






Paranatural Category Theory

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CT Octoberfest

($\mathbb{C}$ any category)

$$\varphi_I \colon \mathrm{Hom}_{\mathbb{C}}(I, I) \to \mathrm{Hom}_{\mathbb{C}}(I, I) \qquad \text{(for each object } I\text{)}$$

$$\begin{aligned} x &\mapsto \mathrm{id}_I \\ x &\mapsto x \\ x &\mapsto x \circ x \\ x &\mapsto x \circ x \circ x \\ &\vdots \\ \overline{n} \colon x &\mapsto x^n \end{aligned}$$

What *kind of thing* is $\bar{n}$?

($\mathbb{C}$ any category)

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Try #1: Natural Transformation

Are $\varphi_I : \mathrm{Hom}_{\mathbb{C}}(I, I) \to \mathrm{Hom}_{\mathbb{C}}(I, I)$ the components of a natural transformation between functors of shape $\mathbb{C} \to \mathrm{Set}$ (or $\mathbb{C}^{\mathrm{op}} \to \mathrm{Set}$)?

No. $\mathrm{Hom}_{\mathbb{C}} : \mathbb{C}^{\mathrm{op}} \times \mathbb{C} \to \mathrm{Set}$, so a natural transform $\psi : \mathrm{Hom}_{\mathbb{C}} \to \mathrm{Hom}_{\mathbb{C}}$ has components

$$\psi_{I,J} : \mathrm{Hom}_{\mathbb{C}}(I, J) \to \mathrm{Hom}_{\mathbb{C}}(I, J)$$

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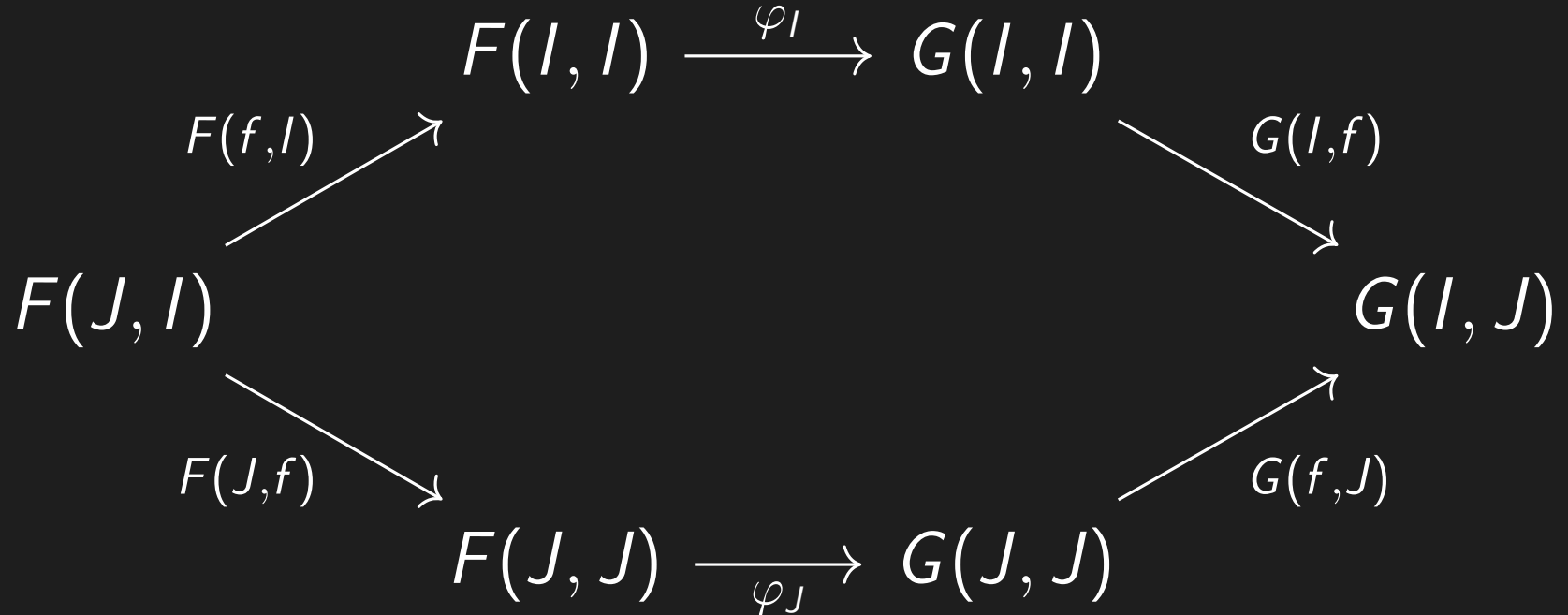
Defn We'll call functors of shape $\mathbb{C}^{\mathrm{op}} \times \mathbb{C} \to \mathrm{Set}$ **difunctors** on $\mathbb{C}$.

