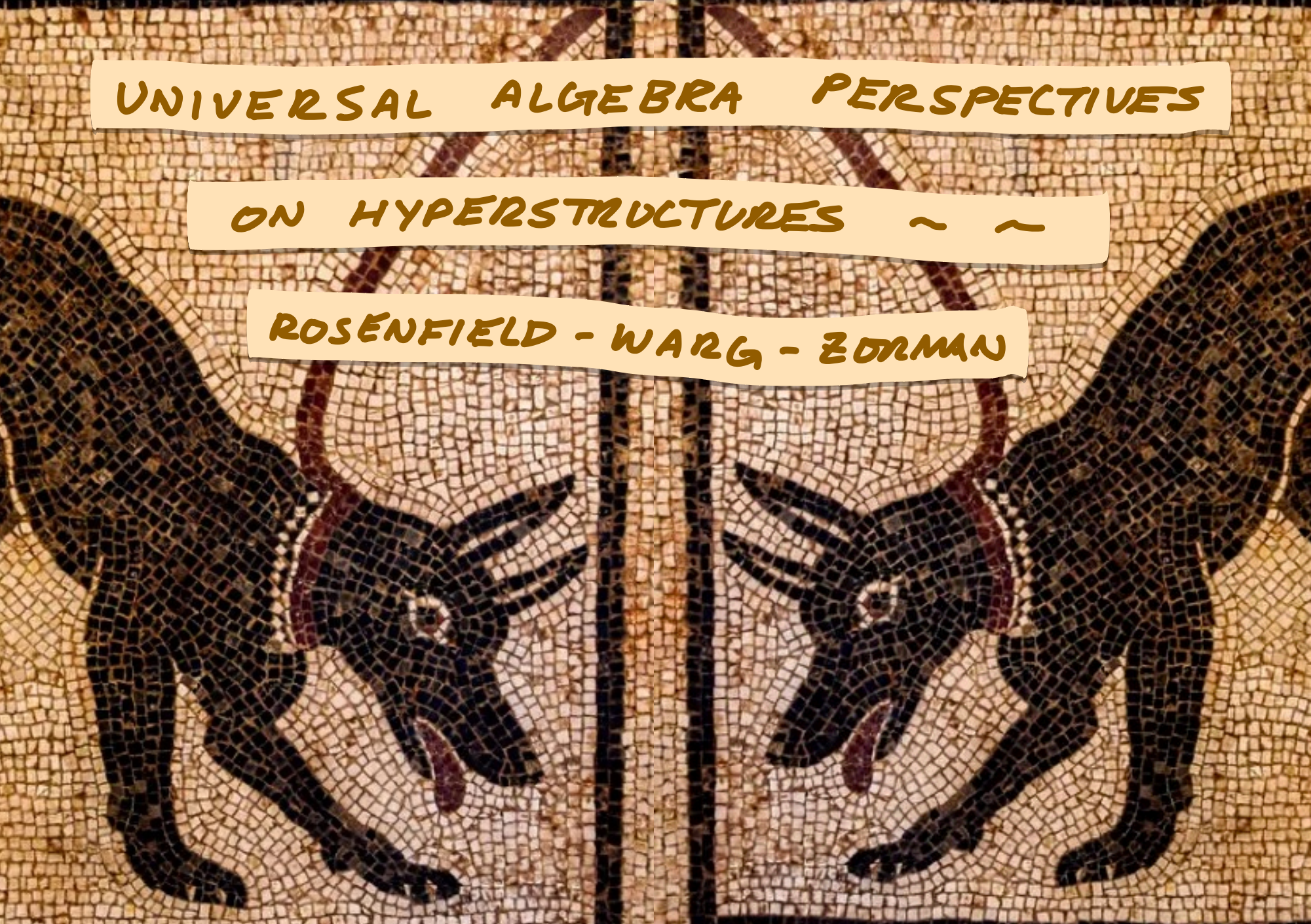
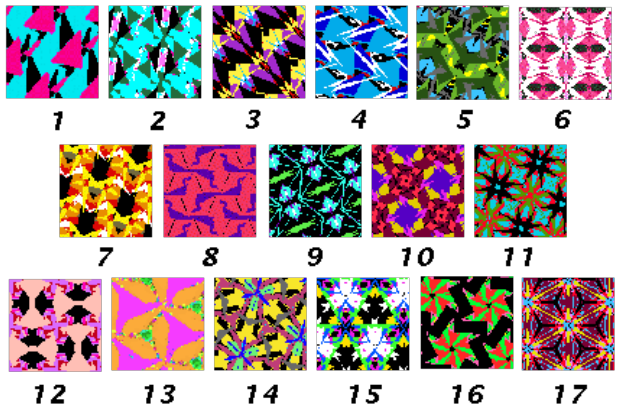

UNIVERSAL ALGEBRA PERSPECTIVES
ON HYPERSTRUCTURES ~ ~
ROSENFIELD - WARG - ZORMAN

# WHAT ARE HYPERSTRUCTURES?

WE KNOW AND LOVE ALGEBRAIC STRUCTURES.

