THE PULLBACK PRODUCT OF MONADS

with Aaron Fairbanks

joint work with Sridhar Ramesh

(paper coming soon)

THE CONSTRUCTION

T and S

are two monads

on a category.

From the monad units we obtain natural transformations:

$$T \xRightarrow{T\eta} T \circ S \quad \text{and} \quad S \xRightarrow{\eta S} T \circ S$$

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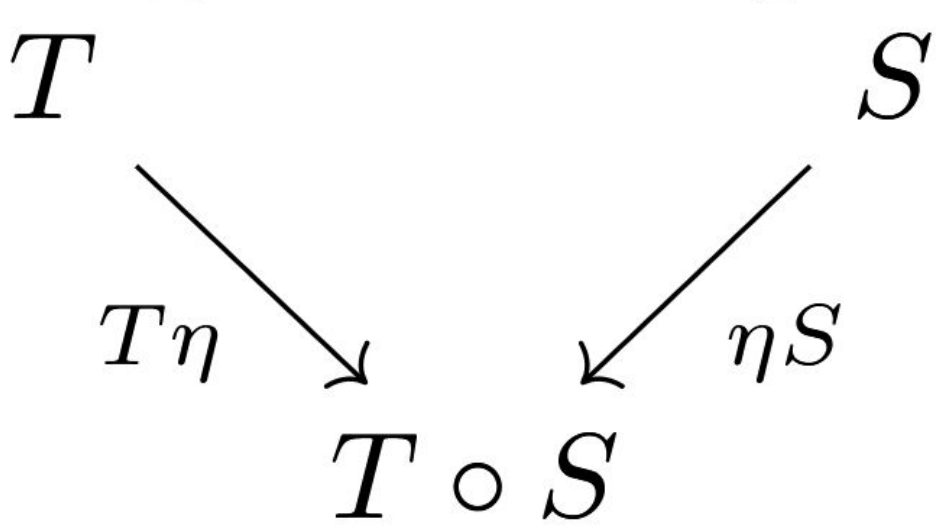
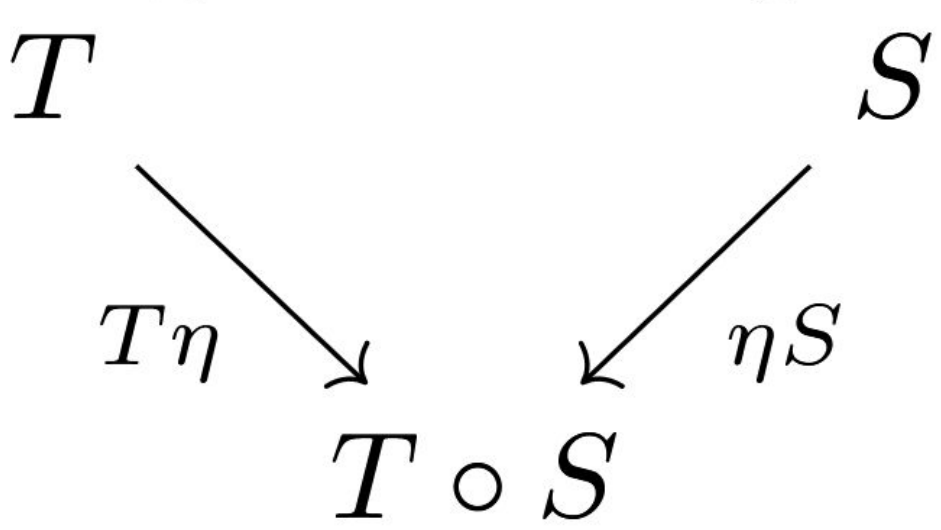
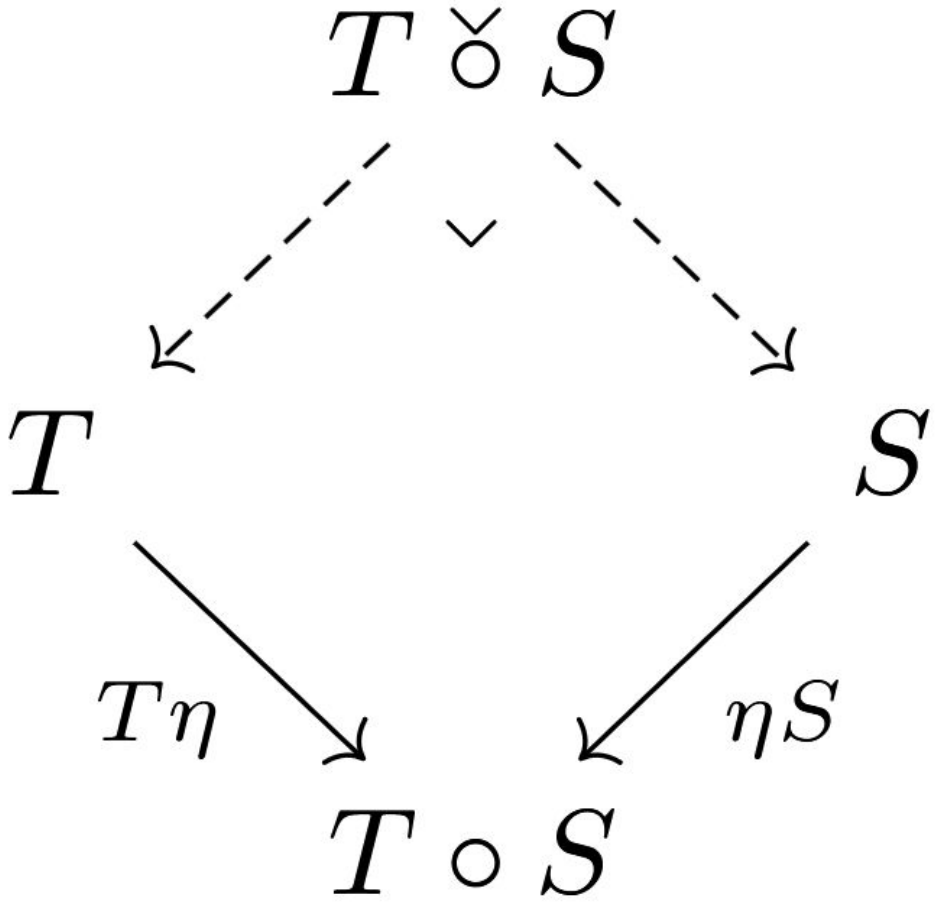
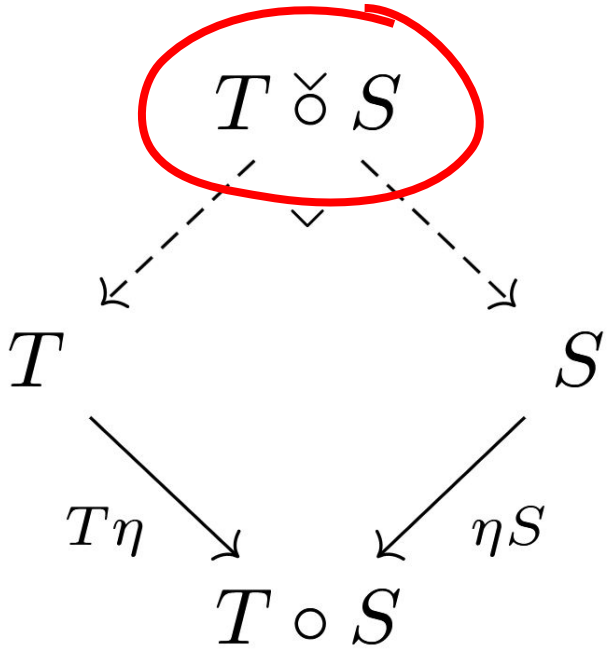



T
S
Tη
ηS
T ∘ S


$T \mathbin{\check{\circ}} S$
T
S
$T\eta$
ηS
$T \circ S$

THE PULLBACK PRODUCT OF MONADS



WARNING

In general, the pullback product $\check{\circ}$ is NOT associative or unital.

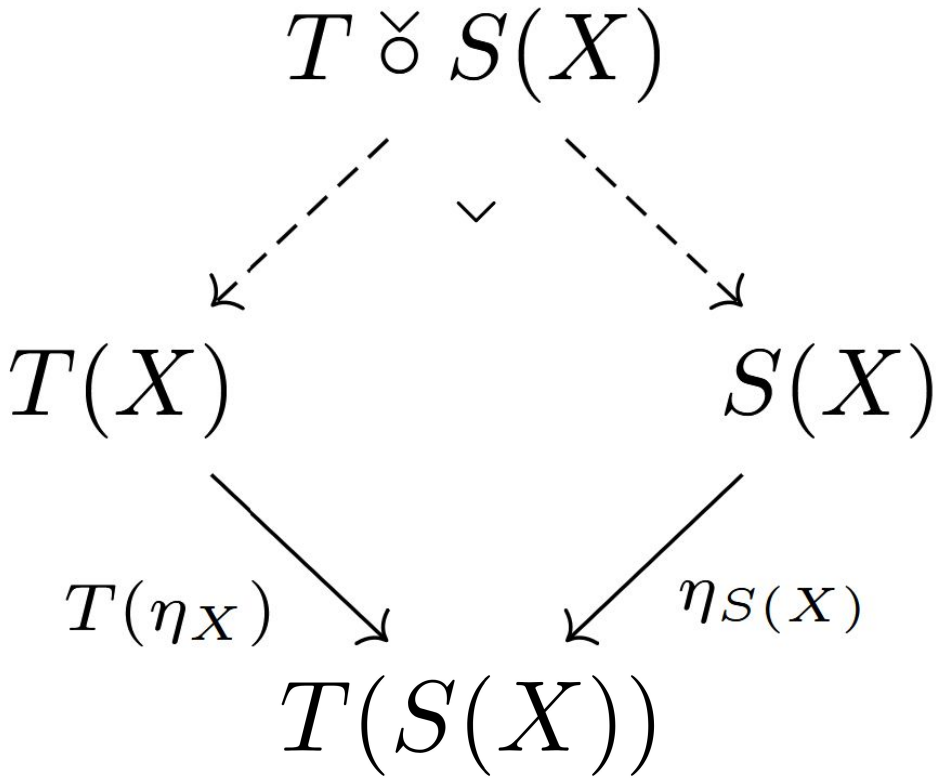
Examples

Let S be the terminal monad:

