Crossed Tiloids and Graded Monoidal Categories

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Part One: Definition and Examples of Crossed Tiloids

A *tiloid* is a strict double catgory with exactly one object (0-cell).

Explicitly, a the data of a tiloid $\mathbb{X}$ is:

- monoids $(\mathbb{X}_H, |, I)$ and $(\mathbb{X}_V, \cdot, I)$.
- for all $A, B \in \mathbb{X}_H$ and $U, W \in \mathbb{X}_V$, a *cell-set* $\mathbb{X}(U \begin{smallmatrix} A \\ B \end{smallmatrix} W)$.
- horizontal and vertical composition operators:

$$(- \mid -) : \mathbb{X}\left(U \begin{smallmatrix} A \\ B \end{smallmatrix} V\right) \times \mathbb{X}\left(V \begin{smallmatrix} A' \\ B' \end{smallmatrix} W\right) \to \mathbb{X}\left(U \begin{smallmatrix} A|A' \\ B|B' \end{smallmatrix} W\right)$$

  and

$$(- \cdot -) : \mathbb{X}\left(U \begin{smallmatrix} A \\ B \end{smallmatrix} W\right) \times \mathbb{X}\left(U' \begin{smallmatrix} B \\ C \end{smallmatrix} W'\right) \to \mathbb{X}\left(U \cdot U' \begin{smallmatrix} A \\ C \end{smallmatrix} W \cdot W'\right)$$

- horizontal and vertical identities:

$$id_U \in \mathbb{X}\left(U \begin{smallmatrix} I \\ I \end{smallmatrix} U\right) \qquad \text{and} \qquad 1_A \in \mathbb{X}\left(I \begin{smallmatrix} A \\ A \end{smallmatrix} I\right)$$

This data must satsify a number of axioms:

Horizontal and vertical composition are associative and unital.

$$(a \mid b) \mid c = a \mid (b \mid c) \qquad a \mid id_W = a = id_U \mid a$$

$$(a \cdot b) \cdot c = a \cdot (b \cdot c) \qquad a \cdot 1_B = a = 1_A \cdot a$$

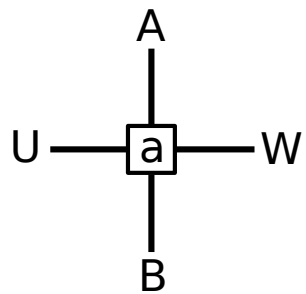
The following coherence conditions are satisfied:

$$(a \cdot b) \mid (c \cdot d) = (a \mid c) \cdot (b \mid d) \qquad 1_I = id_I$$

$$1_{A|B} = 1_A \mid 1_B \qquad id_{U \cdot W} = id_U \cdot id_W$$

Tiloids admit a calculus of string diagrams:

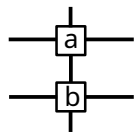
$$a \in \mathbb{X}\begin{pmatrix} & A & \\ U & & W \\ & B & \end{pmatrix} \qquad \leftrightsquigarrow \qquad$$

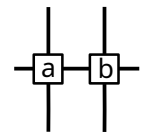


$\leftrightsquigarrow$

$$\begin{array}{ccc} * & \xrightarrow{A} & * \\ {\scriptstyle U}\downarrow & a & \downarrow{\scriptstyle W} \\ * & \xrightarrow[B]{} & * \end{array}$$

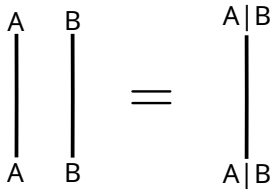
Composition and identities are depicted as in:

$a \cdot b \leftrightsquigarrow$ 

$1_A \leftrightsquigarrow$
$a \mid b \leftrightsquigarrow$ 

$id_U \leftrightsquigarrow$

The equations ensure the diagrams make sense. Also coherences:



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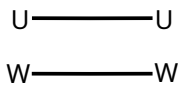

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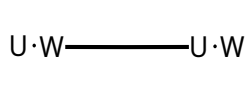