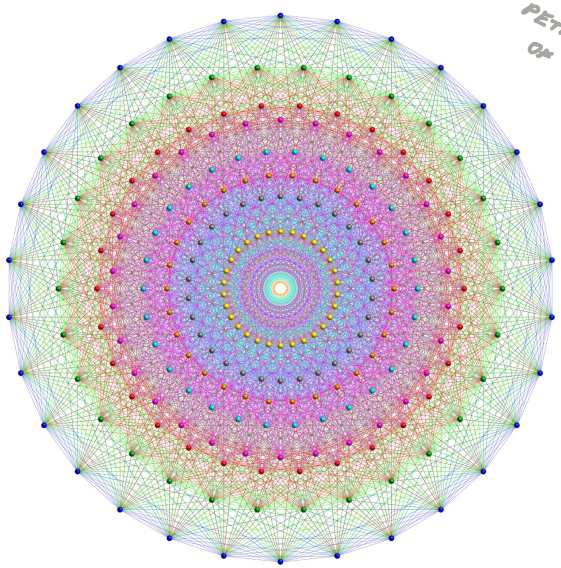
e.g.:

GROUPS




WALLPAPER GRPS.

RINGS



PETRIE PROJECTION OF A LIE ALGEBRA

THEY'RE DEFINED IN TERMS OF **BINARY OPERATIONS**,

FUNCTIONS $G \times G \longrightarrow G$.

$(a, b) \longmapsto a + b$ ← 2 INPUTS, 1 OUTPUT.

# WHAT ARE HYPERSTRUCTURES?

ONE COULD ASK: "WHAT KIND OF MATHEMATICAL THING DO I GET IF I ALLOW THE ADDITION ON MY MONOID/GROUP/RING/FIELD TO BE MULTI-VALUED?"

IN OTHER WORDS: "WHAT IF MY 'ADDITION LAW' HAS THE FORM

$$G \times G \longrightarrow \mathcal{P}(G)$$

INSTEAD OF $G \times G \longrightarrow G$?"

# WHAT ARE HYPERSTRUCTURES?

OKAY, ONE COULD ASK THAT, BUT WHY WOULD THEY? FIRST, TERMINOLOGY:

A SET $X$ EQUIPPED WITH A FUNCTION $X \times X \xrightarrow{*} \mathcal{P}(X)$ IS CALLED A HYPERMAGMA.

# WHAT ARE HYPERSTRUCTURES?

e.g., A **HYPERMONOID** IS AN ASSOCIATIVE, UNITAL HYPERMAGMA.

**ASSOC:** $X^2 \xrightarrow{*} \mathcal{P}(X)$ DEFINES $\mathcal{P}(X)^2 \xrightarrow{*} \mathcal{P}(X)$

VIA $A * B := \bigcup_{\substack{a \in A \\ b \in B}} (a * b)$

$(X, *)$ ASSOC. $\iff$ $(\mathcal{P}(X), *)$ ASSOC.

$\in$ HMag AS A MAGMA.

# WHAT ARE HYPERSTRUCTURES?

e.g., A **HYPERMONOID** IS AN ASSOCIATIVE, UNITAL HYPERMAGMA.

**UNITAL.** $(X, *, e)$ UNITAL $\in$ HMag

$\Longleftrightarrow$

$(\mathcal{P}(X), *, \{e\})$ UNITAL AS A MAGMA.

e.g., ANY MONOID $(X, +, e)$:

TAKE $a * b := \{a + b\}$.

# WHAT ARE HYPERSTRUCTURES?

e.g., A **HYPERGROUP** IS AN ASSOCIATIVE, UNITAL, **REVERSIBLE** HYPERMAGMA.