Defn Given difunctors F, G on $\mathbb{C}$, a **dinatural transformation** $\varphi\colon F \overset{\cdot\cdot}{\to} G$ is a family of maps

$$\varphi_I\colon F(I,I) \to G(I,I)$$

for each object I, such that for every morphism $f : \mathrm{Hom}(I,J)$

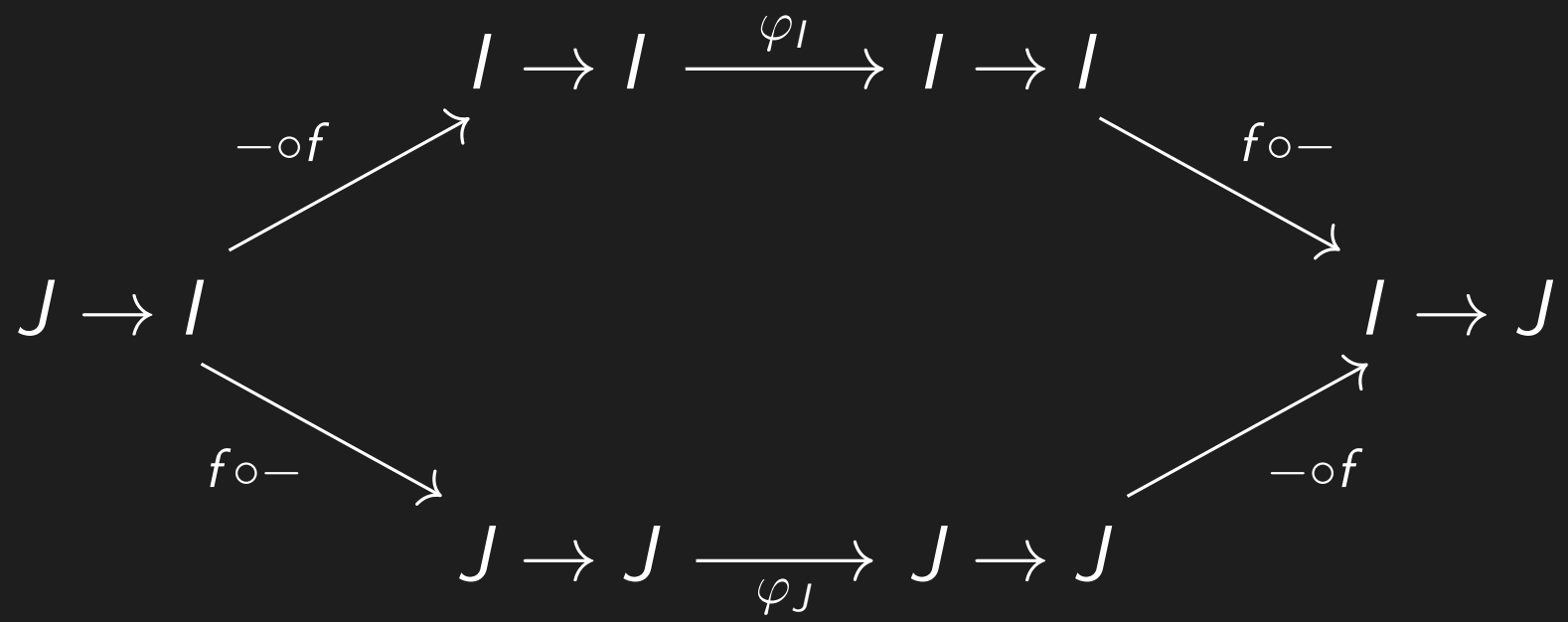


Good News Every $\overline{n}$ is indeed a dinatural transformation $\mathrm{Hom}_{\mathbb{C}} \overset{\cdot\cdot}{\to} \mathrm{Hom}_{\mathbb{C}}$

Bad News The dinaturality condition (the hexagon) is a weird property. E.g. take $\mathbb{C} = \mathsf{Set}$, $F = G = \mathrm{Hom}_{\mathsf{Set}}$; the hexagon diagram for $\varphi \colon \mathrm{Hom}_{\mathsf{Set}} \overset{\cdot\cdot}{\to} \mathrm{Hom}_{\mathsf{Set}}$ and $f : I \to J$ says

$$\text{for all } r \colon J \to I, \quad f \circ \varphi_I(r \circ f) = \varphi_J(f \circ r) \circ f.$$

(Spoiler: Not right!)