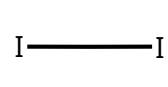

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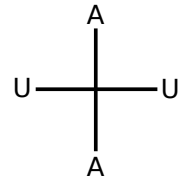
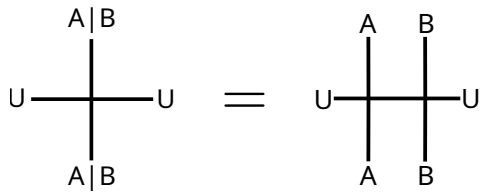
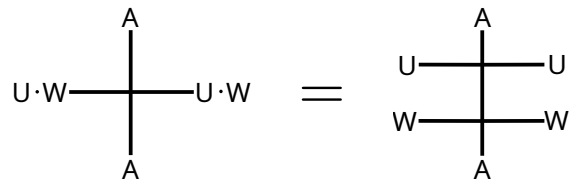
I'm interested in tiloids with *crossing cells* ("crossed tiloids"):

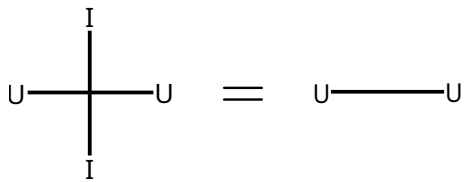
$$\chi_{A,U} \in \mathbb{X}\begin{pmatrix} & A & \\ U & & U \\ & A & \end{pmatrix} \qquad \leftrightsquigarrow \qquad$$

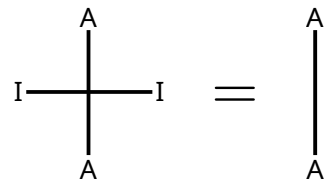


for all $A \in \mathbb{X}_H$ and $U \in \mathbb{X}_V$, which are coherent:

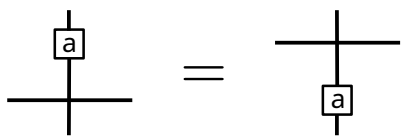


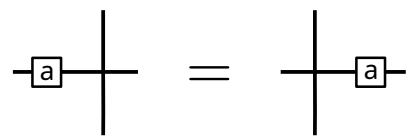






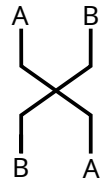
and appropriately natural:



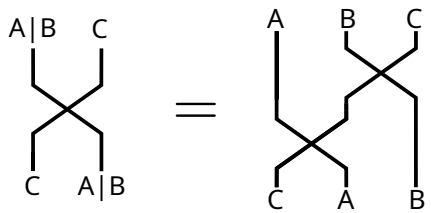


Crossed tiloids are often *vertically symmetric*:

$$\sigma^V_{A,B} \in \mathbb{X}\begin{pmatrix} & A|B & \\ I & & I \\ & B|A & \end{pmatrix} \qquad \leftrightsquigarrow \qquad$$



for all $A, B \in \mathbb{X}_H$, which are coherent as in:



and satisfy:



Transposed, this yields a definition of *horizontally symmetric* crossed tiloid, with cells $\sigma^H_{U,W} \in \mathbb{X}(_{U\cdot W}{}^{I}_{I}{}_{W\cdot U})$.

A crossed tiliod is *doubly symmetric* in case it is both.

Example: Let $\mathbb{C}$ and $\mathbb{D}$ be (strict) monoidal categories.