↑ "CORRECT" NOTION OF INVERSES

**REVERS.** THERE IS A FUNCTION $X \xrightarrow{(-)'} X$

WITH $e \in [x * x'] \cap [x' * x] \quad \forall x \in X$

SUCH THAT $\forall x, y, z \in X$,

$$x \in y * z \iff y \in x * z' \iff z \in y' * x.$$

# WHAT ARE HYPERSTRUCTURES?

e.g., A MOSAIC IS A NON-ASSOCIATIVE HYPERGROUP.

↑ NOT NECESSARILY

IF $G$ IS A GROUP AND $H \leq G$, THE DOUBLE COSETS $HgH$ ARE A MOSAIC WITH

$$HgH * HhH := \{HjH : j \in HgHhH\}.$$

$$\mathbb{1} := He_GH, \quad (HgH)^{-1} := Hg^{-1}H.$$

# WHAT ARE HYPERSTRUCTURES?

GENERIC "STRUCTURED" HYPERMAGMA: WILL SAY HYPERSTRUCTURE.

TURNS OUT THAT THESE APPEAR NATURALLY WHEN STUDYING NUMBER THEORY, REPRESENTATION THEORY, QUANTUM ERROR CORRECTION:

# HYPERSTRUCTURES IN THE WILD

ELSEVIER

Journal of Number Theory
Volume 131, Issue 2, February 2011, Pages 159-194



## The hyperring of adèle classes ☆

Alain Connes [a b c], Caterina Consani [d]

IDEA: DO ALGEBRAIC GEOMETRY OVER "$\mathbb{F}_1$" TOWARDS A PROOF OF THE RIEMANN HYPOTHESIS — THE KRASNER HYPERFIELD

$\mathbb{K} = \{0,1\}$ ⟷ $0+0=\{0\}$, $1+0=0+1=\{1\}$, $1+1=\{0,1\}$

ENDS UP BEING A GOOD CANDIDATE FOR "$\mathbb{F}_1$" IN THAT $\mathrm{Hom}_{\mathrm{HRing}}(X, \mathbb{K}) \cong \mathrm{Spec}\, X$.

# HYPERSTRUCTURES IN THE WILD

IDEA: DO REPRESENTATION THEORY OF COMPACT QUANTUM GROUPS — ANY GEL'FAND PAIR

$$(G, H), \quad H \leq G$$

GIVES RISE TO A COMMUTATIVE HYPERGROUP WHOSE UNDERLYING SET IS THE DOUBLE COSETS OF $H$ IN $G$.

LEONID VAINERMAN

**Hypergroup structures associated with Gel'fand pairs of compact quantum groups**

*Astérisque*, tome 232 (1995), p. 231-242

# HYPERSTRUCTURES IN THE WILD

IDEA: CHARACTERIZE CSS CODE STATES AS A CLASS OF MATROIDS

**Local equivalence, surface-code states, and matroids**

Pradeep Sarvepalli* and Robert Raussendorf

Show more

Phys. Rev. A **82**, 022304 – **Published 5 August, 2010**




Research Paper

Categories of hypermagmas, hypergroups, and related hyperstructures ☆

So Nakamura ✉, Manuel L. Reyes ✉

FACT: ANY MATROID CAN BE REALIZED CANONICALLY AS A MOSAIC.

# THE PROBLEM

HYPERSTRUCTURES ARE ALGEBRAIC STRUCTURES, SO IN A JUST WORLD, WE COULD REALIZE THEM AS "OBJECTS WITH STRUCTURE" IN SOME CAT.

# THE PROBLEM

HOWEVER: $X^n \xrightarrow{*} \mathcal{P}(X)$ IS NOT AN $n$-ARY OPERATION ON $X$!

SO, THESE ARE NOT "SETS WITH EXTRA STRUCTURE." (NOT IN THE SENSE OF GROUPS/RINGS/ETC.)

# THE PROBLEM

THE GOAL BECOMES: SHOW THAT

HMag, HSemi, HMon, Msc, HGrp

CAN BE REALIZED AS "OBJECTS WITH STRUCTURE" IN SOME OTHER CATEGORY.

# THE PROBLEM

TWO APPROACHES:

1. MINE! POSTED TO ARXIV AND GOT AN EMAIL FROM...

2. TONY AND FLORIAN!

NOW WE HAVE JOINED FORCES.



## THE PROBLEM

(1) MY APPROACH: MAYBE HYPERSTRUCTURES ARE "OBJECTS WITH STRUCTURE" IN CABA.