I → I
φ_I
I → I → I
−∘f
f∘−
J → I
I → J
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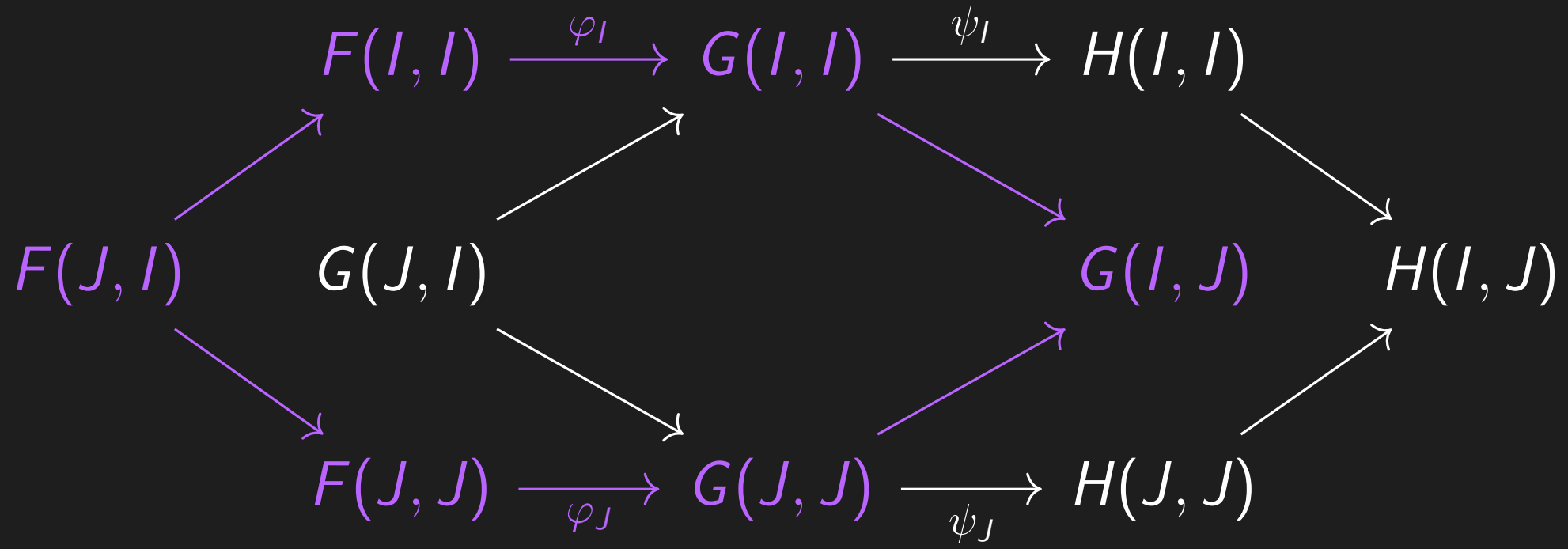
$$\varphi \colon F \dddot{\to} G \qquad \psi \colon G \dddot{\to} H$$

$$\{\psi_I \circ \varphi_I\}_{I \in \mathbb{C}} \colon F \dddot{\to} H?$$



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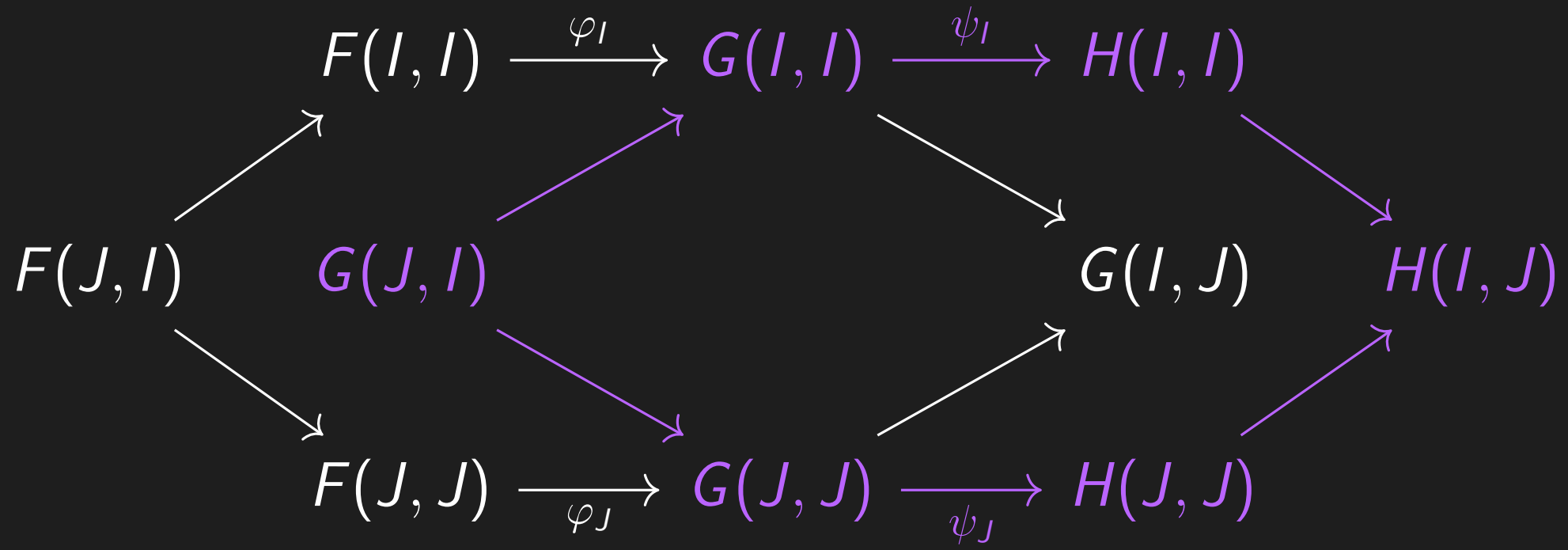