Their *perpendicular product* $\mathbb{C} \boxtimes \mathbb{D}$ is the tiloid with:

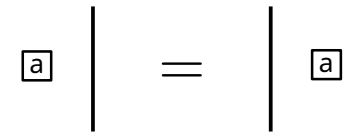
- $(\mathbb{C} \boxtimes \mathbb{D})_H = (\mathbb{C}_0, \otimes, I)$ and $(\mathbb{C} \boxtimes \mathbb{D})_V = (\mathbb{D}_0, \otimes, I)$.
- $(a_0, a_1) \in (\mathbb{C} \boxtimes \mathbb{D})(U \begin{smallmatrix} A \\ B \end{smallmatrix} W)$ is $a_0 \in \mathbb{C}(A, B)$ and $a_1 \in \mathbb{D}(U, W)$.
- vertical and horizontal composition/identities as in:

$$(a_0, a_1) \cdot (b_0, b_1) = (a_0 b_0, a_1 \otimes b_1) \qquad 1_A = (1_A, 1_I)$$

$$(a_0, a_1) \mid (b_0, b_1) = (a_0 \otimes b_0, a_1 b_1) \qquad id_U = (1_I, 1_U)$$

Recall that a monoidal category $\mathbb{C}$ is *spatial* in case:

$$a \otimes 1_A = 1_A \otimes a \qquad \leftrightsquigarrow \qquad$$



for all $a \in \mathbb{C}(I, I)$. Every symmetric monoidal category is spatial.

We have:

- If $\mathbb{C}$ and $\mathbb{D}$ are spatial then $\mathbb{C} \boxtimes \mathbb{D}$ is crossed with:

$$\chi_{A,U} = (1_A, 1_U)$$

- If $\mathbb{C}$ is in fact symmetric then $\mathbb{C} \boxtimes \mathbb{D}$ is vertically symmetric:

$$\sigma^V_{A,B} = (\sigma_{A,B}, 1_I)$$

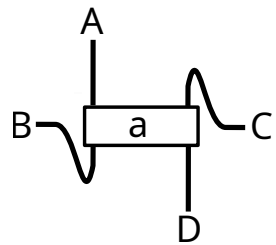
- If $\mathbb{D}$ is in fact symmetric then $\mathbb{C} \boxtimes \mathbb{D}$ is horizontally symmetric:

$$\sigma^H_{U,W} = (1_I, \sigma_{U,W})$$

Example: Let $\mathbb{C}$ be a (strict) monoidal category.

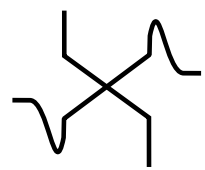
The tiloid $Q(\mathbb{C})$ of *quintets over* $\mathbb{C}$ is defined as in:

- $Q(\mathbb{C})_H = Q(\mathbb{C})_V = (\mathbb{C}_0, \otimes, I)$.
- a cell $(A, B, C, D, a) \in Q(\mathbb{C})({}_B{}^A_D C)$ is given by objects $A, B, C, D \in \mathbb{C}_0$ and $a \in \mathbb{C}(A \otimes C, B \otimes D)$, as in:



- vertical/horizontal composition/identities as suggested by this.

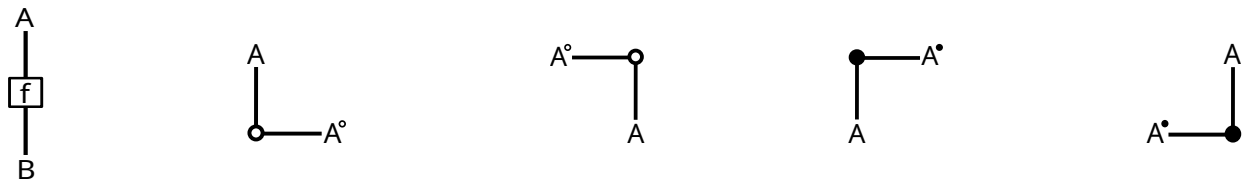
If $\mathbb{C}$ is symmetric then $Q(\mathbb{C})$ is crossed and doubly symmetric:

$\chi_{A,B}$ $\quad\leftrightsquigarrow\quad$ 

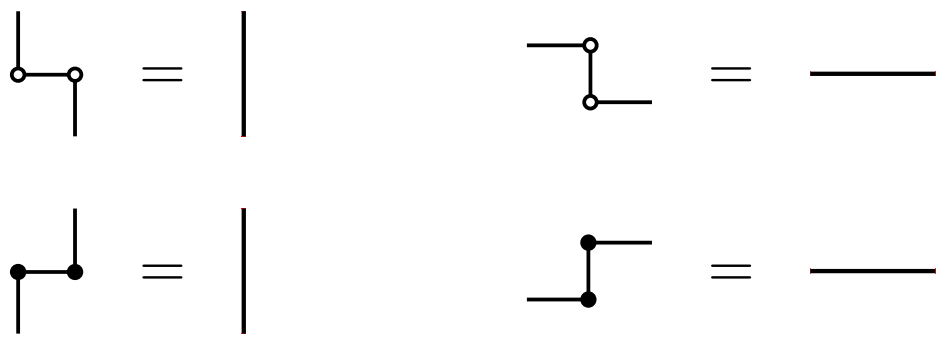
Example: Let $\mathbb{C}$ be a (strict) monoidal category.

The *free cornering* $[\mathbb{C}]$ of $\mathbb{C}$ is the tiloid defined by:

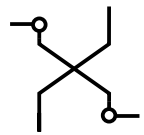
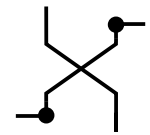
- $[\mathbb{C}]_H = (\mathbb{C}_0, \otimes, I)$
- $[\mathbb{C}]_V = (\mathbb{C}_0 \times \{\circ, \bullet\})^*$. E.g., $A^\bullet \cdot B^\circ \cdot (A \otimes B)^\bullet \in [\mathbb{C}]_V$.
- Cells are generated by (for all $f \in \mathbb{C}(A, B)$ and $A \in \mathbb{C}_0$):



Satisfying a number of equations, in particular:

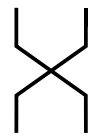


If $\mathbb{C}$ is symmetric, then $[\mathbb{C}]$ is crossed, as in:

χ_{A,B° $\leftrightsquigarrow$  $\chi_{A,B^\bullet}$ $\leftrightsquigarrow$ 

with $\chi_{A,I} = 1_A$ and $\chi_{A,U\cdot W} = \chi_{A,U} \cdot \chi_{A,W}$

$[\mathbb{C}]$ is moreover vertically symmetric, as in:

$\sigma^V_{A,B}$ $\leftrightsquigarrow$ 

We have one more interesting example, built from functors with tensorial strength, but it is more involved and I won't present it.

Part Two: Connection to Graded Monoidal Categories

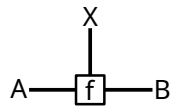
Let $\mathbb{V}$ be a (strict) monoidal category.

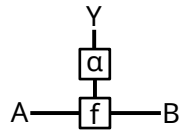
A $\mathbb{V}$-*graded category* $\mathbb{C}$ consists of:

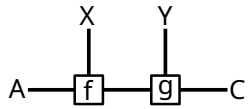
- a set C_0 of *objects*.
- for each $A, B \in \mathbb{C}_0$ and $X \in \mathbb{V}_0$ a set $\mathbb{C}_X(A, B)$ of *morphisms* with domain A, codomain B, and *grade* X.
- for each $\alpha \in \mathbb{V}(Y, X)$, and $f \in \mathbb{C}_X(A, B)$, a morphism $\alpha^*(f) \in \mathbb{C}_Y(A, B)$ called the *regrading of* f *along* α.
- for each $f \in \mathbb{C}_X(A, B)$ and $g \in \mathbb{C}_Y(B, C)$, a *composite* $fg \in \mathbb{C}_{X \otimes Y}(A, C)$.
- *identities* $1_A = \mathbb{C}_I(A, A)$ for each $A \in \mathbb{C}_0$.

Satisfying sensible axioms.

Consider:

$f \in \mathbb{C}_X(A, B)$ $\leftrightsquigarrow$ 

$\alpha^*(f)$ $\leftrightsquigarrow$ 

fg $\leftrightsquigarrow$ 

1_A $\leftrightsquigarrow$ 

In this way, every tiloid has an underlying graded category.

