SYMMETRIC MONOIDAL (∞, n) CATEGORIES

Arghan Dutta

University of Ottawa

OCTOBERFEST 26

Symmetric monoidality — the classical picture

A triple $(\mathrm{Vect}_k, \otimes, k)$

Underlying category

- **Objects** : Vector spaces $U, V, W, \dots$
- **Morphisms** : linear maps $f, g, h, \dots$
- **Composition** : Composition of linear maps $U \xrightarrow{f} V \xrightarrow{g} W$
- **Identity** : Identity linear map $V \xrightarrow{id_V} V$

Tensor product

- A **functor** $\otimes : \mathrm{Vect} \times \mathrm{Vect} \to \mathrm{Vect}$

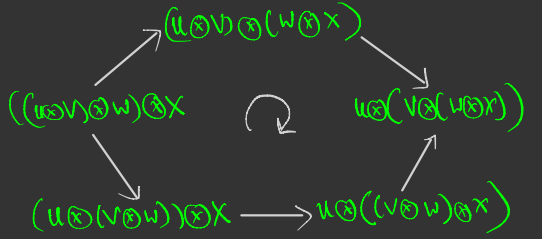
$(V, W) \longmapsto V \otimes W$

$(V \xrightarrow{f} V', W \xrightarrow{g} W') \longmapsto (V \otimes V' \xrightarrow{f \otimes g} W \otimes W')$

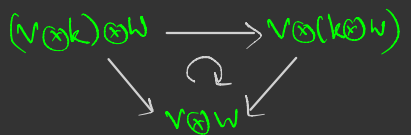
- A **unit** $k \in \mathrm{Vect}$
- **Associators** : $\alpha_{U,V,W} : (U \otimes V) \otimes W \xrightarrow{\cong} U \otimes (V \otimes W)$
- **Unitors** : $l_V : (k \otimes V) \xrightarrow{\cong} V \quad r_V : V \otimes k \xrightarrow{\cong} V$
- **Braiding** : $\beta_{V,W} : V \otimes W \xrightarrow{\cong} W \otimes V$

Coherence axioms

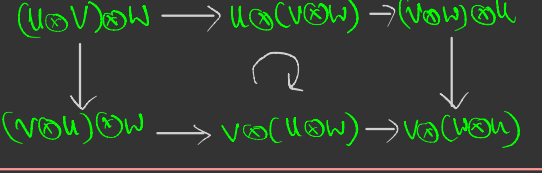
- **Pentagon** :



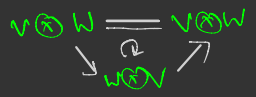
- **Triangle** :



- **Hexagon** :



- **Symmetry** :



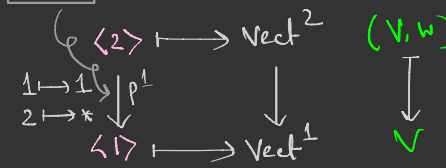
SYMMETRIC MONOIDALITY — THE SPECIAL Γ OBJECT PICTURE

A (special) pseudofunctor $F : Fin_* \cong \Gamma^{op} \longrightarrow Cat$, $\langle m \rangle \longmapsto Vect^m$ ($\langle m \rangle = \{*, 1, 2, \dots, m\}$)

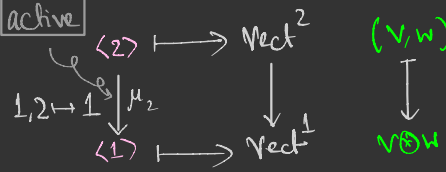
$F(\langle m \rangle) \simeq F(\langle 1 \rangle)^m$

