Bistellar Data in Arbitrary Dimension

Aleksandar Ivanov

University of Vienna

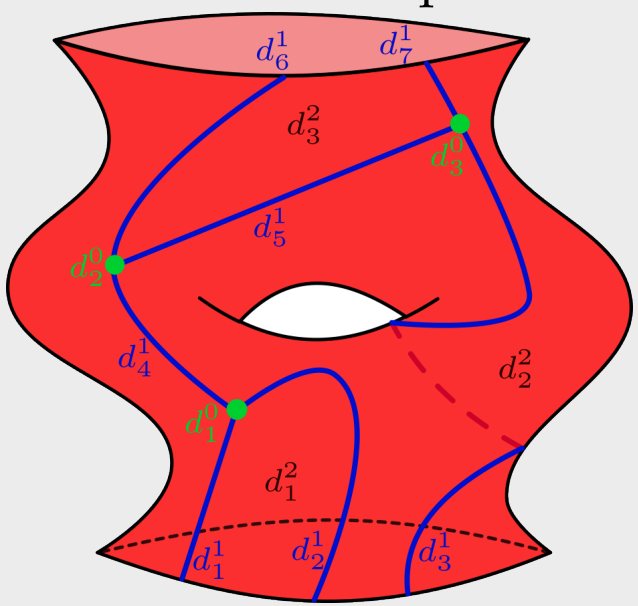
Category Theory October/Novemberfest, 31.10.2026

Motivation

Orbifold Construction

[Fröhlich, Fuchs, Runkel, Schweigert, ...]

Bordism Aspect

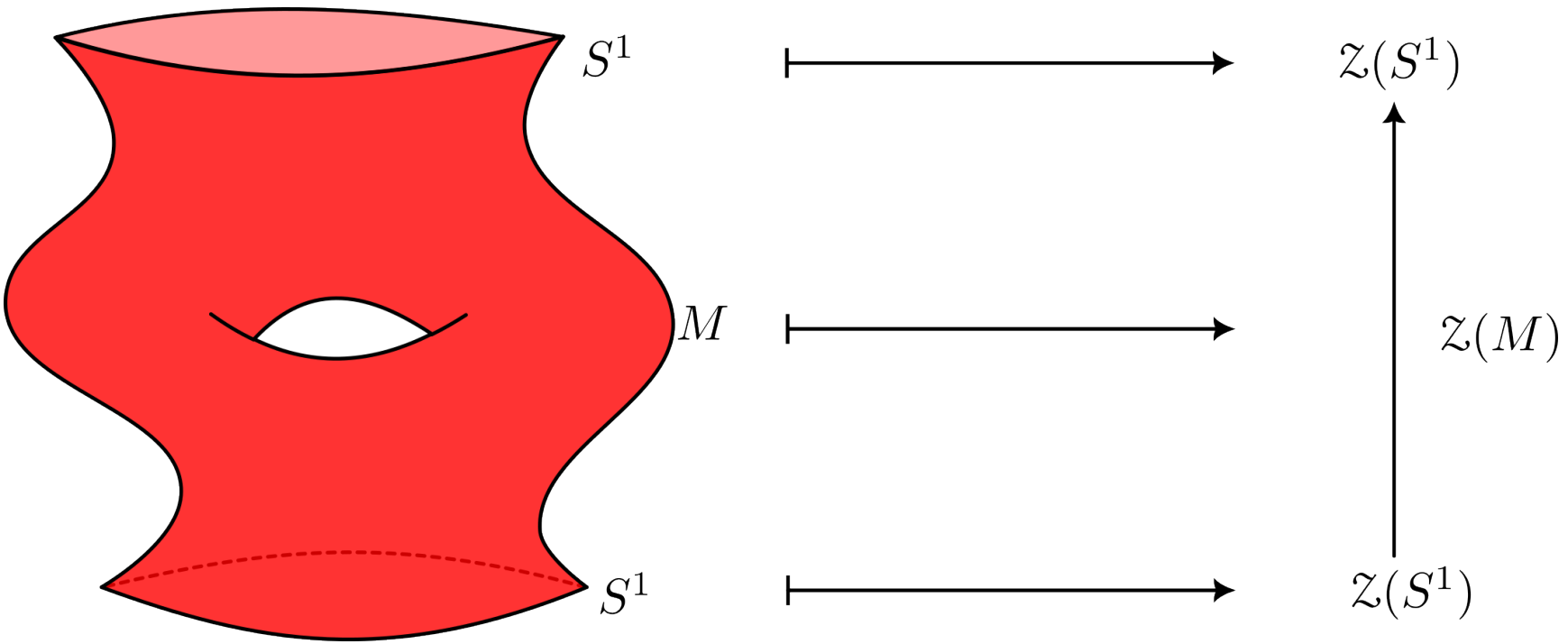


Categorical Aspect

$$\begin{tikzcd} \mathcal{C} \arrow[rr, hook] \arrow[dr, hook] & & \mathsf{Mor}_{\mathcal{P}}(\mathcal{C}) \arrow[dl, hook, dashed] \\ & \mathcal{T}\mathsf{QFT}_n & \end{tikzcd}$$