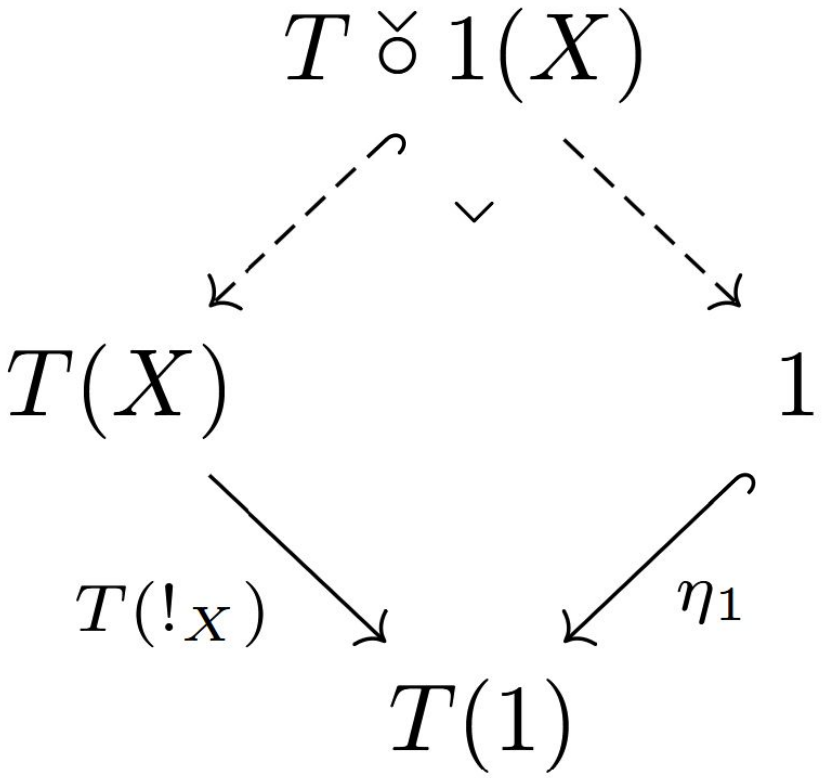
$$S(X) = 1$$

(on a category with finite limits)

Calculating the pullback product $T \mathbin{\breve{\circ}} S$:



Calculating the pullback product $T \mathbin{\breve{\circ}} 1$:



$T \mathbin{\breve{\circ}} 1$ is the *"affine part"* T_{aff} of T

(the co-free 1-preserving monad on T)

$$\begin{array}{ccc} T_{\mathrm{aff}}(X) & \dashrightarrow & 1 \\ \big\downarrow & \lrcorner & \big\downarrow{\scriptstyle \eta_1} \\ T(X) & \xrightarrow[T(!_X)]{} & T(1) \end{array}$$

Affine parts of monads

By

HARALD LINDNER

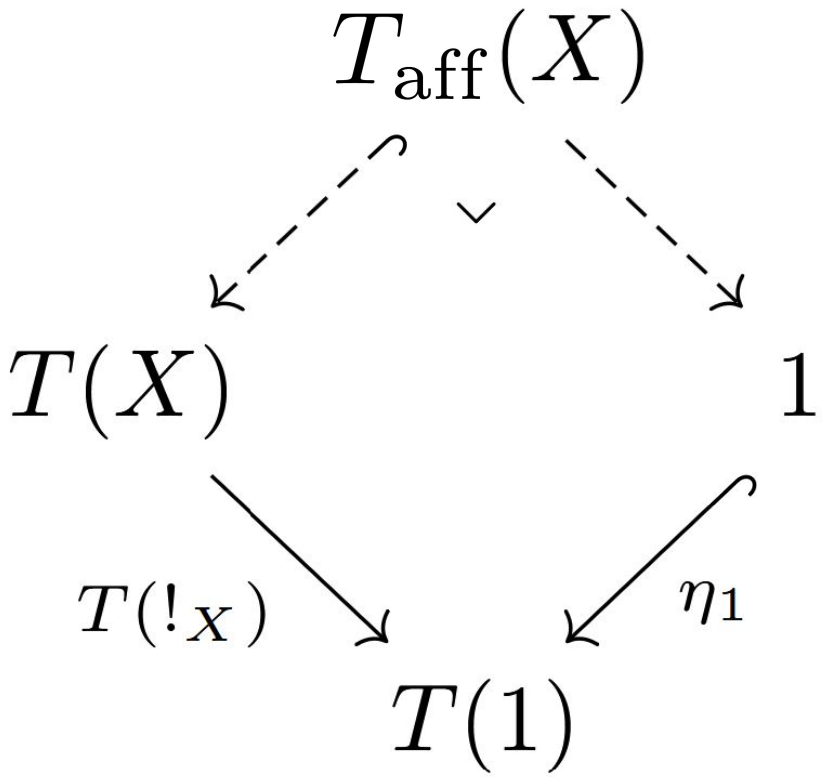
Lindner, H. Affine parts of monads. *Arch. Math* **33**, 437–443 (1979).

Example of this example:

Let T be the
free group monad on $\mathbf{Set}$

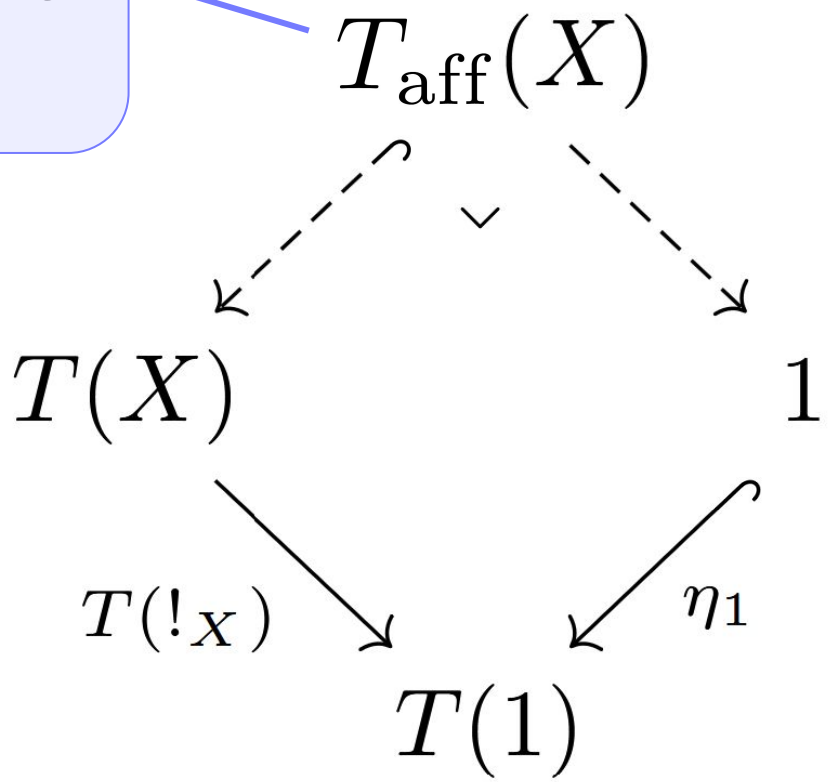
$$T(X) = \left\{ \text{group words in } X \right\}$$

group words in X



group words in X with one more non-inverted element than inverted elements

group words in X



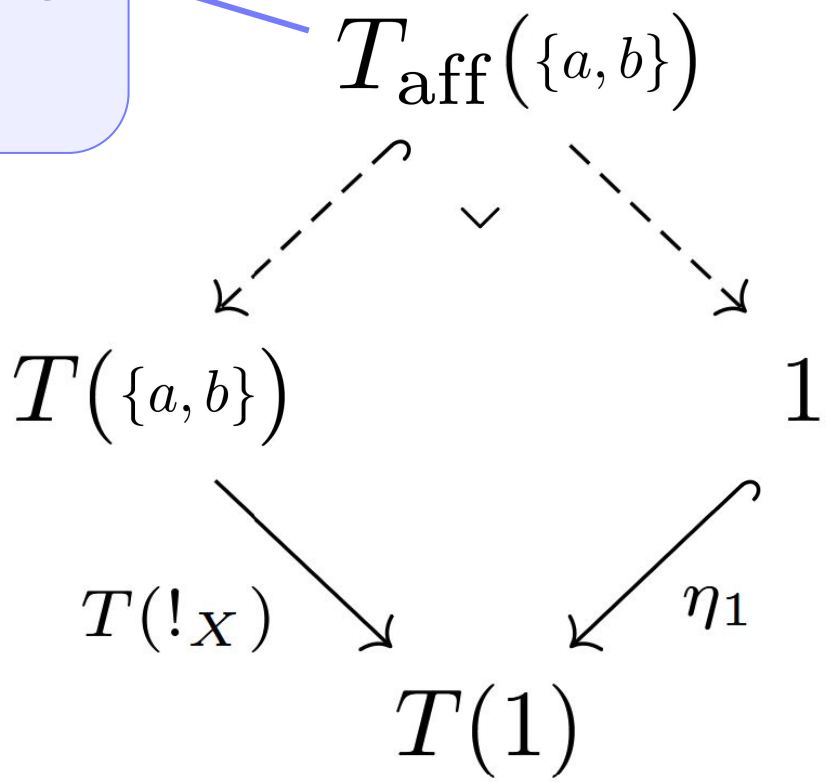
group words in $\{a, b\}$ with one more non-inverted element than inverted elements