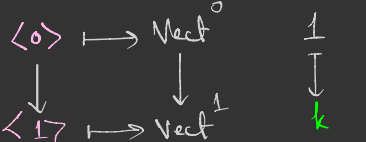
OPERATIONS

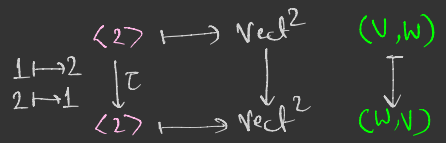
inert



active

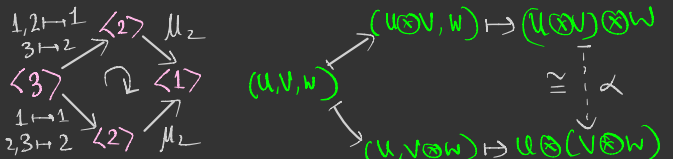




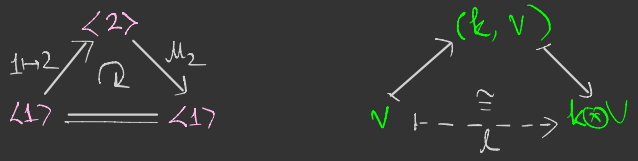


COHERATORS

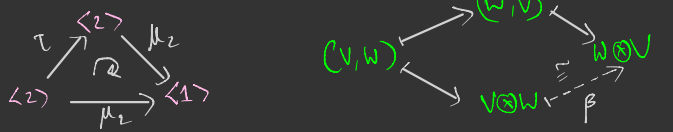
ASSOCIATORS



UNITORS



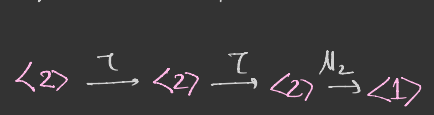
BRAIDING

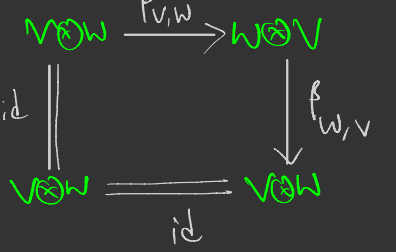


AXIOMS

SYMMETRY

$\mu_2 \tau = \mu_2$, $\tau^2 = id$





$\beta_{W,V} \circ \beta_{V,W} = id_{V \otimes W}$

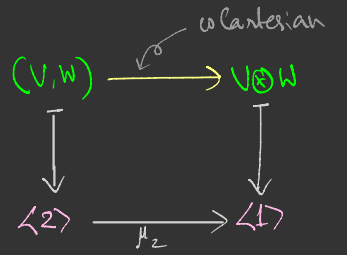
SYMMETRIC MONOIDALITY — THE coCARTESIAN FIBRATION PICTURE

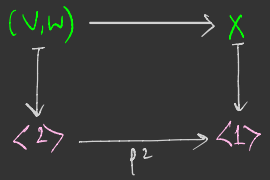
A (special) coCartesian fibration $p: \mathrm{Vect}^{\otimes} \longrightarrow \mathrm{Fin}_*$

$\mathrm{Vect}^{\otimes}_{\langle m\rangle} \simeq (\mathrm{Vect}^{\otimes}_{\langle 1\rangle})^m$