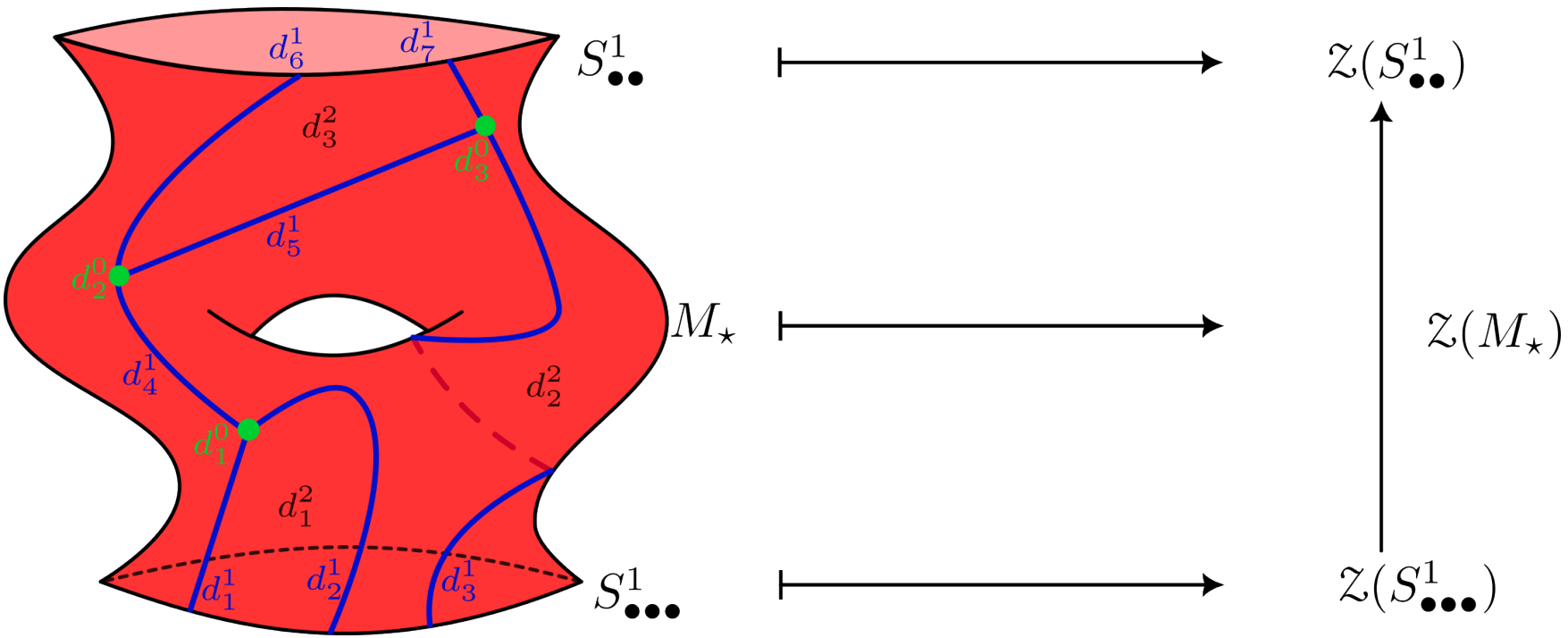
Defect TQFTs

$$\mathcal{Z} : \mathcal{B}\mathsf{ord}^{\mathrm{or}}_{n,n-1} \longrightarrow \mathcal{V}\mathsf{ect}_{\Bbbk}$$