↑ COMPLETE ATOMIC BOOLEAN ALGEBRAS

REMINDER ON CABAs:

## THE PROBLEM

A CABA IS A COMPLETE DISTRIBUTIVE LATTICE WITH COMPLEMENTATION AND WHERE EVERY NON-BOTTOM ELEMENT LIES ABOVE SOME ATOM.

IN OTHER WORDS:

# THE PROBLEM

FACT. A CABA IS A SUPLATTICE ISOMORPHIC TO $\mathcal{P}(X)$ FOR SOME $X \in \mathbf{Set}$.

OK. BUT WHY WOULD YOU THINK HYPERSTRUCTURES ARE CABAs ?

# THE PROBLEM

**FACT.** FOR SETS $X, Y$,

SuplatMor. IN EACH VARIABLE

$$\mathrm{Hom}_{\mathrm{Set}}\left(Y^n, \mathcal{P}(X)\right) \cong \mathrm{Mult}_{\mathrm{Suplat}}\left(\mathcal{P}(Y)^n, \mathcal{P}(X)\right)$$

$$= \mathrm{Mult}_{\mathrm{CABA}}\left(\mathcal{P}(Y)^n, \mathcal{P}(X)\right).$$

i.e., THERE IS AN $n$-ARY HYPEROPERATION ON $X \in \mathrm{Set}$ $\Leftrightarrow$ THERE IS AN $n$-ARY OPERATION ON $\mathcal{P}(X) \in \mathrm{CABA}$.

# THE PROBLEM

(2) TONY'S AND FLORIAN'S APPROACH:

MAYBE HYPERSTRUCTURES ARE "OBJECTS WITH STRUCTURE" IN Rel.

WHY WOULD YOU THINK THAT?

# THE PROBLEM

**FACT.** BIALGEBRA OBJECTS IN THE KLEISLI CATEGORY $\mathsf{Set}_{\mathcal{P}^*}$ ARE EXACTLY MONOIDS.

($\mathcal{P}^*$: NONEMPTY POWERSET MONAD)

**FACT.** $\mathsf{Rel} \cong \mathsf{Set}_{\mathcal{P}}$. ← POSSIBLY-EMPTY POWERSET MONAD.

SO, WHAT DID Y'ALL PROVE?

## THE SOLUTION: CABAs

ON THE CABA SIDE:

**THEOREM.** (R.) THERE ARE BIJECTIONS

i) $\{$ MOSAIC STRUCTURES ON $X \in \mathrm{Set}\}$

$\Longleftrightarrow$

$\{$ NON-ASSOC. RELATION ALG. STRUCTURES ON $\mathcal{P}(X) \in \mathrm{CABA}\}$

(RELATION ALGEBRA: "UNITAL RESIDUATED" MAGMA IN CABA.)

## THE SOLUTION: CABAs

ON THE CABA SIDE:

**THEOREM.** (R.) THERE ARE BIJECTIONS

ii) $\{$ HGROUP STRUCTURES ON $X \in \mathrm{Set}\}$

$\Longleftrightarrow$

$\{$ RELATION ALG. STRUCTURES ON

$\mathcal{P}(X) \in \mathrm{CABA}\}$

# THE SOLUTION: CABAs

SIMILAR FOR HMag, HMon, etc.

THIS CORRESPONDENCE ON OBJECTS EXTENDS TO AN EQUIV. OF CAT's IN EACH CASE, RESTRICTING TO ATOM-PRESERVING MON's IN CABA.

# THE SOLUTION: CABAs

HEY, WHY DO YOU CARE SO MUCH ABOUT Msc?

FACT. Msc IS BICOMPLETE AND CLOSED SYM. MONOIDAL; $H\mathrm{Gr}_\infty$ IS NEITHER.

# THE SOLUTION: Rel

RECALL: Rel IS A DAGGER CAT.

**THEOREM.** (W., Z.) FOR A SET $H$, THERE IS A BIJECTION

$$\{ \text{HGROUP STRUCTURES ON } H \in \mathbf{Set} \}$$

$$\Longleftrightarrow$$

$$\{ \text{'TOTAL' DAGGER FROBENIUS ALG. STRUCTURES ON } H \in \mathbf{Rel} \}$$

(TOTAL: $\mu$, $\eta$ ARE TOTAL RELATIONS.)

# THE SOLUTION: Rel

'SPECIAL' CASE. GROUPS ARE

SPECIAL TOTAL DAGGER FROBENIUS

ALG's IN Rel.