for example…

$$a, \quad b, \quad ab^{-1}a, \quad a^{-2}b^3$$

but not…

$$a^{-1}, \quad ab, \quad ab^{-1}$$



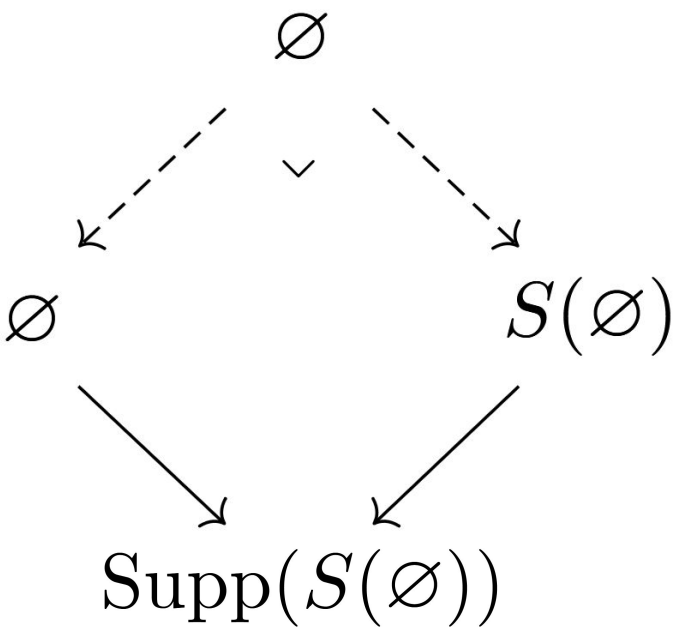
Different example

Let T be the

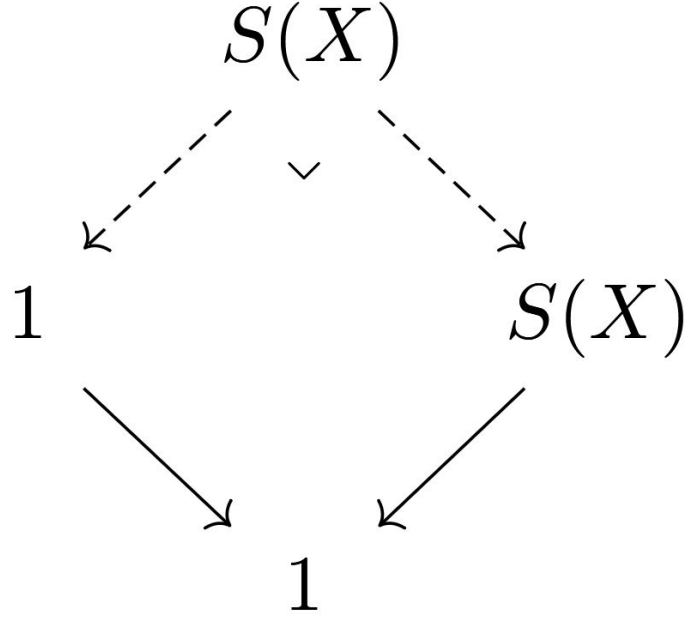
"support monad" on $\mathbf{Set}$

$$\mathrm{Supp}(X) = \begin{cases} \varnothing & \text{if } X = \varnothing \\ 1 & \text{otherwise} \end{cases}$$

Calculating the pullback product $\mathrm{Supp} \mathbin{\breve{\circ}} S$:



if $X = \varnothing$



otherwise

$\mathrm{Supp}^{\breve{\circ}}\, S$ is the *"coaffine part"* S_{coaff} of S

(the co-free $\varnothing$-preserving monad on S)

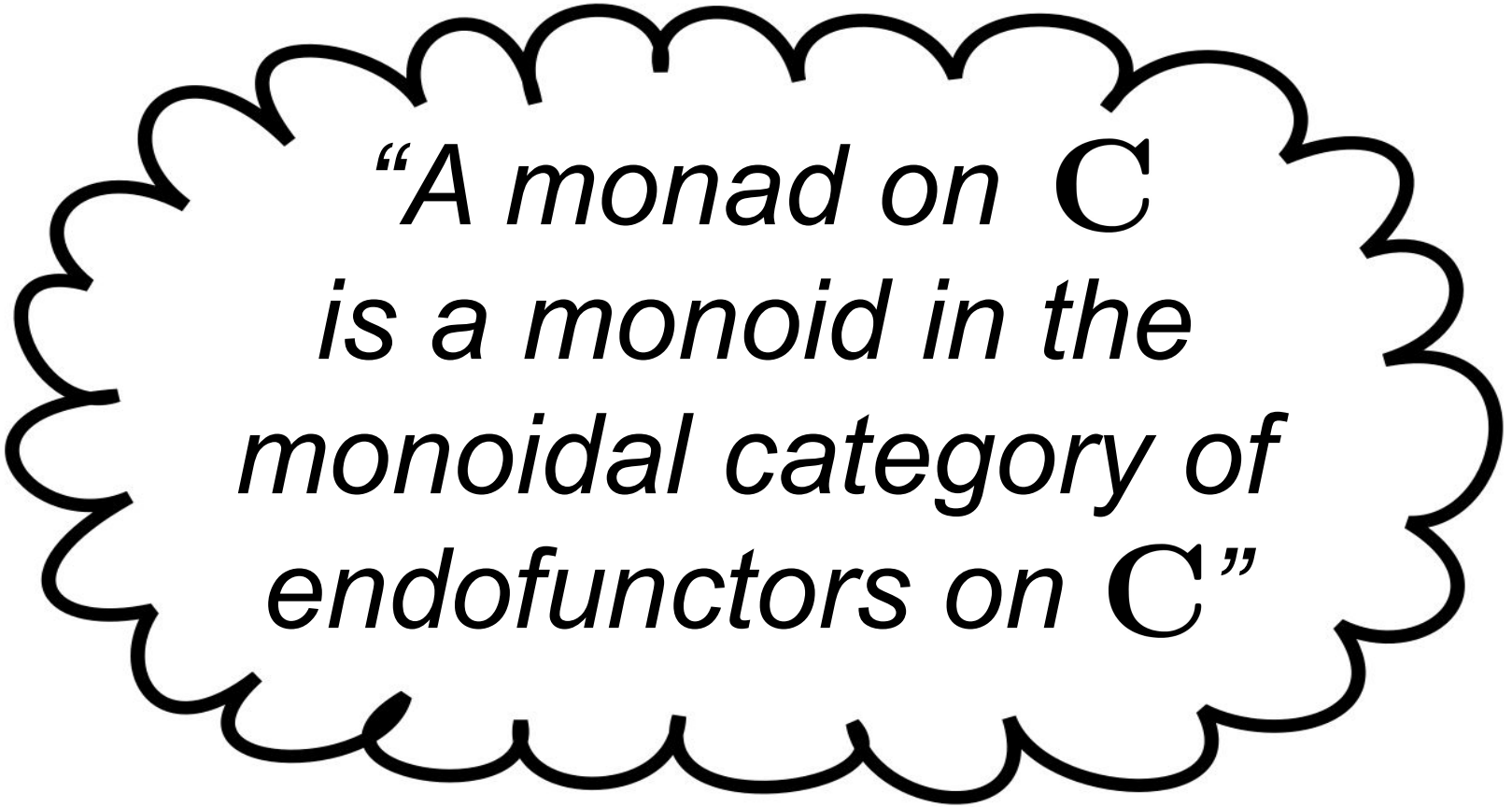
$$S_{\mathrm{coaff}}(X) = \begin{cases} \varnothing & \text{if } X = \varnothing \\ S(X) & \text{otherwise} \end{cases}$$

An S_{coaff}-algebra is an S-algebra OR the empty set.