Defect TQFTs

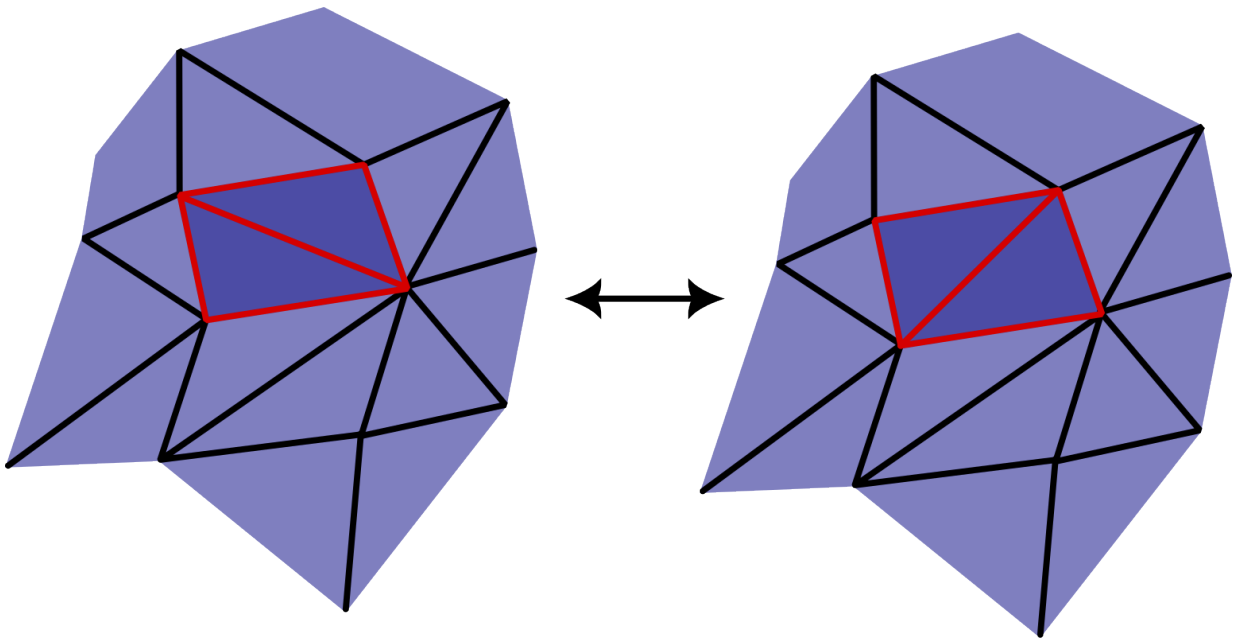
$$\bar{\mathcal{Z}} : \mathcal{B}\mathsf{ord}^{\mathrm{or}}_{n,n-1}(\mathbb{D}) \longrightarrow \mathcal{V}\mathsf{ect}_{\Bbbk}$$



The Orbifold Construction

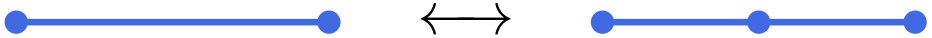
Theorem (Pachner)

One can turn any triangulation of a manifold into any other by some sequence of local moves chosen from a finite collection.



The Orbifold Construction

- in dimension 1:

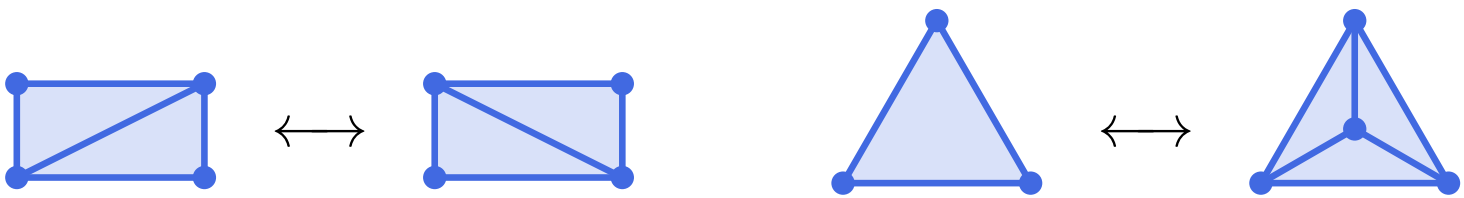


The Orbifold Construction

- in dimension 1:



- in dimension 2:



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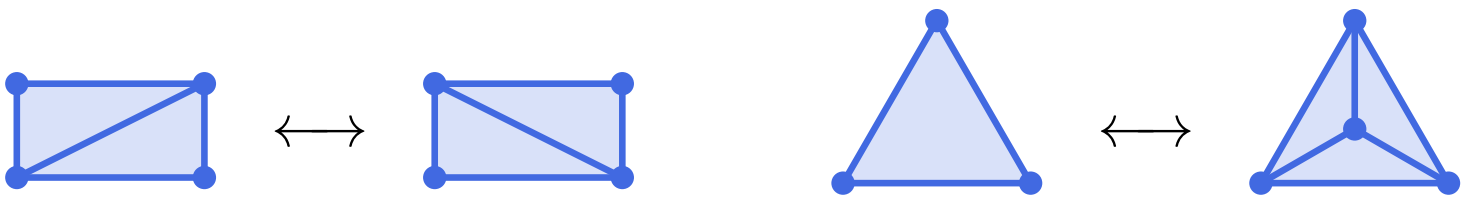

Alternatively, we have the Poincaré dual pictures.