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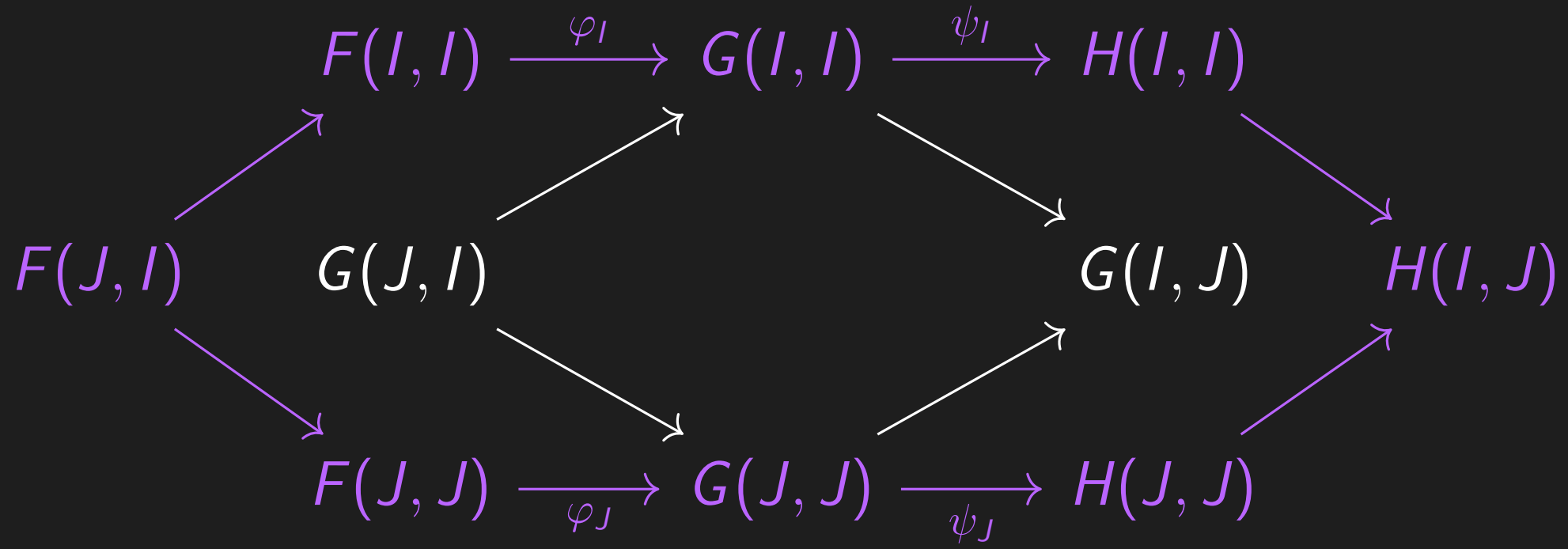

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Starting point of paranatural CT:

Dinaturals ain't it

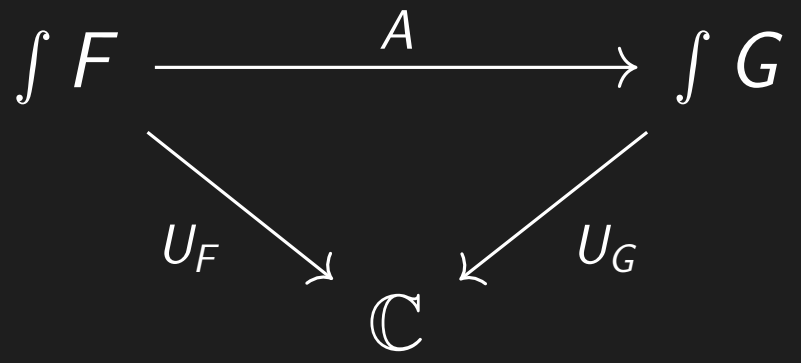
0

Strong Dinaturality

Naturality

Fact Given functors $F, G : \mathbb{C} \to \mathrm{Set}$, the following data are equivalent:

- Natural transformations $\alpha : F \to G$
- Functors $A : \int F \to \int G$ such that



commutes, where $U_F : \int F \to \mathbb{C}$ is the forgetful functor from $\int F$, the *category of elements* of F.