$(V_1, V_2, \dots, V_m) \longmapsto \langle m\rangle$

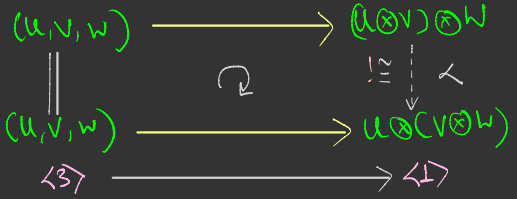
OPERATIONS



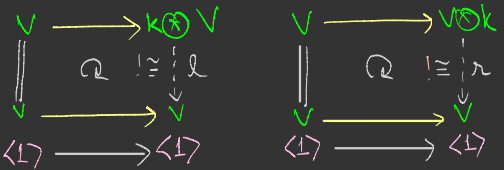


COHERATORS

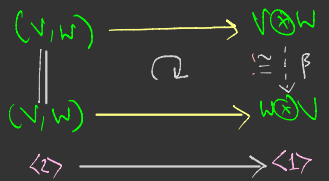
ASSOCIATORS



UNITORS

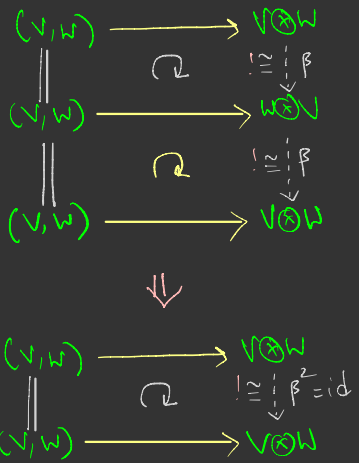


BRAIDING



AXIOMS

SYMMETRY



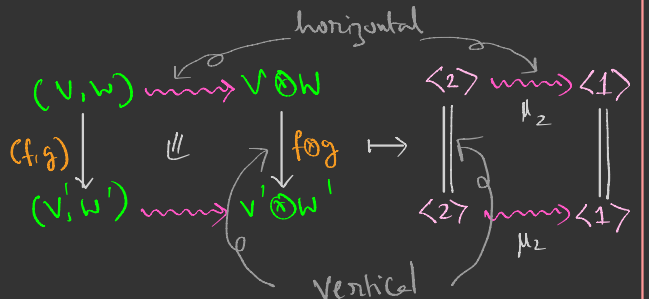
$$\beta_{W,V} \circ \beta_{V,W} = \mathrm{id}_{V\otimes W}$$

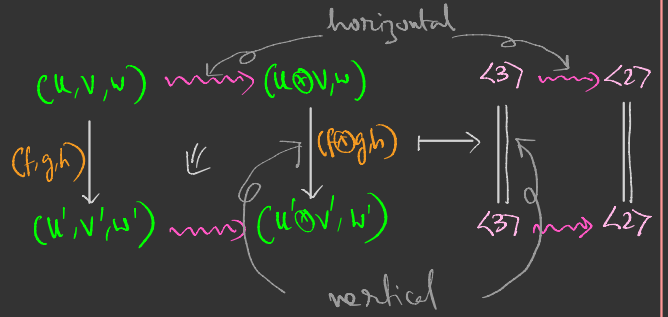
SYMMETRIC MONOIDALITY — THE LEFT DOUBLE FIBRATION PICTURE

A (special) left double fibration $P : \mathrm{Vect}^{\otimes} \longrightarrow \mathrm{H}_1(\mathrm{Fin}_*)$

$\mathrm{Vect}^{\otimes}_{\langle m \rangle} \simeq (\mathrm{Vect}^{\otimes}_{\langle 1 \rangle})^m$

DOUBLE STRUCTURE

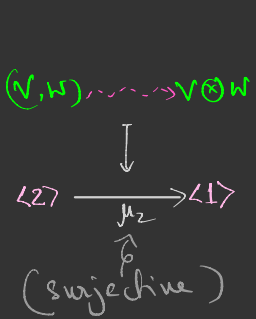


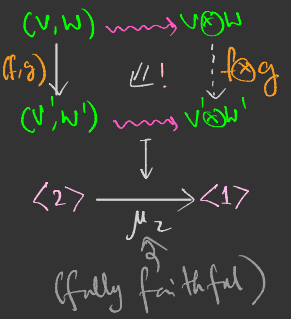


Idea: ① Horizontal : Tensor operations

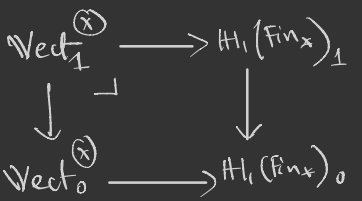
② Vertical : Infibre maps

OPERATIONS









is a pseudo pullback in Cat.

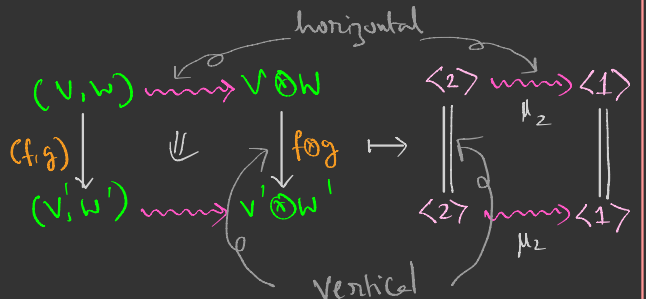
AXIOMS

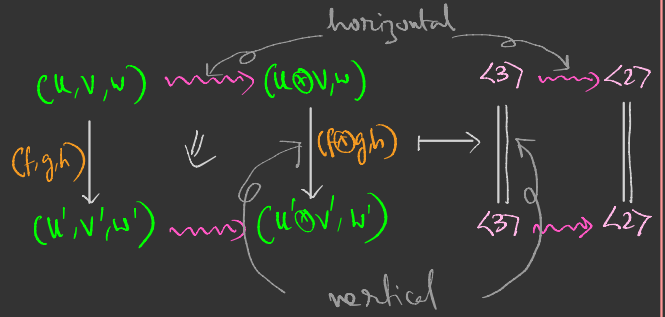
SYMMETRIC MONOIDALITY — THE LEFT DOUBLE FIBRATION PICTURE

A (special) left double fibration $P : \mathrm{Vect}^{\otimes} \longrightarrow \mathbb{H}_1(\mathrm{Fin}_*)$