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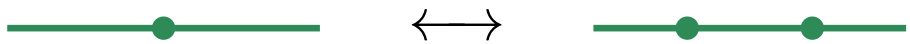
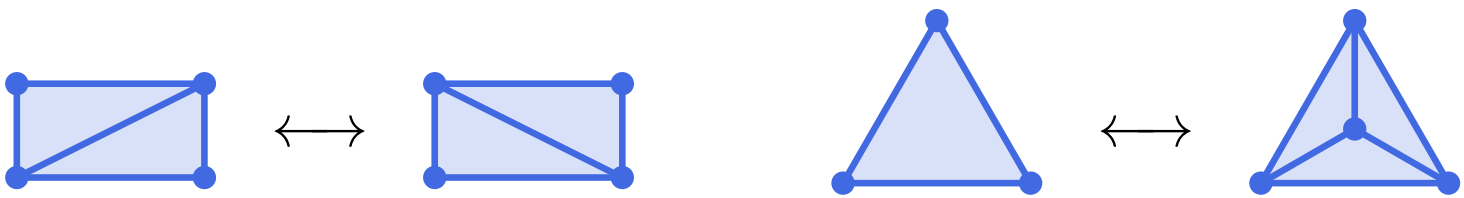


- in dimension 2:



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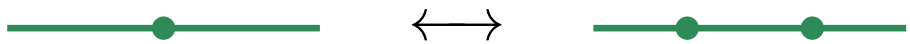

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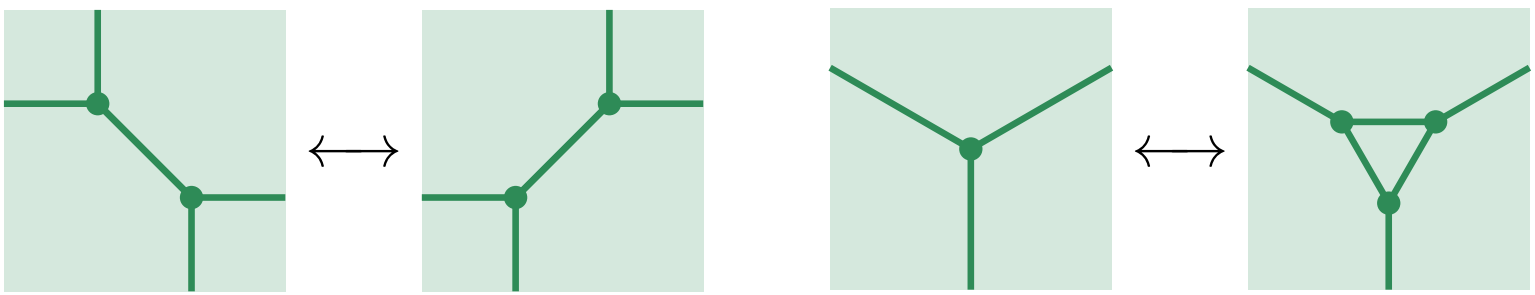

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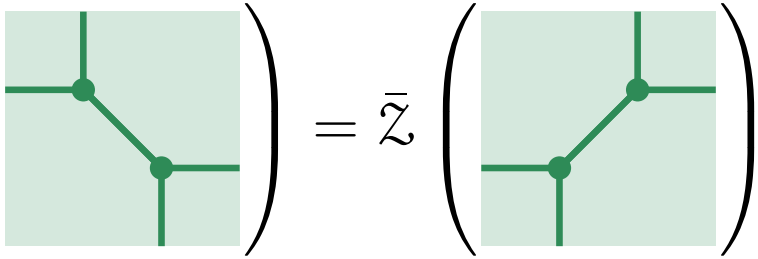
The Orbifold Construction

Theorem (Pachner)

One can turn any triangulation of a manifold into any other by some sequence of local moves chosen from a finite collection.

If we can find a subcollection $\mathcal{A}$ of the defect data of a defect TQFT $\bar{\mathcal{Z}}$ which:

- can label the Poincaré duals of all triangulations, and
- is invariant under all dual bistellar moves:

e.g. $\bar{\mathcal{Z}}\left(\right) = \bar{\mathcal{Z}}\left(\right)$ 

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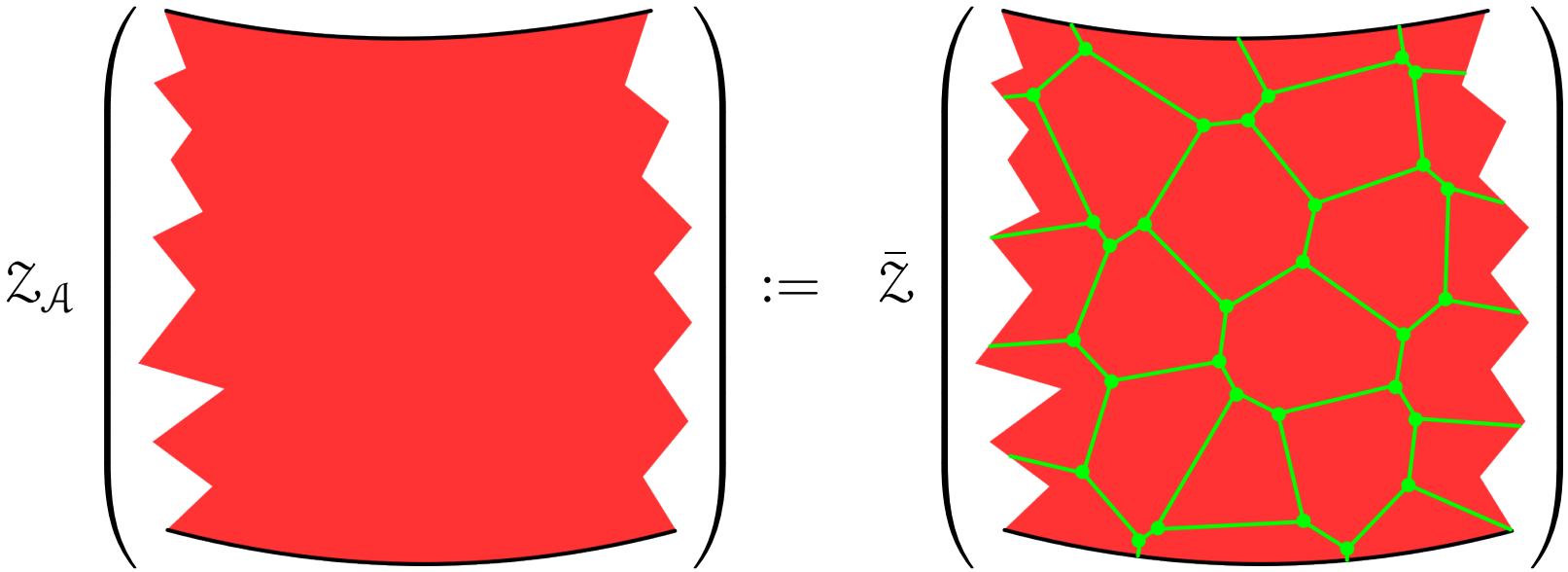

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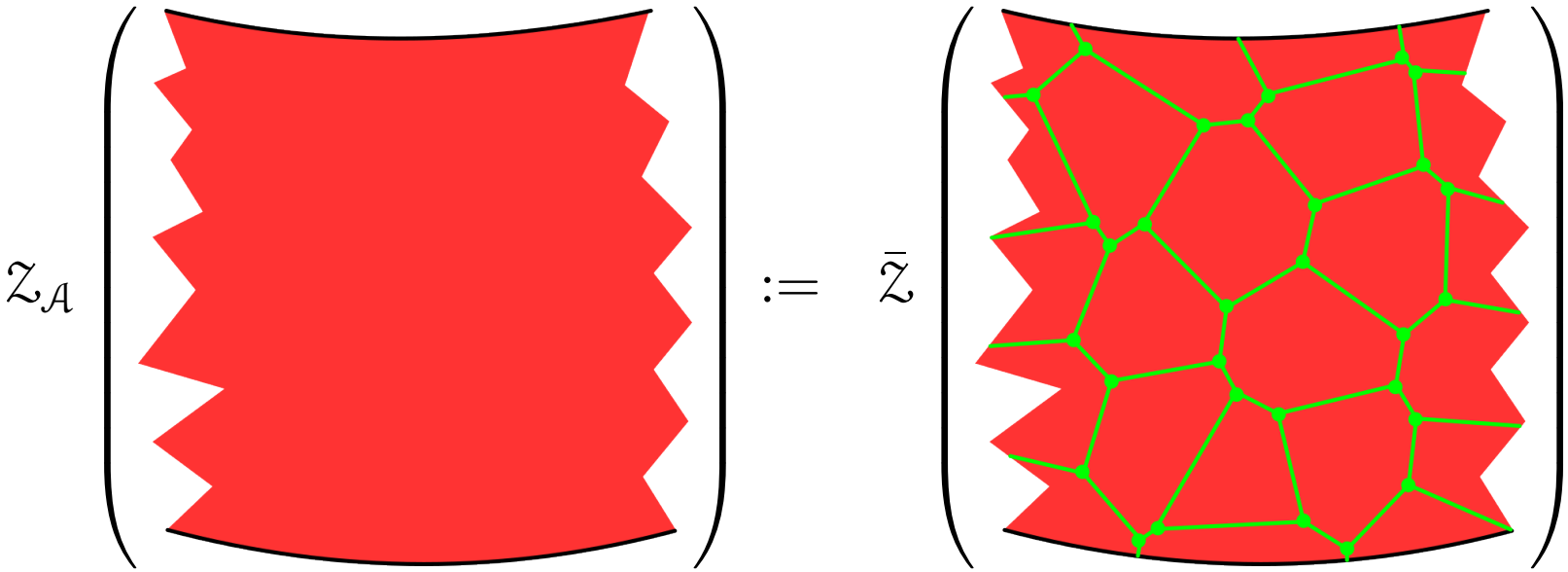

And with some more work we can also define a whole defect theory $\bar{\mathcal{Z}}_{\mathcal{A}}$, not just a closed one, where, for example, the codimension-1 defects are labeled by bimodule-like data.