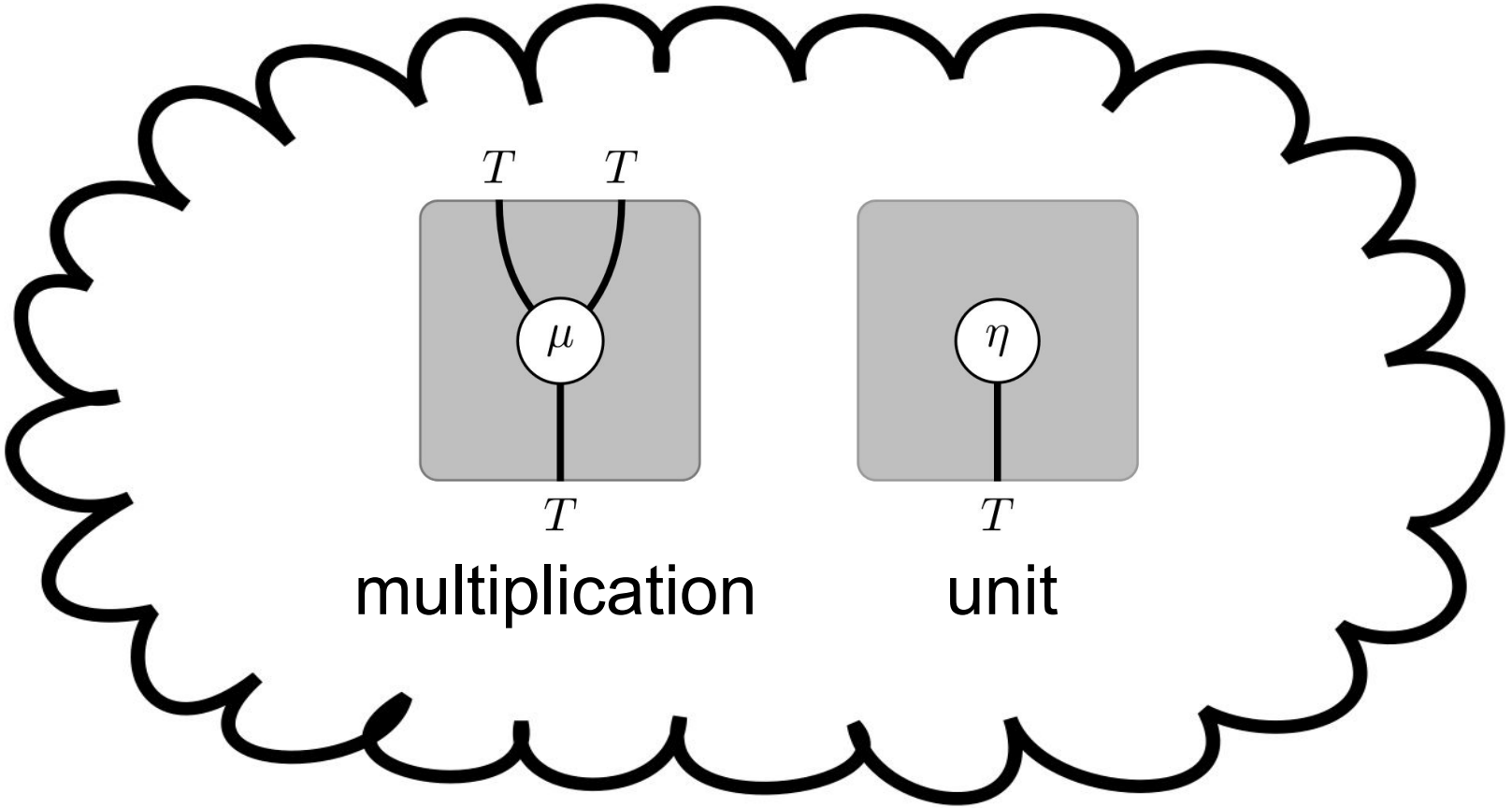
$\bar{\circ}$	1	Supp	Other S
1	1	Supp	S
Supp	Supp	Supp	S_{coaff}
Other T	T_{aff}	T_{aff}	id

TABLE 1. All pullback products of monads on **Set**

Remember…

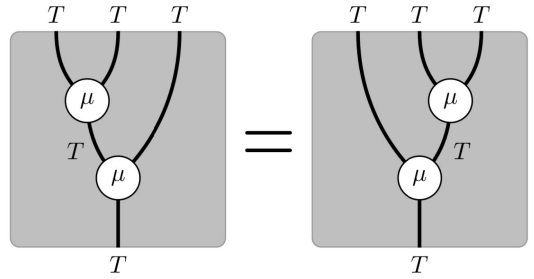

"A monad on C
is a monoid in the
monoidal category of
endofunctors on C"

Remember, a *monoid* consists of…

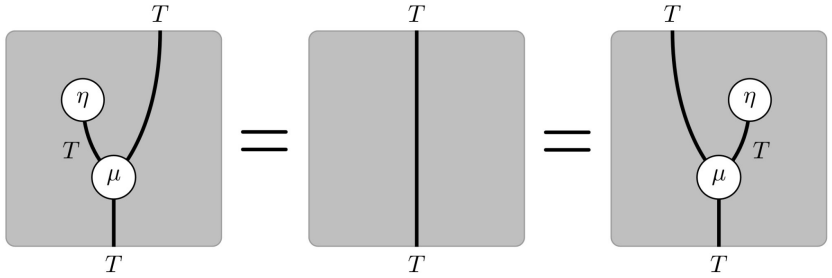


Remember, a *monoid* consists of…

associativity



identity



The pullback product construction works not only for **monads**, but for **monoids** in any monoidal category

(assuming the desired pullback exists)

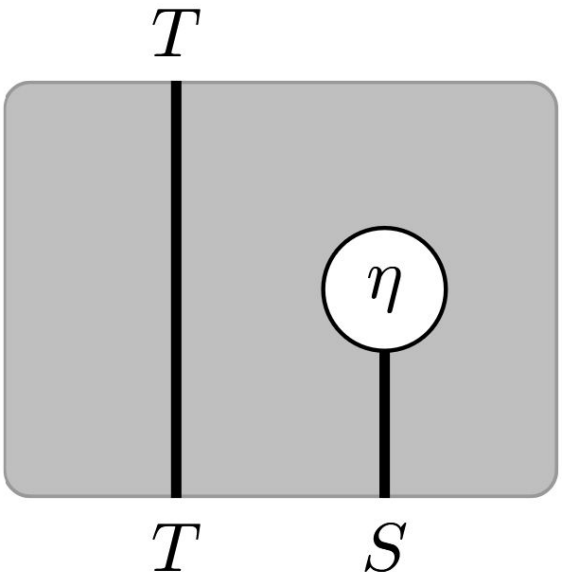
THE CONSTRUCTION

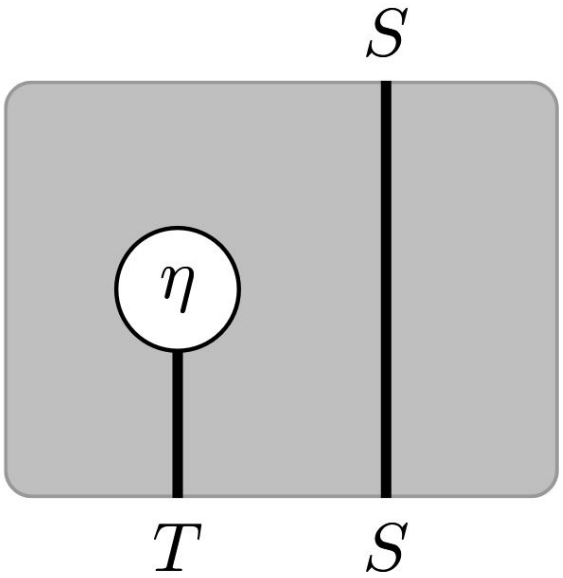
T and S

are two monoids

in a monoidal category.

From the monoid units we obtain arrows:





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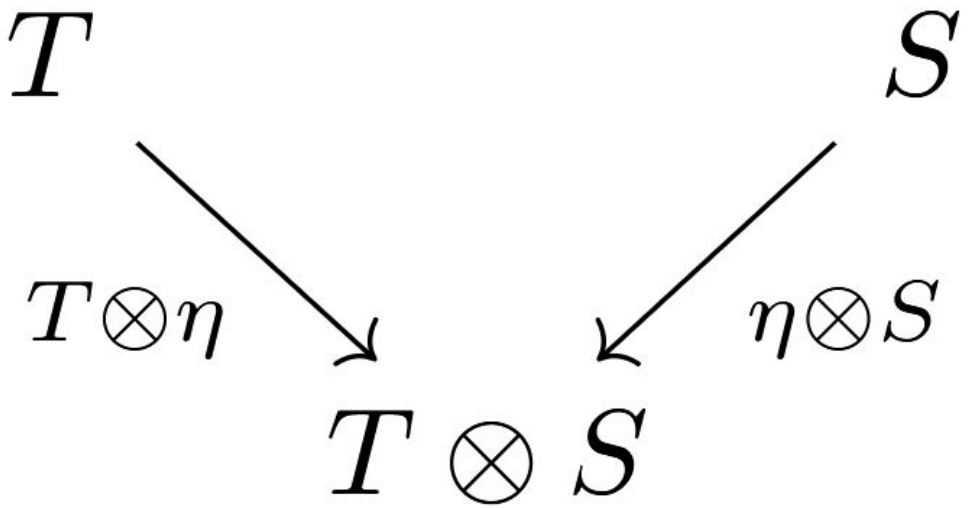
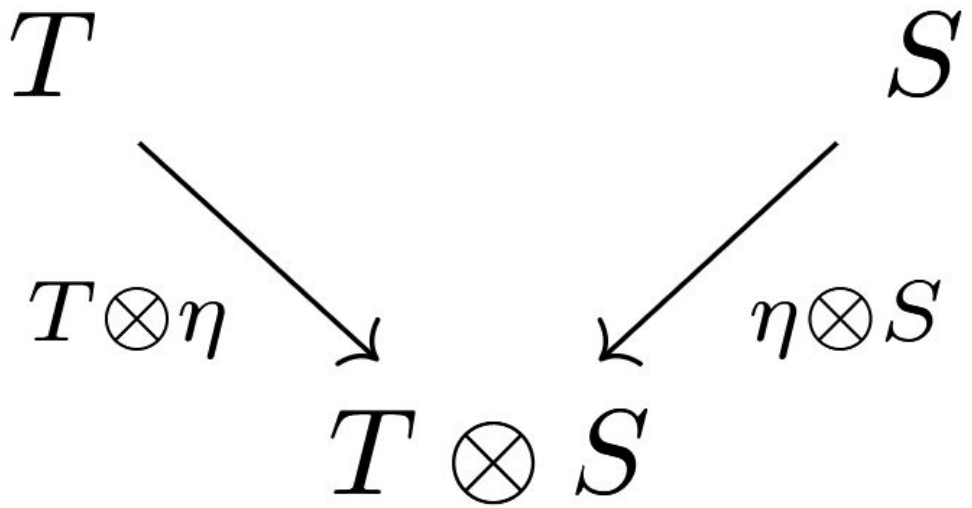
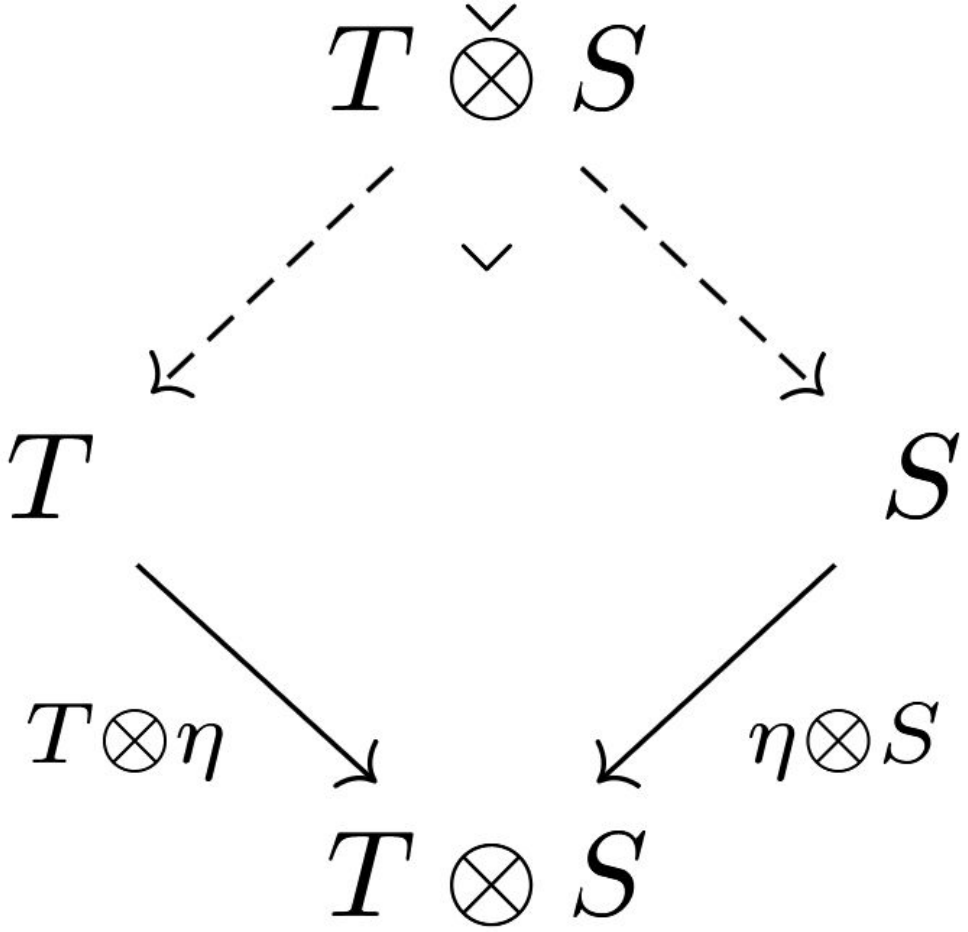
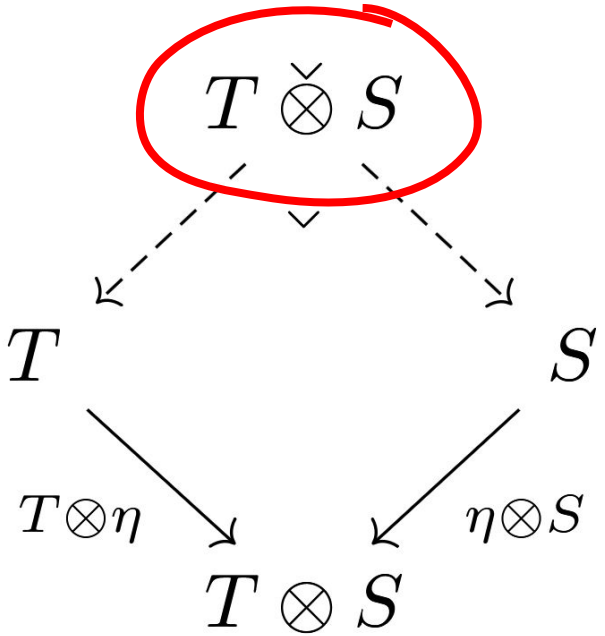



T
S
T⊗η
η⊗S
T ⊗ S


$T \check{\otimes} S$
T
S
$T \otimes \eta$
$\eta \otimes S$
$T \otimes S$

THE PULLBACK PRODUCT OF MONOIDS



Examples?

The pullback product of monoids in $\mathbf{Set}$ (actual monoids) is always trivial.

$$T \mathbin{\check{\times}} S = 1 \quad \text{always}$$

More generally:

The pullback product of monoids in any *cartesian* monoidal category is always trivial.

$$T \mathbin{\check{\times}} S = 1 \quad \text{always}$$

Nontrivial examples?

Take the category of commutative rings with coproduct.

Every object has a unique monoid structure

(as in any co-cartesian monoidal category).

Nontrivial pullback products abound.

Example: $\mathbb{Z}/m \mathbin{\check{\otimes}} \mathbb{Z}/n \cong \mathbb{Z}/\mathrm{LCM}(m,n)$

Enough examples.

Why is $T \check{\otimes} S$ a monoid?

If $\otimes$ is *symmetric* (or *braided*), the result follows from some standard categorical facts…

If $\otimes$ is *symmetric* (or *braided*), a tensor product of monoids is itself a monoid.

$$T \otimes S$$

monoid

In this case, the pullback product arises as a limit of monoids.

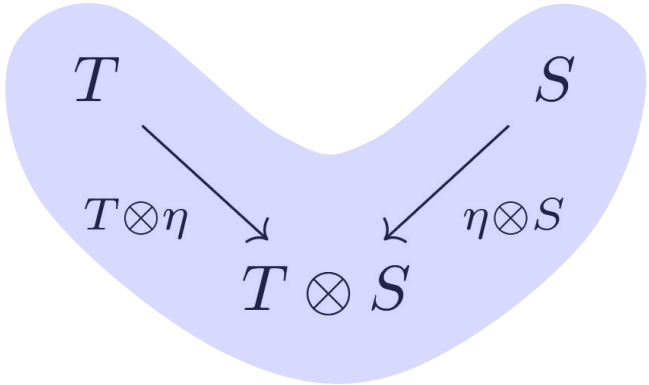


diagram of monoid morphisms

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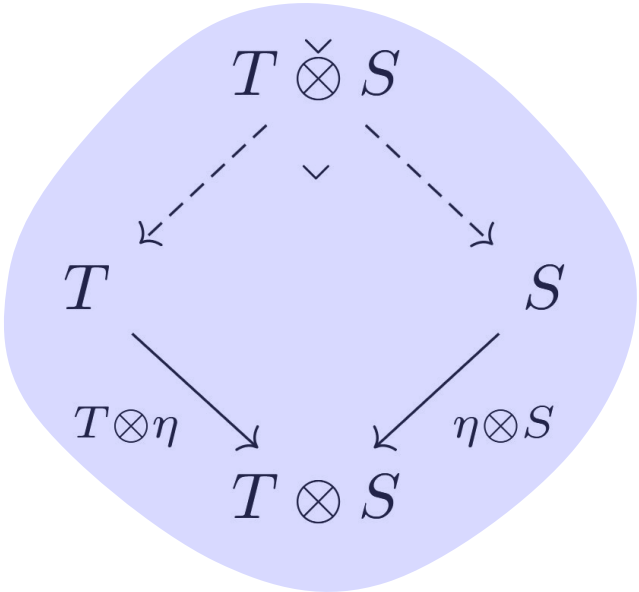

diagram of monoid morphisms

In general,
a limit of monoids
(and monoid morphisms)
is itself a monoid.

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(and monoid morphisms)
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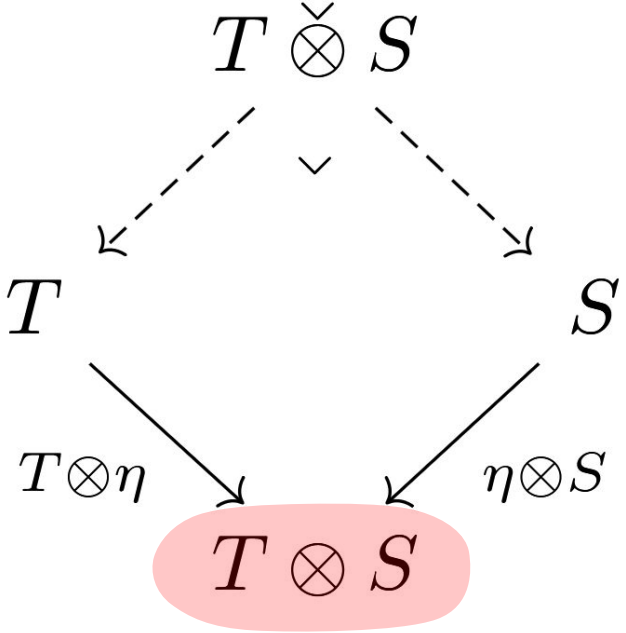
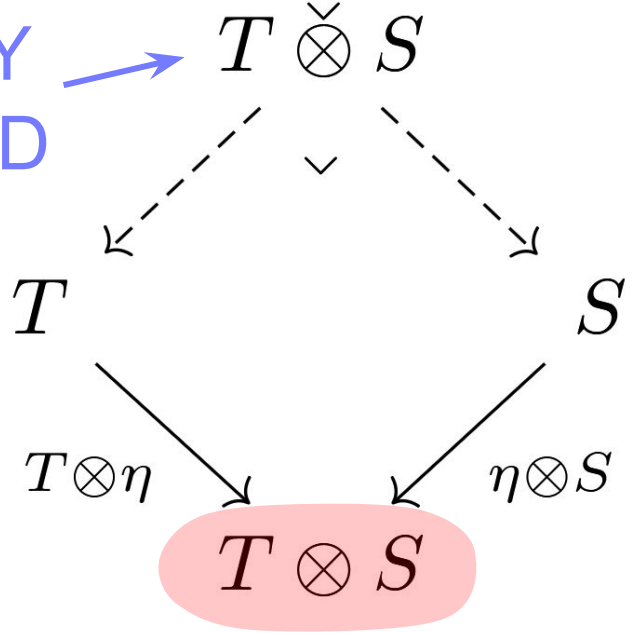
The non-symmetric case is more interesting.



not a monoid

The non-symmetric case is more interesting.

