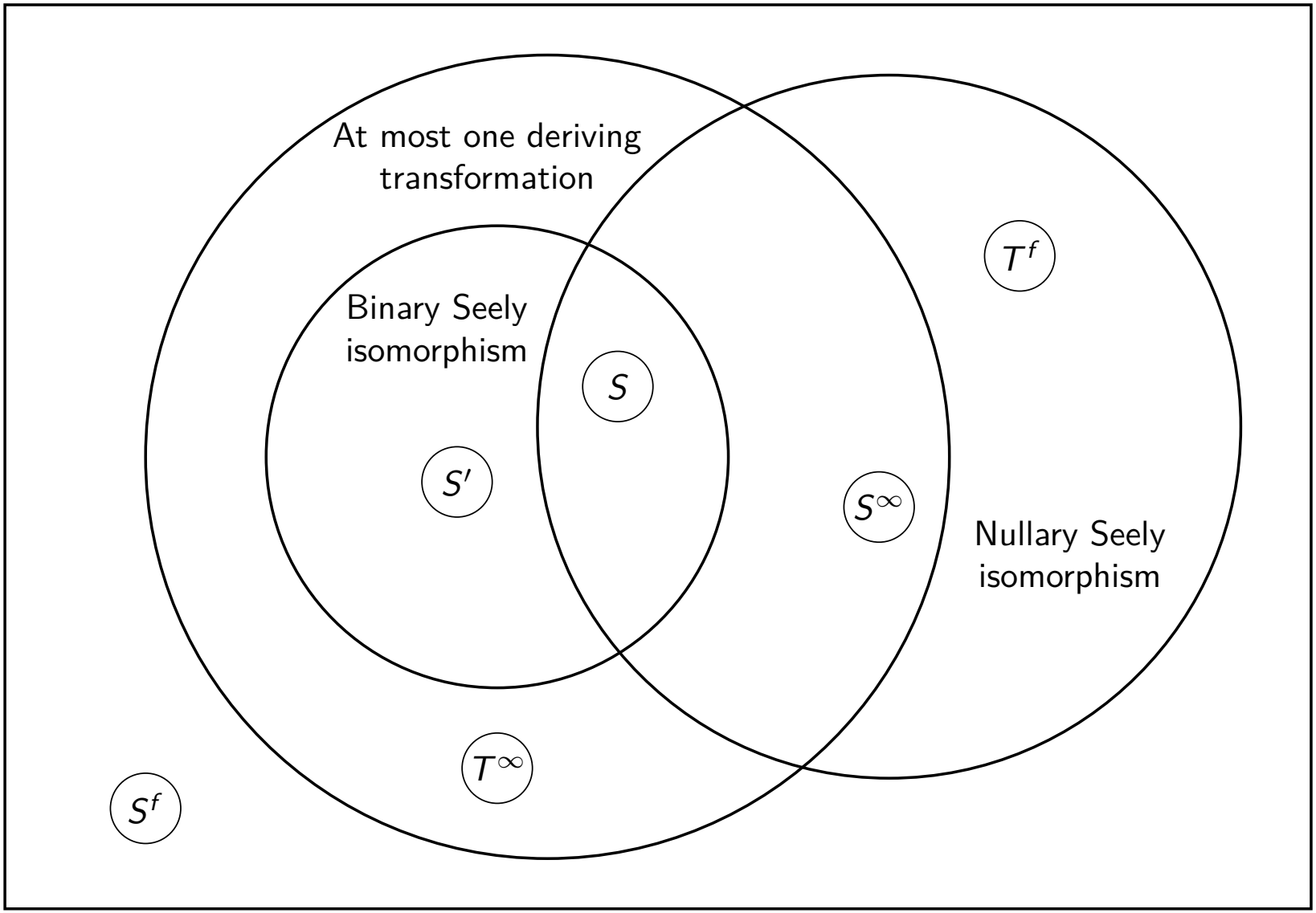
Theorem
We have bijections $\mathrm{Der}(S^f) \simeq S^f\mathbb{N}$ *and* $\mathrm{Der}(T^f) \simeq \mathbb{N}$ *given by* $\mathrm{d} \mapsto \mathrm{d}_{\mathbb{N}}(f(x))$.

Intuition: It suffices to know the derivative of $f(x) \in S^f(\mathbb{N})$ where $x := \eta_{\mathbb{N}}(1) \in S^f(\mathbb{N})$ to be able to compute the derivitative of any element in $T^f A$.

In the case of T^f, the equations $f(a+b) = f(a) + f(b)$, $f(0) = 0$, and $f(1) = 1$ force the derivative of $f(x)$ to be an integer.

Corollary: Both S^f and T^f admit infinitely many deriving transformations.

Conclusion



Note: All the algebra modalities here admit at least one der. trans.