The Orbifold Construction

Describe 'theory with symmetries' as $\mathrm{End}(Q) \subset \mathcal{TQFT}_n$, or more generally, 'family of theories' as a subcategory $\mathcal{C} \subseteq \mathcal{TQFT}_n$.

- construct pivotal strict 2-category for an oriented defect TQFT in $n = 2$ [Davydov, Kong, Runkel].

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- construct pivotal strict 2-category for an oriented defect TQFT in $n = 2$ [Davydov, Kong, Runkel].

Then we would like to see the data as internal to $\mathcal{C}$.

- $(n = 1)$ the 1-bistellar moves give idempotents.
- $(n = 2)$ special Frobenius algebras satisfy the 2-bistellar move relations [Carqueville, Runkel]

Furthermore, from the categorical data we should produce a Morita-like higher category $\mathrm{Mor}_{\mathcal{P}}(\mathcal{C}) \supseteq \mathcal{C}$, which can see more theories than $\mathcal{C}$.

Plan for the Remainder of the Talk

Observe that:

- the data is presented strictly
- generators + relations only at top level

Goal: Give generators and relations for a higher category whose *representations* are data for the orbifold construction.

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Results in Low Dimensions

Strict Computads

Atoms and Molecules

Proof Sketch

Bistellar Data

Theorem (I.)

For each integer n there is a strict n-category $\mathcal{P}_n$, the walking n-bistellar datum, such that:

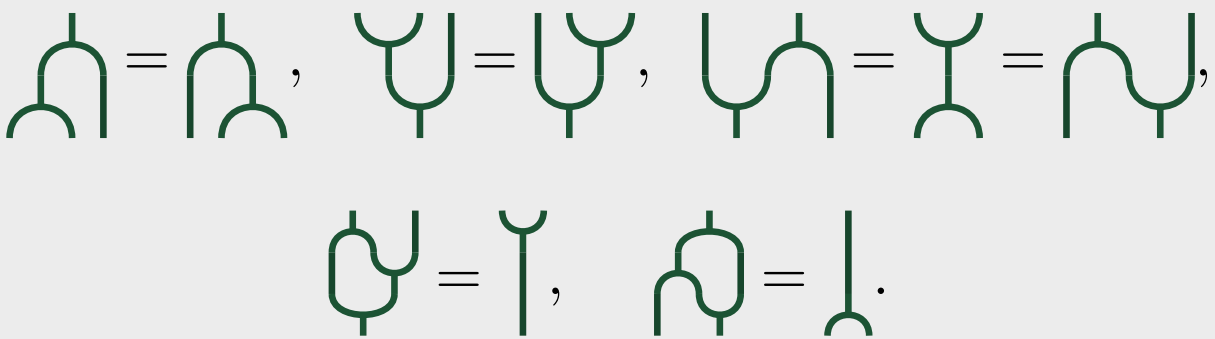
- $\mathcal{P}_1$ *is the walking idempotent*

$$\mathcal{P}_1 = \{a, \quad e : a \to a; \quad e^2 = e\}$$

- $\mathcal{P}_2$ *has a presentation with generators*

$$a, \quad A : a \to a, \quad \mu : A \otimes A \to A, \quad \delta : A \to A \otimes A$$

and relations



n-Computads

Definition

A 0-computad is a set, and the adjunction $F_0 : \mathfrak{Cptd}_0 \leftrightharpoons \mathfrak{Cat}_\omega$ is the discrete category functor. An $(n+1)$-computad A_{n+1} consists of

- an n-computad A_n
- a set of $(n+1)$-cells a_{n+1}
- two maps $b^{\pm} : a_{n+1} \to F_n(A_n)_n$ whose outputs are parallel.

The adjunction $F_{n+1} : \mathfrak{Cptd}_n \leftrightharpoons \mathfrak{Cat}_\omega$ is such that F_{n+1} freely adds all necessary composites at level $n+1$ and acts as F_n at lower levels.

An ω-computad A is a family $\{A_n\}_{n\in\mathbb{N}}$, and the adjunctions combine to give an adjunction $F_\omega : \mathfrak{Cptd}_\omega \leftrightharpoons \mathfrak{Cat}_\omega$.