SURPRISINGLY STILL A MONOID



not a monoid

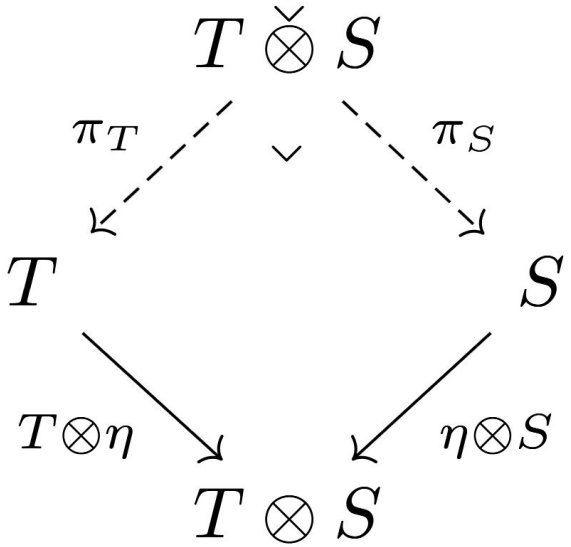
(again)

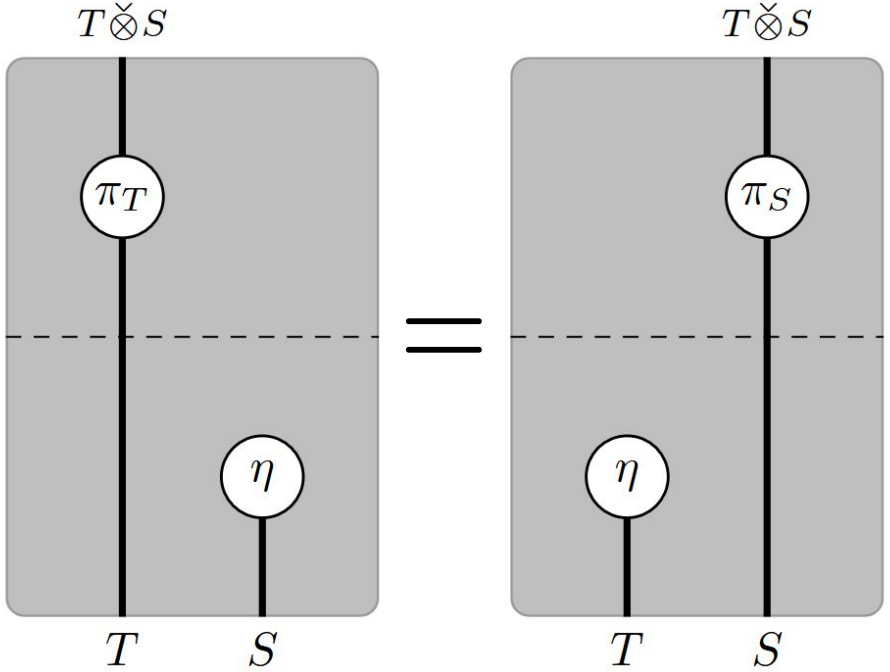
Why is $T \check{\otimes} S$ a monoid?

Sketch of proof

(DIRECT VERSION)

We have the defining pullback square





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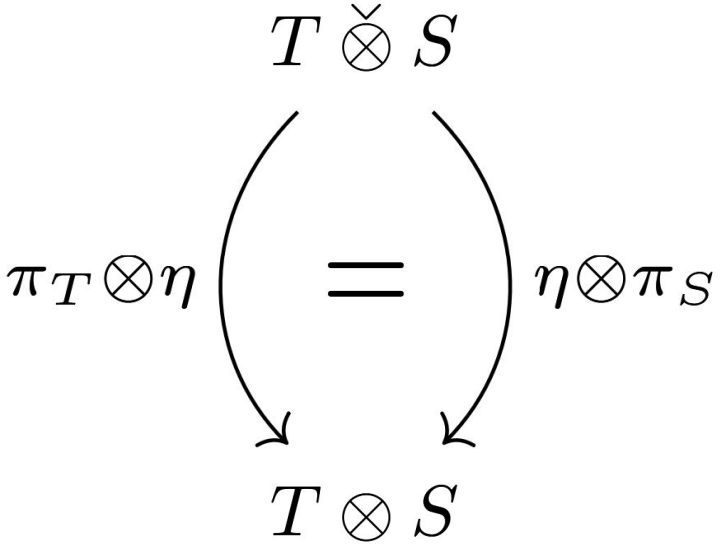
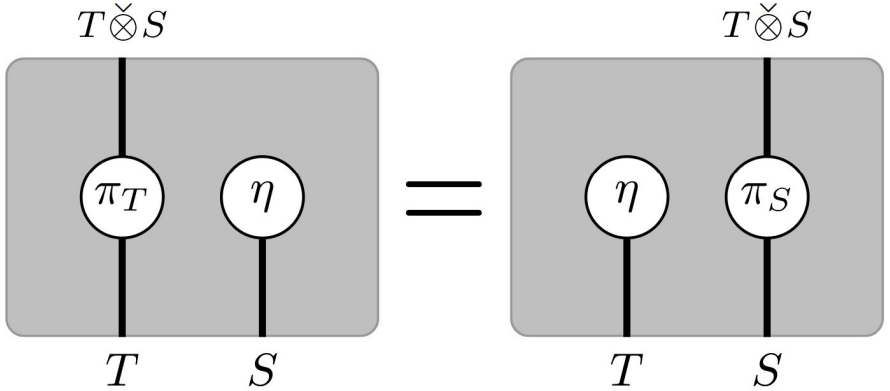


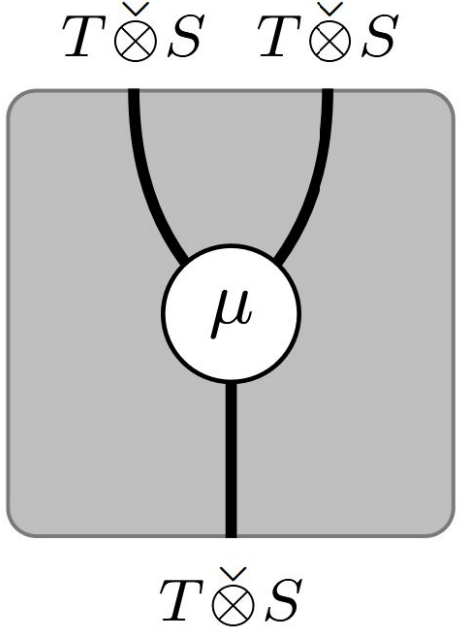


We want a “multiplication” map

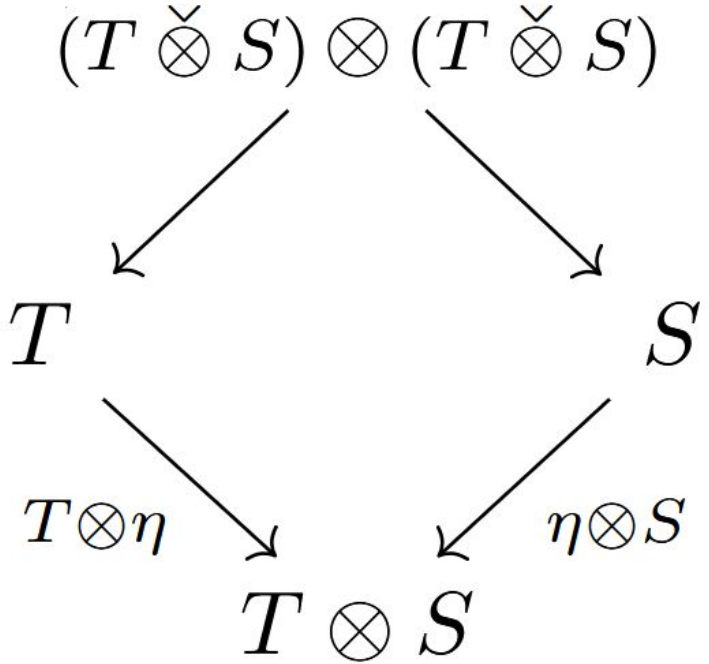
$$(T \mathbin{\check{\otimes}} S) \otimes (T \mathbin{\check{\otimes}} S)$$

$$\downarrow \mu$$

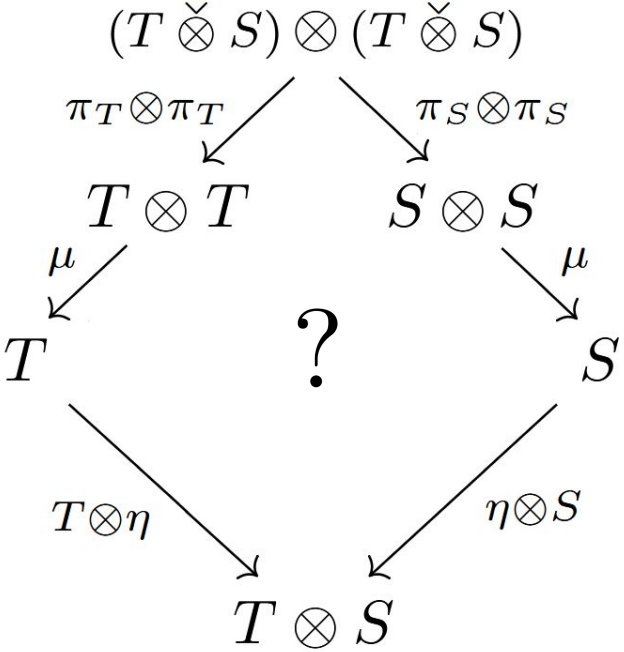
$$T \mathbin{\check{\otimes}} S$$



By the universal property of the pullback, this corresponds to a commutative square