$\mathrm{Vect}^{\otimes}_{\langle m \rangle} \simeq (\mathrm{Vect}^{\otimes}_{\langle 1 \rangle})^m$

DOUBLE STRUCTURE



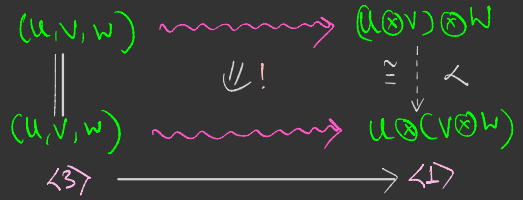


Idea: ① Horizontal : Tensor operations

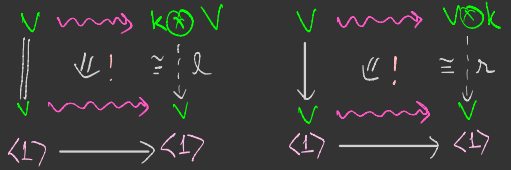
② Vertical : Infibre maps

COHERATORS

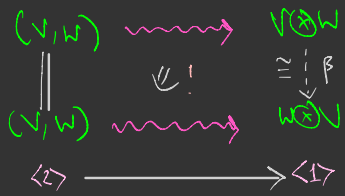
ASSOCIATORS



UNITORS



BRAIDING



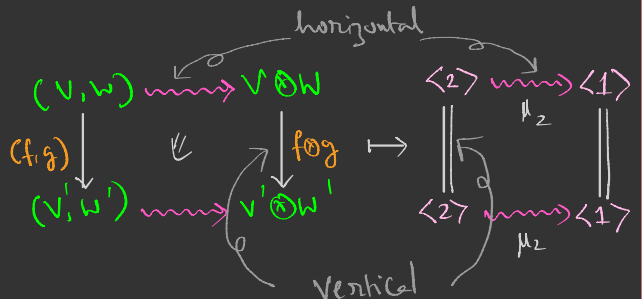
AXIOMS

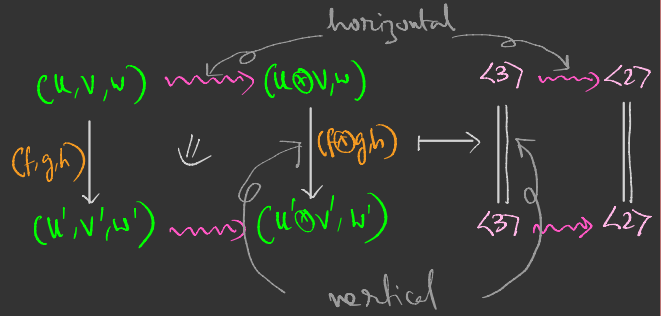
SYMMETRIC MONOIDALITY — THE LEFT DOUBLE FIBRATION PICTURE

A (special) left double fibration $P : \mathrm{Vect}^{\otimes} \longrightarrow \mathbb{H}_1(\mathrm{Fin}_*)$

$\mathrm{Vect}^{\otimes}_{\langle m \rangle} \simeq (\mathrm{Vect}^{\otimes}_{\langle 1 \rangle})^m$