Oriented Graded Posets

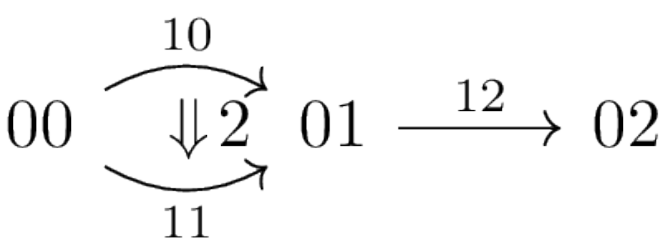
Definition

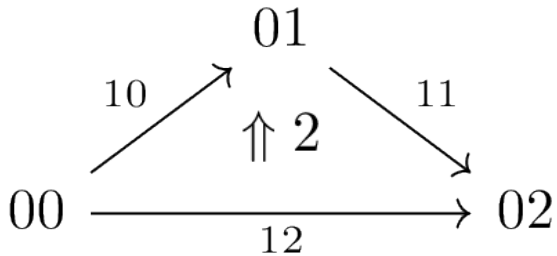
An *oriented graded poset* is a graded poset $\dim : P \to \mathbb{N}$, together with, for each element x in the poset, a bipartition of its faces

$$\Delta x = \Delta^- x \amalg \Delta^+ x.$$

A map of such is a map of graded posets that preserves the bipartitions. We get the category ogPos. Oriented graded posets have a positive and a negative boundary

$$\partial_k^+ P, \ \partial_k^- P \quad \subseteq \quad P$$





Atoms and Molecules

Definition ([Hadzihasanovic])

Atoms and molecules are such that $* \in \mathsf{Atm} \subset \mathsf{Mol} \subset \mathsf{og}\mathcal{P}\mathsf{os}$ and:

atom: Given two (round) molecules $U, V \in \mathsf{Mol}$ of the same dimension, such that $\partial^{\pm} U = \partial^{\pm} V$ there is an atom

$$(U \Rightarrow V)$$

glue: Given two molecules $U, V \in \mathsf{Mol}$ such that $\partial_k^+ U = \partial_k^- V$ then the pushout

$$\begin{array}{ccc} \partial_k^+ U & = & \partial_k^- V \hookrightarrow V \\ \downarrow & & \downarrow \\ U & \longrightarrow & U \#_k V \end{array} \quad \text{(in } \mathsf{og}\mathcal{P}\mathsf{os}\text{)}$$

is again a molecule.

Simplicial Molecules

Definition

A *simplicial n-molecule* is a molecule whose underlying graded poset is the face poset of a simplicial complex of dimension n. We get the families of sets sAtm_n and sMol_n.

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Theorem (I.)

The sets $\mathsf{sAtm}_n^{(fa)}$ *assemble into an* ω*-computad* $\mathsf{sAtm}^{(fa)}$*, where at level* $n+1$ *we have the* n*-computad* $\mathsf{sAtm}_{\leq n}^{(fa)}$ *and the source and target maps are*

$$b_n^{\pm} : \mathsf{sAtm}_{n+1}^{(fa)} \xrightarrow{\partial^{\pm}} \mathsf{sMol}_n^{(fa)} \longrightarrow F_n(\mathsf{sAtm}_{\leq n}^{(fa)})_n.$$

Definition

In each dimension n, the *walking n-bistellar datum* is the strict n-category defined as $\mathcal{P}_n := L_n F_\omega(\mathsf{sAtm}^{(fa)})$.

Proof Sketch

Theorem

For each integer there is a strict ω-category $\mathcal{P}_n$, the n-bistellar data, such that:

- $\mathcal{P}_1$ *is the walking idempotent*

$$\mathcal{P}_1 = \{a, \quad e : a \to a; \quad e^2 = e\}$$

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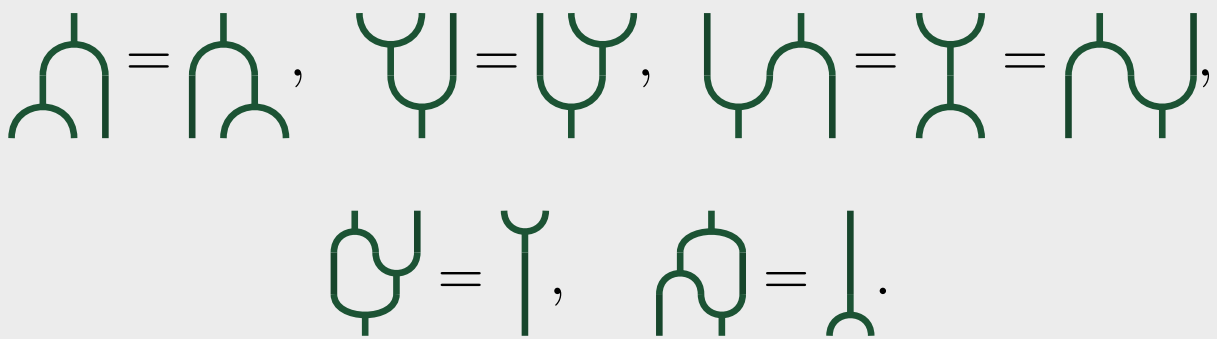

Proof Sketch of Results

Lemma

Every simplicial n-atom's underlying simplicial complex is Δ^n.