How about…


$(T \check{\otimes} S) \otimes (T \check{\otimes} S)$
$\pi_T \otimes \pi_T$
$\pi_S \otimes \pi_S$
$T \otimes T$
$S \otimes S$
μ
μ
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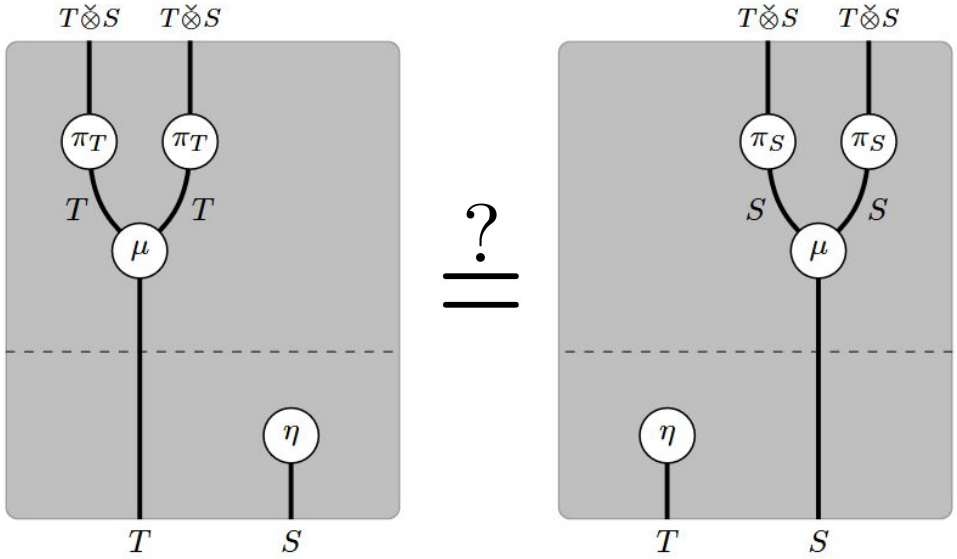
?
S
$T \otimes \eta$
$\eta \otimes S$
$T \otimes S$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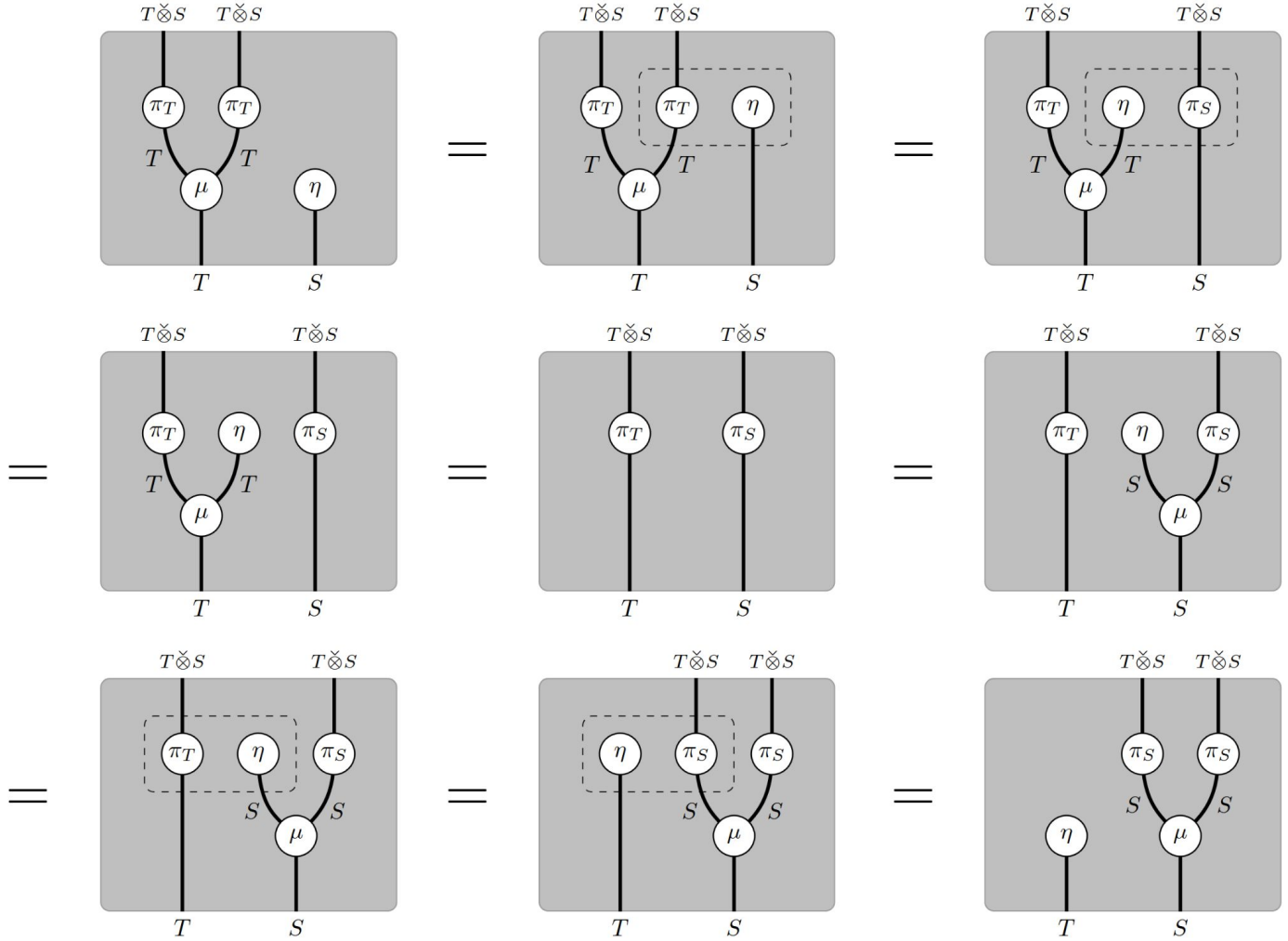
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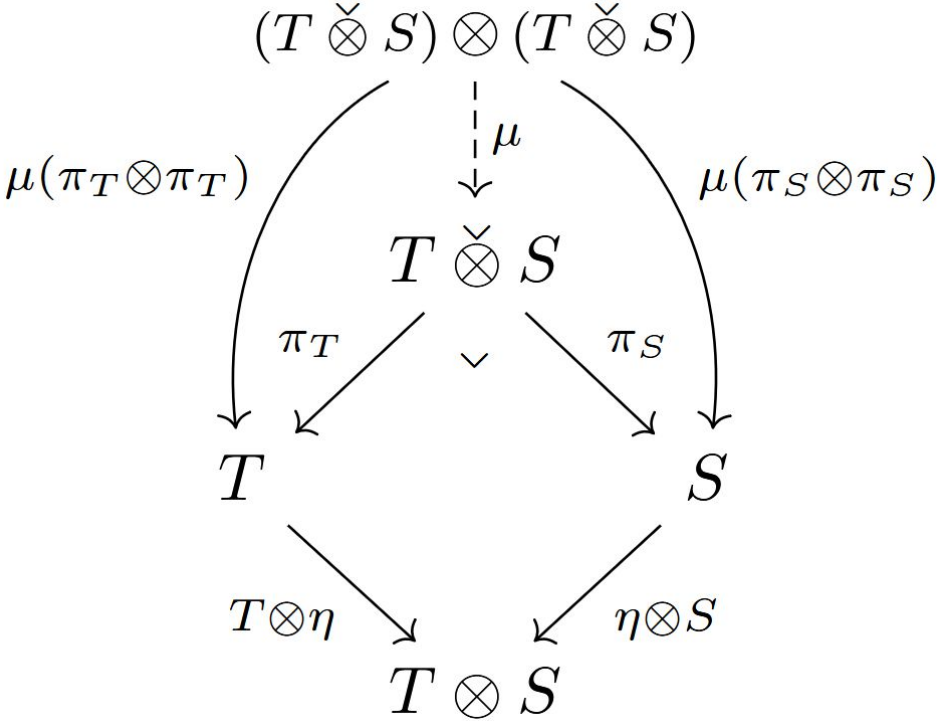
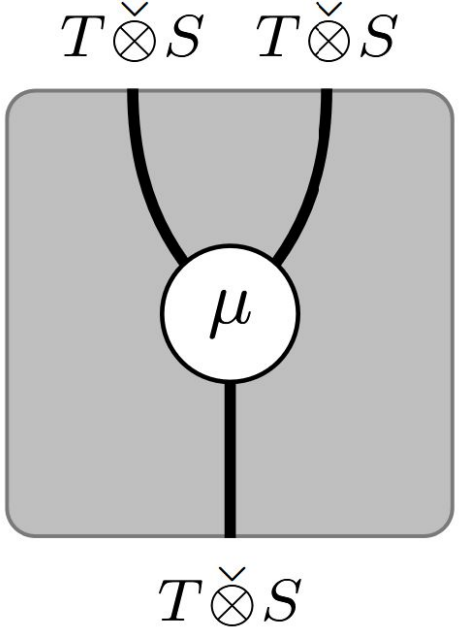
Thus we obtain the desired multiplication.





The unit map is similar (easier).

Exercise:
associativity and identity laws.

The unit map is similar (easier).

Exercise:
associativity and identity laws. □

(again)

Why is $T \check{\otimes} S$ a monoid?

Sketch of proof
(ABSTRACT VERSION)

Without loss of generality, assume our monoidal category has pullbacks.

Justification:

Every monoidal category $\mathbf{C}$ admits a limit-preserving, monoidal Yoneda embedding into $\mathrm{Psh}(\mathbf{C})$ (with Day convolution).