Applying this to difunctors

Now, if $F, G : \mathbb{C}^{\mathrm{op}} \times \mathbb{C} \to \mathrm{Set}$, if we take $\int F$ to be the category of elements of F *as a functor from* $\mathbb{C}^{\mathrm{op}} \times \mathbb{C}$ *to* Set, then we'll get that a transformation $F \to G$ is a natural transformation $\{\psi_{I,J} : F(I, J) \to G(I, J)\}_{I,J \in \mathbb{C}}$. Instead, consider:

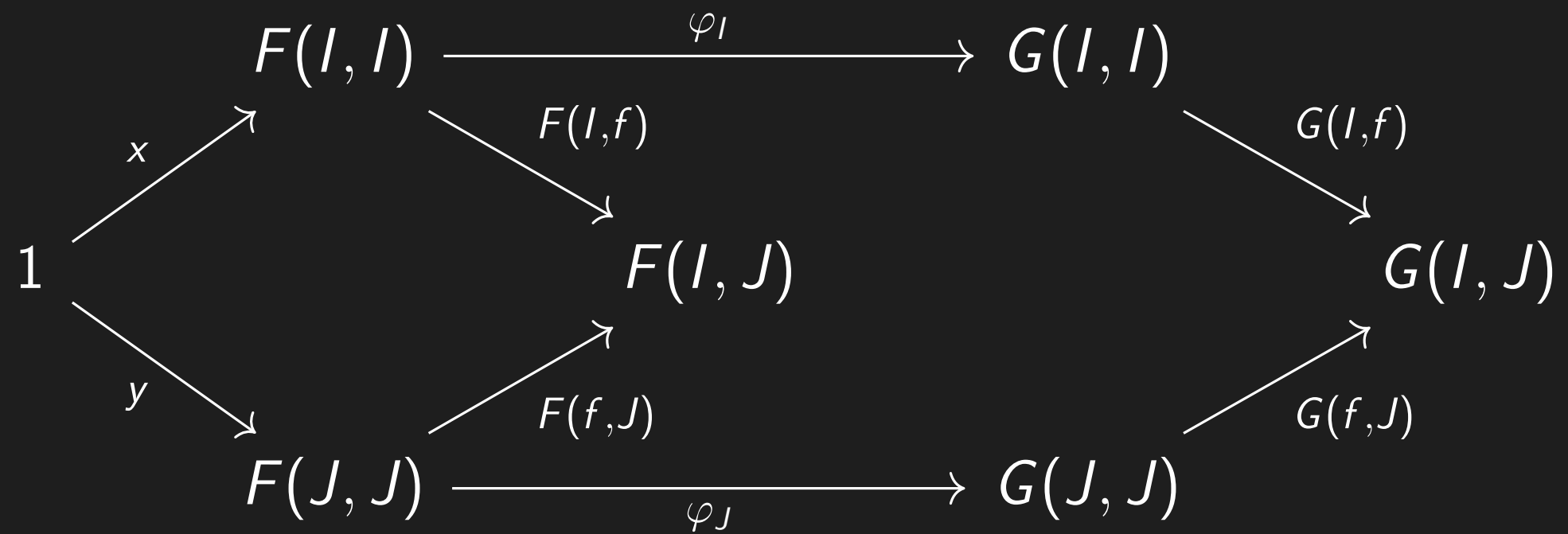
Defn For a difunctor F, the **diagonal category of elements** of F, denoted F-Struct, is the category

- whose objects are pairs (I, x), where I is an object of $\mathbb{C}$ and $x \in F(I, I)$
- whose morphisms $(I, x) \to (J, y)$ are $\mathbb{C}$-morphisms $f : \mathrm{Hom}_{\mathbb{C}}(I, J)$ such that

$$F(I, f)\, x = F(f, J)\, y.$$

Defn Between difunctors $F, G \colon \mathbb{C}^{\mathrm{op}} \times \mathbb{C} \to \mathrm{Set}$, a **strong dinatural transformation** $\varphi \colon F \overset{\diamond}{\to} G$ is a family of maps $\varphi_I \colon F(I,I) \to G(I,I)$ over objects I of $\mathbb{C}$, such that for all $f \in \mathrm{Hom}_{\mathbb{C}}(I,J)$, all $x \in F(I,I)$, $y \in F(J,J)$,

$$F(I,f)\,x = F(f,J)\,y \qquad \text{implies} \qquad G(I,f)\,(\varphi_I\,x) = G(f,J)\,(\varphi_J\,y)$$