(SPECIAL: $\mu \circ \mu^{+} = id$ )

WORK IN PROGRESS

CAN WE GENERALIZE THIS LATTER THEOREM? IF $\mathcal{E}$ IS A REGULAR CAT., WE CAN DEFINE $\mathrm{Rel}(\mathcal{E})$.

THERE SHOULD BE A NOTION OF 'HYPERGROUP INTERNAL TO $\mathcal{E}$.'

WORK IN PROGRESS

**CONJECTURE.** LET $\mathcal{E}$ BE A REGULAR CAT. THERE IS A BIJECTION

$$\{\mathcal{E}\text{-HGROUP STRUCTURES ON } H \in \mathcal{E}\}$$

$$\Longleftrightarrow$$

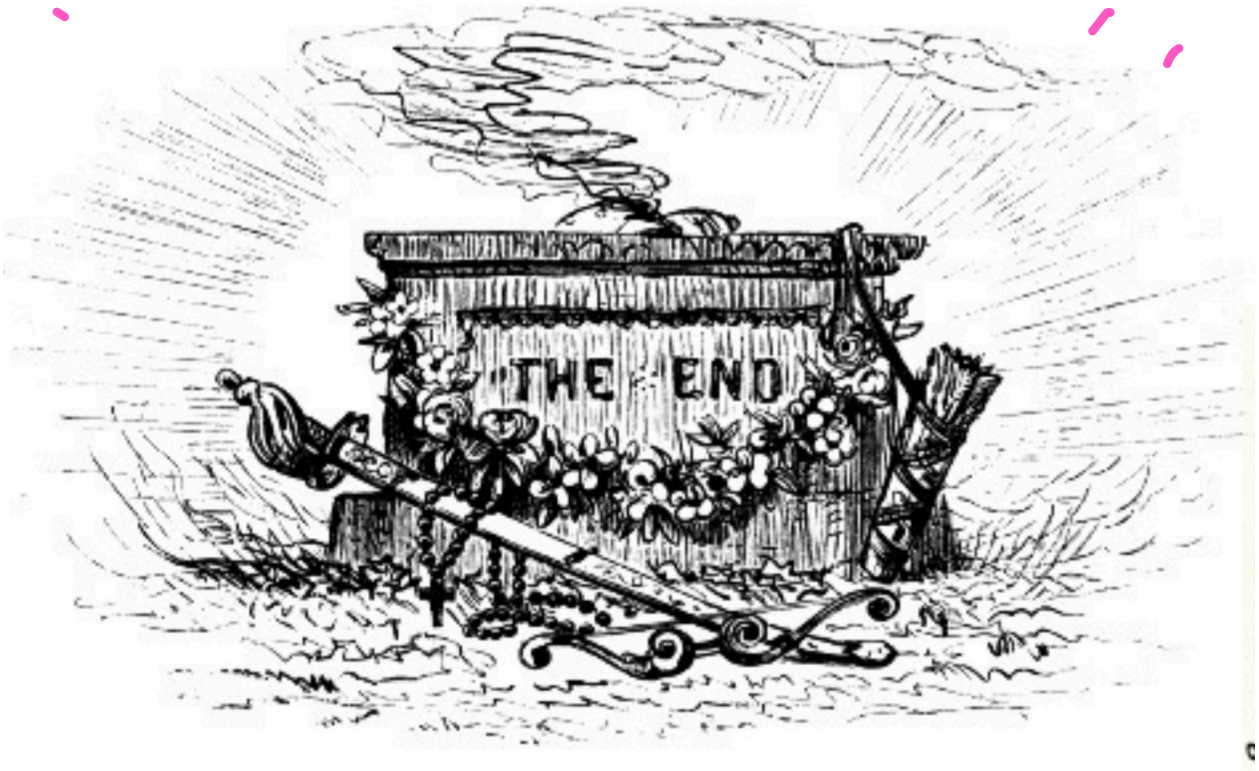
$$\{\text{'TOTAL' DAGGER FROBENIUS ALG. STRUCTURES ON } H \in \mathbf{Rel}(\mathcal{E})\}$$

(MODIFY NOTION OF 'TOTAL' APPROPRIATELY.)

WORK IN PROGRESS

**CONJECTURE.** THERE IS A VERSION OF THIS THEOREM FOR THE $\mathcal{E}$-ANALOGUE OF MOSAICS.

thanks ~ for ~ listening!




arXiv:
2607.26165

slides:
ARI-ROSENFIELD.
GITHUB.IO