We obtain an adjunction:

$$\mathsf{G} \underset{R}{\overset{L}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathsf{T}$$

where T is the category of tiloids and tiloid functors, and G is a category of graded monoidal categories.

Let $\mathbb{V}$ be a *symmetric* (strict) monoidal category.

A $\mathbb{V}$-*graded monoidal category* $\mathbb{C}$ consists of:

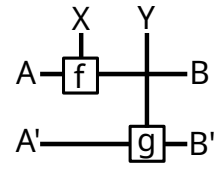
- a *monoid* $(\mathbb{C}_0, \otimes, I)$ of objects.
- the rest of the data of a $\mathbb{V}$-graded category.
- for each $f \in \mathbb{C}_X(A, B)$ and $g \in \mathbb{C}_Y(A', B')$, a *tensor product* $f \otimes g \in \mathbb{C}_{X \otimes Y}(A \otimes A', B \otimes B')$

satisfying sensible axioms.

This definition appears in a recent paper by Sarkis and Zanasi.

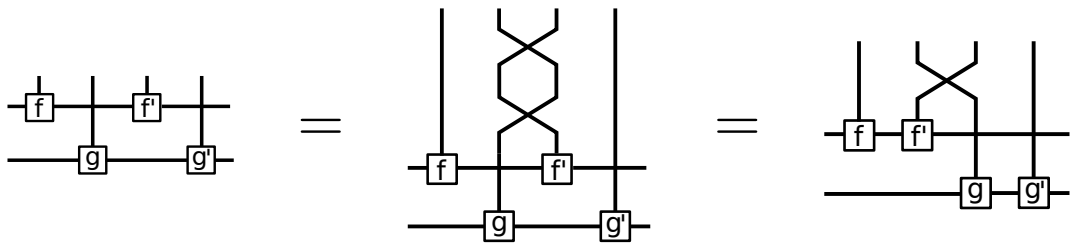
Consider:

$$f \otimes g \qquad \leftrightsquigarrow$$



In this way, every vertically symmetric crossed tiloid has an underlying graded monoidal category.

The vertical braiding is required for one of the axioms:



There is again an adjunction $L : \mathsf{GM} \dashv \mathsf{VCT} : R$.

A $\mathbb{V}$-graded monoidal category $\mathbb{C}$ is *symmetric* in case it has:

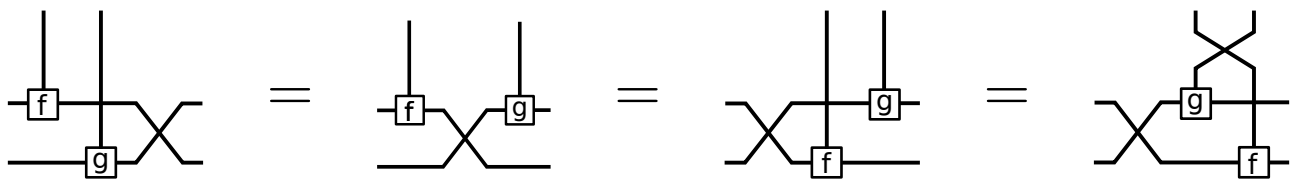
$$\sigma_{A,B} \in \mathbb{C}_X(A \otimes B, B \otimes A) \quad \leftrightsquigarrow \quad$$



for all $A, B \in \mathbb{C}_0$, satisfying sensible axioms.

Every doubly symmetric crossed tiloid has an underlying graded symmetric monoidal category.

The only interesting axiom holds as in:



We again obtain an adjunction $L : \mathsf{GSM} \dashv \mathsf{DCT} : R$.

Thanks for listening!

Preprint coming soon!

References:

Graded String Diagrams for Imprecise Probability and Causal Intervention. Sarkis and Zanasi, 2026.

Protocol Choice and Iteration for the Free Cornering. Nester and Voorneveld, 2024.