Without loss of generality, assume our monoidal category has an initial unit object I.

Justification:

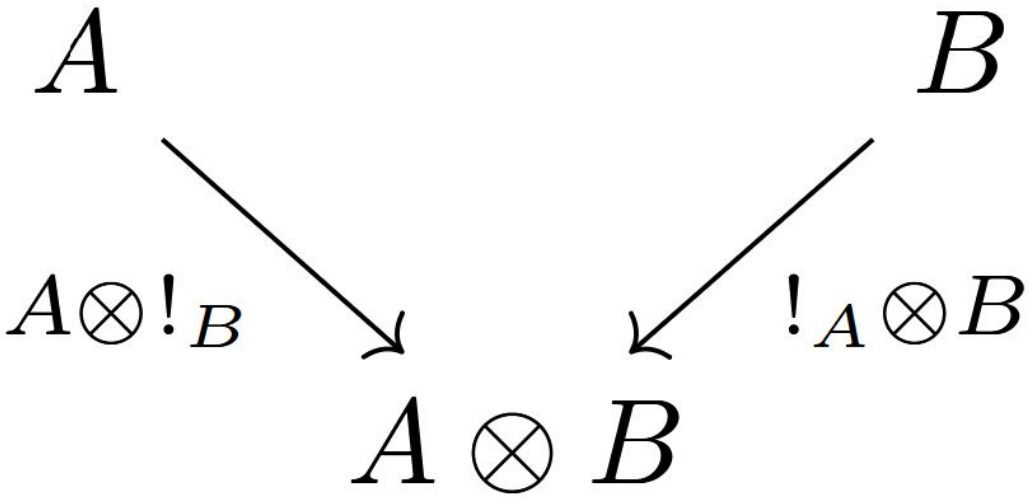
Replace $\mathbf{C}$ with the coslice category $I/\mathbf{C}$.

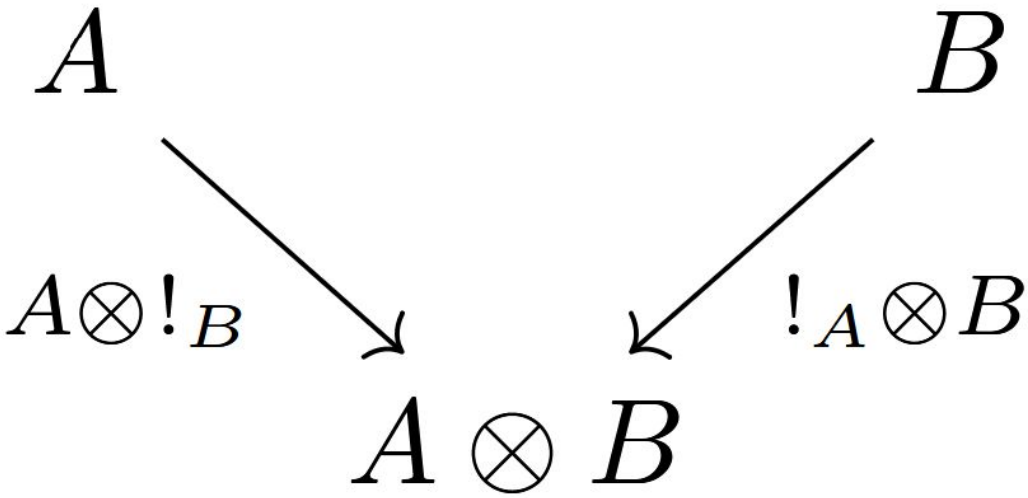
$I/\mathbf{C}$ inherits monoidal structure from $\mathbf{C}$.

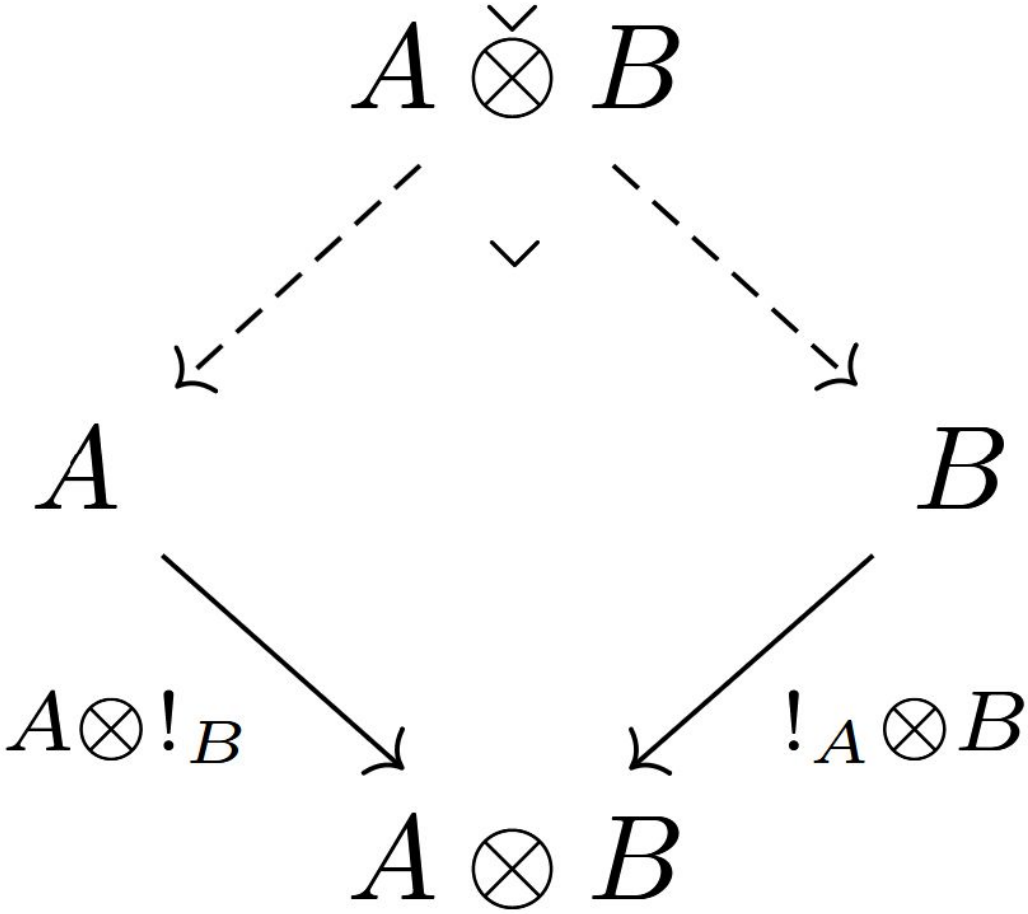
Monoids in $I/\mathbf{C}$ are identified with monoids in $\mathbf{C}$.

Moreover, the projection $I/\mathbf{C} \to \mathbf{C}$ creates limits.

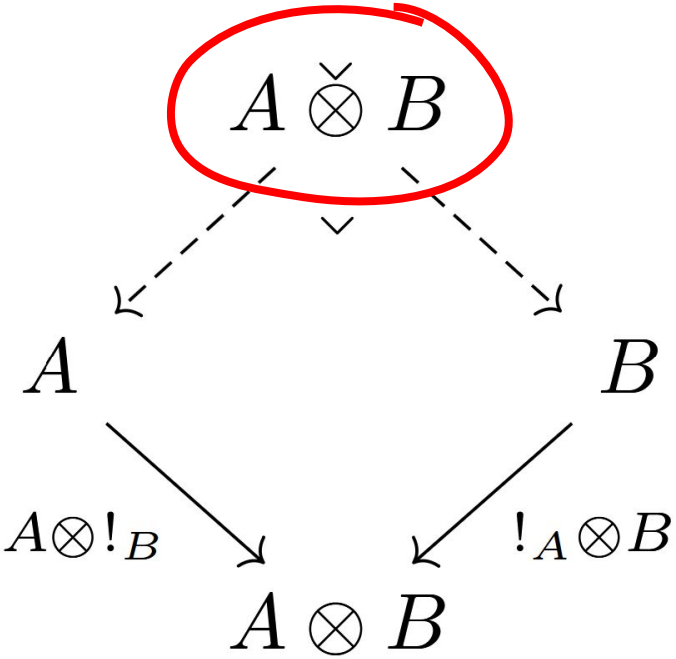
In such a monoidal category, we can form “pullback products” of arbitrary objects A and B.




A
B
A⊗!B
!A⊗B
A ⊗ B


$A \mathbin{\check{\otimes}} B$
A
B
$A \otimes !_B$
$!_A \otimes B$
$A \otimes B$

THE PULLBACK PRODUCT OF OBJECTS



Theorem

Let $\mathbf{C}$ be a monoidal category with pullbacks and initial unit.

The pullback product constitutes a *lax monoidal functor*

$$\check{\otimes} : \mathbf{C} \times \mathbf{C} \to \mathbf{C}.$$

Lax monoidal functors
take monoids to monoids.

Therefore, pullback products
of monoids are monoids.

Lax monoidal functors
take monoids to monoids.

Therefore, pullback products
of monoids are monoids. □

This abstract proof generalizes beyond monoids to some other structures.

A monoid is a model of
the *terminal multicategory*.

More generally, we obtain
pullback products of models
of any multicategory $\mathbf{X}$ with
a unique constant at every sort.

Examples

- $\mathbf{X} =$ the theory of a *monoid*
- $\mathbf{X} =$ the theory of a *pointed object*
- $\mathbf{X} =$ the theory of a *pair of monoids with a pointed bimodule*
- $\mathbf{X} =$ the theory of a *magma with a chosen idempotent*
- $\mathbf{X} =$ the theory of a *pointed* Y (most general case)

Equivalently:

We obtain pullback products
of *pointed* models
of any multicategory.

(*"point"* = model morphism from I)

The end