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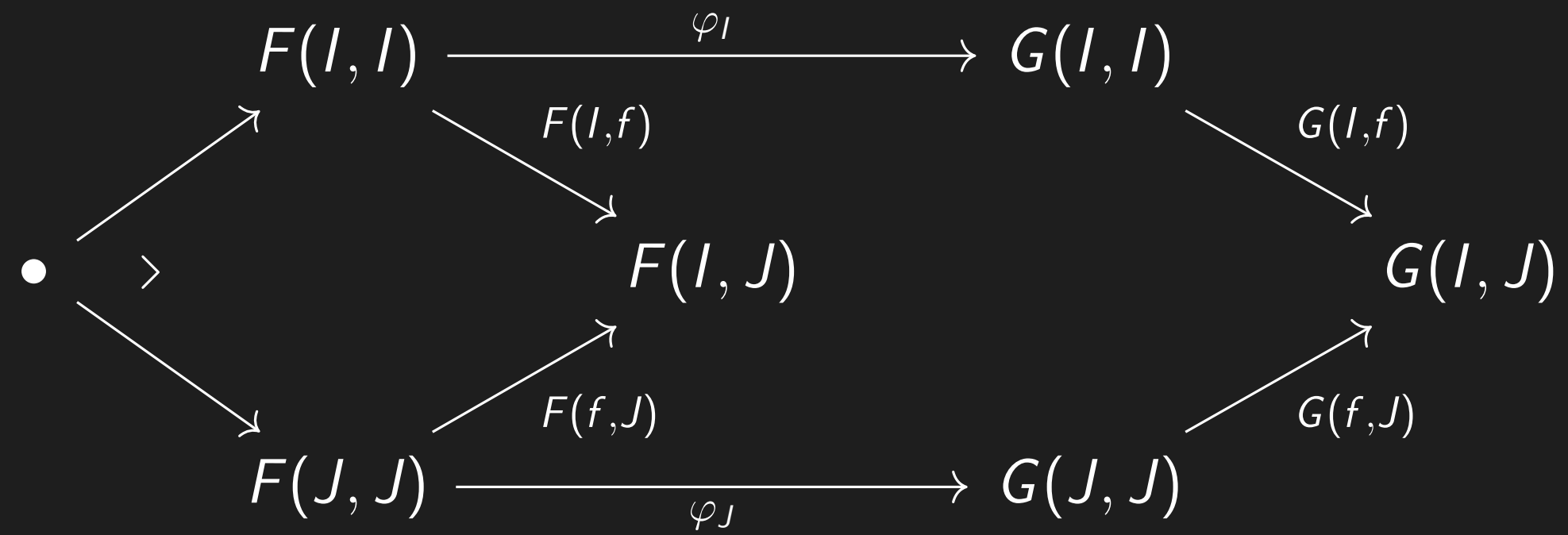


commutes.

What is a strong dinatural transformation $\varphi\colon \mathrm{Hom}_{\mathsf{Set}} \overset{\diamond}{\to} \mathrm{Hom}_{\mathsf{Set}}$?

A family of functions $\varphi_I\colon (I \to I) \to (I \to I)$ for each set I, such that, for all $f\colon I \to J$, $x\colon I \to I$, and $y\colon J \to J$,

$$f \circ x = y \circ f \qquad \text{implies} \qquad f \circ (\varphi_I\, x) = (\varphi_J\, y) \circ f$$

Prop [PR98, Proposition 1] Every strong dinatural transformation $\varphi\colon \mathrm{Hom}_{\mathsf{Set}} \overset{\diamond}{\to} \mathrm{Hom}_{\mathsf{Set}}$ is of the form $x \mapsto x^n$ for some $n \in \mathbb{N}$.

One tangent about notation

We copy [ML78]'s use of *integral notation* for *ends*, taking it even further:

write $\varphi\colon \int_{I:\mathbb{C}} F(I,I)\,\mathbf{d}\,G(I,I)$ for $\varphi\colon F \overset{\diamond}{\to} G$