Proof.

For conciseness $A_n := \mathsf{sAtm}_n$.

$$A_0 = \{*\}, \quad A_1 = \{\longrightarrow\}, \quad A_2 = \left\{ \; , \; \right\},$$

$$A_3 = \left\{ \; , \; , \; , \; , \; , \; , \; , \; , \; , \; \right\}$$

□

Labeled Molecules

Theorem ([Hadzihasanovic])

There is a functor

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Definition

For C a strict ω-category, the category of C-labeled simplicial n-molecules is the pullback

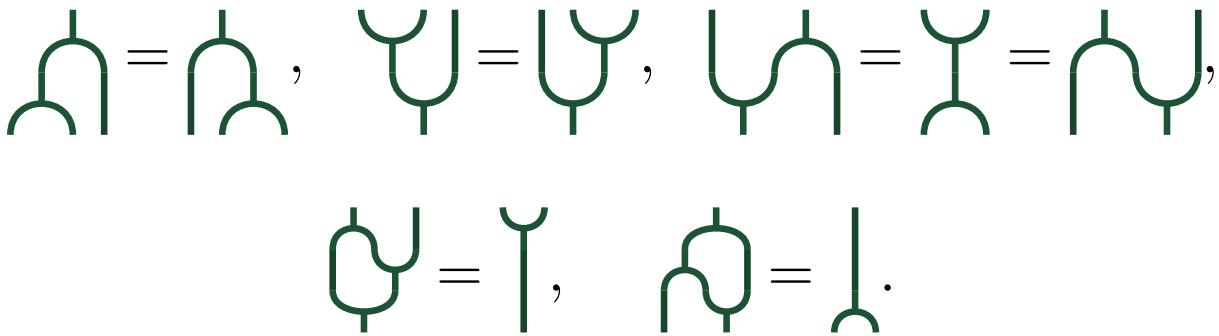
$$\begin{array}{ccc} \mathsf{sMol}_n(C) & \longrightarrow & \mathsf{sMol}_n \\ \downarrow & \ulcorner & \downarrow{\scriptstyle \mathsf{Mol}_{/-}} \\ \mathcal{C}\mathsf{at}_{\omega\,/\,C} & \longrightarrow & \mathcal{C}\mathsf{at}_\omega \end{array} .$$

Higher Bistellar Cells

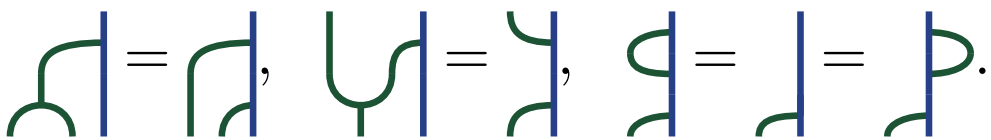
Theorem (I.)

Let $\mathbb{G}_n$ be the n-globe.

- $\mathcal{P}_2(\mathbb{G}_0)$ *has a presentation by* a, A, μ, δ *and relations*



- $\mathcal{P}_2(\mathbb{G}_1)$ *has two copies of* $\mathcal{P}_2(\mathbb{G}_0)$ *inside and the remainder is presented by* $M, \lambda, \rho, \check{\lambda}, \check{\rho}$ *and relations including*



- $\mathcal{P}_2(\mathbb{G}_2)$ *has two copies of* $\mathcal{P}_2(\mathbb{G}_1)$ *and the remainder is the data of a bimodule map between them.*

Outlook

- Consider tangential structures, like SO, properly:
  1. Put extra labels on the simplices to encode the tangential structure from topological considerations, and
  2. Construct higher dagger categories instead of just plain higher categories.
- Comparison with the condensation monads of [Gaiotto, Johnson-Freyd/Reutter, Zetto], which should also encode some framed version of bistellar data.
- Can the framework deal with composition of bimodules and higher cells to actually form a Morita-like higher category?

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Thank you.

2-Bistellar is Not That Interesting

Proposition

Let C be a 2-category that is enriched in idempotent complete additive categories. Then, every 2-bistellar datum P decomposes as

$$P = A \oplus B,$$

where A carries a special Frobenius algebra structure and B has trivial multiplication and comultiplication.