$\int$trong $\mathbf{d}$inatural

One tangent about notation

We copy [ML78]'s use of *integral notation* for *ends*, taking it even further:

write $\varphi : \int_{I:\mathbb{C}} F(I,I)\,\mathbf{d}\,G(I,I)$ for $\varphi : F \overset{\diamond}{\to} G$

$$\mathbb{N} \cong \int_{I\,:\,\mathsf{Set}} (I \to I)\,\mathbf{d}\,(I \to I)$$

$$\emptyset \cong \int_{I\,:\,\mathsf{Set}} 1\,\mathbf{d}\,I$$

$$1 \cong \int_{I\,:\,\mathsf{Set}} I\,\mathbf{d}\,I$$

$$2 \cong \int_{I\,:\,\mathsf{Set}} I \times I\,\mathbf{d}\,I$$

Lemma [Uus10, Proposition 1] Let $T : \mathbb{C} \to \mathbb{C}$ be a functor with initial algebra $(\mu_T \in \mathbb{C}, \mathsf{in}_T \in \mathsf{Hom}_{\mathbb{C}}(T(\mu_T), \mu_T))$ and $K : \mathbb{C} \to \mathsf{Set}$ any functor. Then

$$K(\mu_T) \cong \int_{I\,:\,\mathbb{C}} \mathsf{Hom}_{\mathbb{C}}(T(I), I) \,\mathbf{d}\, K(I).$$

$$\mathbb{N} \cong \int_{I\,:\,\mathsf{Set}} (I \to I) \times I \,\mathbf{d}\, I$$

$$\mathtt{List}\ A \cong \int_{I\,:\,\mathsf{Set}} (I \times A \to I) \times I \,\mathbf{d}\, I$$

So what's the catch?

[Neu23] (old version of [Neu26]) asserts that the category $\mathrm{Difun}(\mathbb{C})$ of difunctors on $\mathbb{C}$ and strong dinatural transformations is a cartesian closed category. Wrong!

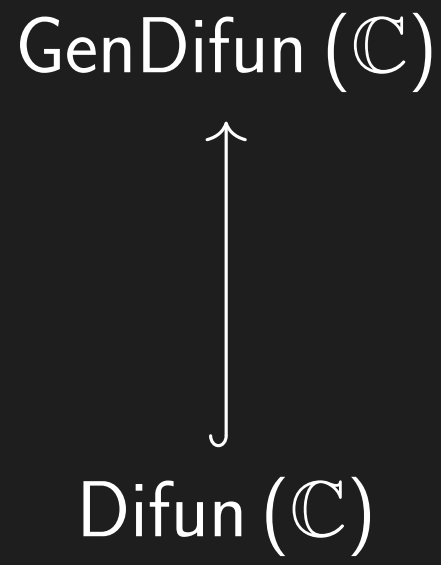
[nLa26] claims that [Uus10] proves that $\mathrm{Difun}(\mathbb{C})$ is *not* a CCC. Probably right!

Not clear how to understand e.g.

$$\int_{I\,:\,\mathrm{Set}} ((I \to I) \to I)\,\mathbf{d}\,G(I, I).$$

1
Paranaturality

General picture



Consider: How do we actually use difunctors?

Defn A **generalized difunctor** $\mathbb{D}$ over $\mathbb{C}$ consists of

- a set $\mathbb{D}_I$ for each object I of $\mathbb{C}$
- for each $f : \mathrm{Hom}_{\mathbb{C}}(I, J)$, a relation $\mathbb{D}_f \subseteq \mathbb{D}_I \times \mathbb{D}_J$

such that $(x, x) \in \mathbb{D}_{\mathrm{id}_I}$ for every $x \in \mathbb{D}_I$.

Defn For any $F : \mathbb{C}^{\mathrm{op}} \times \mathbb{C} \to \mathrm{Set}$, view F as a *generalized* difunctor via

- $F_I := F(I, I)$
- $(x, y) \in F_f$ just in case $F(I, f)\, x = F(f, J)\, y$

Defn Between generalized difunctors $\mathbb{D}, \mathbb{E}$, a **paranatural transformation** $\alpha : \mathbb{D} \xrightarrow{\diamond} \mathbb{E}$ is a map

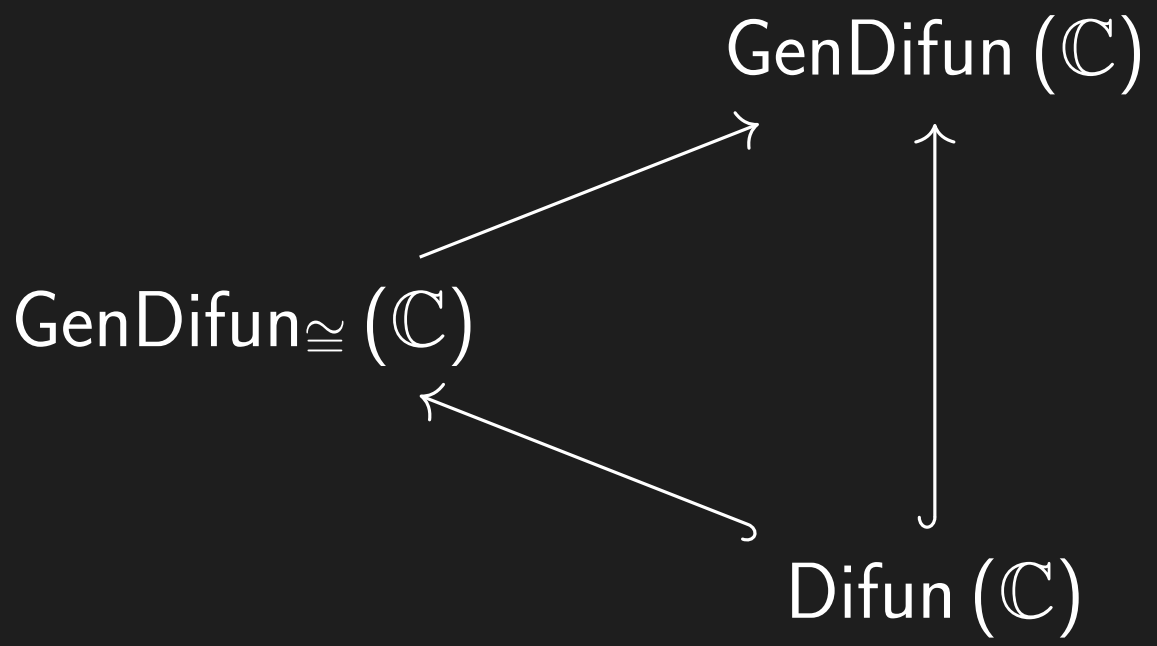
$$\alpha_I : \mathbb{D}_I \to \mathbb{E}_I$$

for each I, such that $(x, y) \in \mathbb{D}_f$ implies $(\alpha_I x, \alpha_J y) \in \mathbb{E}_f$.

Prop For generalized difunctors $\mathbb{D}, \mathbb{E}$ over $\mathbb{C}$, there is a generalized difunctor $\mathbb{E}^{\mathbb{D}}$ and paranatural transformation $\epsilon_{\mathbb{D},\mathbb{E}} \colon \mathbb{E}^{\mathbb{D}} \times \mathbb{D} \xrightarrow{\diamond} \mathbb{E}$ satisfying the universal mapping property of the exponential.

2
Isofibrant Paranaturality

General picture



Defn An **isofibrant generalized difunctor** $\mathbb{D}$ over $\mathbb{C}$ consists of

- a set $\mathbb{D}_I$ for each object I of $\mathbb{C}$
- for each $f : \mathrm{Hom}_{\mathbb{C}}(I, J)$, a relation $\mathbb{D}_f \subseteq \mathbb{D}_I \times \mathbb{D}_J$
- for each iso $e : I \cong J$, a function $\mathrm{coe}_{\mathbb{D}} : F(I, I) \to F(J, J)$

such that

- $(x, x) \in \mathbb{D}_{\mathrm{id}_I}$ for every $x \in \mathbb{D}_I$;
- $\mathrm{coe}_{\mathbb{D}}(\mathrm{id}, x) = x$ and $\mathrm{coe}_{\mathbb{D}}(e \circ d, x) = \mathrm{coe}_{\mathbb{D}}(e, \mathrm{coe}_{\mathbb{D}}(d, x))$
- $(x, \mathrm{coe}_{\mathbb{D}}(e, x)) \in \mathbb{D}_e$

Defn For any $F : \mathbb{C}^{\mathrm{op}} \times \mathbb{C} \to \mathrm{Set}$, view F as a *generalized* difunctor via

- $F_I := F(I, I)$
- $(x, y) \in F_f$ just in case $F(I, f)\ x = F(f, J)\ y$
- define coe_F: for $e : I \cong J$

Results [Neu26]

- (di)Yoneda Lemma
- Exponentials
- Generalization of Uustalu-Yoneda Lemma

[ML78] Saunders Mac Lane.
Categories for the Working Mathematician.
Springer New York, 1978.

[Neu23] Jacob Neumann.
Paranatural category theory.
https://arxiv.org/abs/2307.09289v1, 2023.

[Neu26] Jacob Neumann.
Paranatural category theory.
https://arxiv.org/abs/2307.09289, 2026.

[nLa26] nLab authors.
strong dinatural transformation.
https://ncatlab.org/nlab/show/strong+dinatural+transformation, jun 2026.
Revision 10.

[PR98] Robert Paré and Leopoldo Román.
Dinatural numbers.
Journal of Pure and Applied Algebra, 128(1):33–92, 1998.

[Uus10] Tarmo Uustalu.
A note on strong dinaturality, initial algebras and uniform parameterized fixpoint operators.
In *FICS*, pages 77–82, 2010.