DOUBLE STRUCTURE



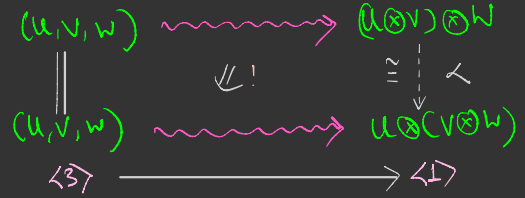


Idea : ① Horizontal : Tensor operations

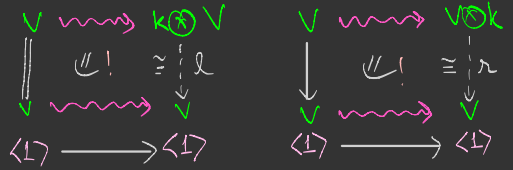
② Vertical : Infibre maps

COHERATORS

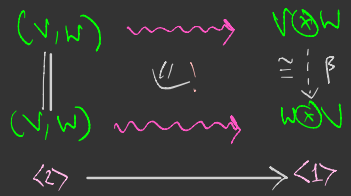
ASSOCIATORS



UNITORS

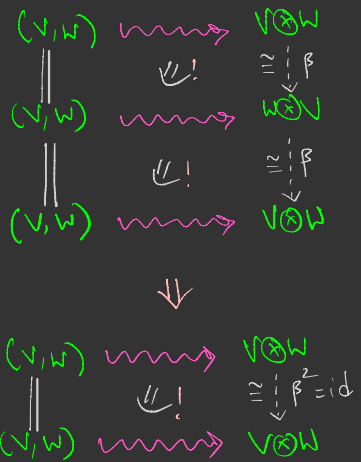


BRAIDING



AXIOMS

SYMMETRY



$$\beta_{W,V} \circ \beta_{V,W} = \mathrm{id}_{V \otimes W}$$

LEFT DOUBLE FIBRATIONS

HORIZONTAL SATURATION

Suppose X is a presentation of $V \otimes W$, i.e

$$V \otimes W \xrightarrow[\cong]{f} X$$

we should be able to complete the square

$$\begin{tikzcd}
(V,W) \arrow[r, squiggly] \arrow[d, equal] & V \otimes W \arrow[d, "\cong"', "f"] \\
(V,W) & X
\end{tikzcd}$$

$$\langle 2 \rangle \xrightarrow{\mu_2} \langle 1 \rangle$$

$$\Updownarrow$$

$$\begin{tikzcd}
{[0]} \arrow[r] \arrow[d, hook] & \mathbb{V}\mathrm{ect}_1^{\otimes} \arrow[d] \\
\mathbb{I} \arrow[r] \arrow[ur, dashed] & (\mathbb{V}\mathrm{ect}_0^{\otimes} \times \mathbb{V}\mathrm{ect}_0^{\otimes}) \times_{\mathbb{H}_1(\mathrm{Fin}_*)_0 \times \mathbb{H}_1(\mathrm{Fin}_*)_0} \mathbb{H}_1(\mathrm{Fin}_*)_1
\end{tikzcd}$$

walking isomorphism

⚠ This is an instance of Reedy fibrancy.

FORMAL DEFINITION

Def^{n}: A left double fibration over $\mathbb{H}_1(\mathrm{Fin}_*)$ is a map of double categories $P : \mathbb{E} \longrightarrow \mathbb{H}_1(\mathrm{Fin}_*)$ s.t

① $$\begin{tikzcd}
\mathbb{E}_1 \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{H}_1(\mathrm{Fin}_*)_1 \arrow[d] \\
\mathbb{E}_0 \arrow[r] & \mathbb{H}_1(\mathrm{Fin}_*)_0
\end{tikzcd}$$ is a pseudo pullback in $\mathcal{C}at$, equivalently homotopy pullback in $(\mathcal{C}at)_{folk}$.

② The horizontal nerve of P

$$N^h P : N^h \mathbb{E} \longrightarrow N^h \mathbb{H}_1(\mathrm{Fin}_*)$$

is a Reedy fibration in $(\mathcal{C}at_{folk})^{\Delta^{op}}$.

COMPARISON RESULTS

coCARTESIAN DESCRIPTION

Thm (Classical): There is an equivalence of categories

$$\{\text{Symmetric monoidal categories}\}$$
$$\cong$$
$$\{\text{(Special) coCartesian fibrations over } \mathrm{Fin}_*\}$$

Idea:

$$(\mathcal{C}, \otimes, \mathbb{1}) \longmapsto \begin{array}{c} \mathcal{C}^{\otimes} \\ \downarrow \\ \mathrm{Fin}_* \end{array} \qquad \begin{array}{c} (X_1, \dots, X_n) \\ \downarrow \\ \langle n \rangle \end{array}$$

$$\begin{array}{c} \mathcal{D} \\ \downarrow \\ \mathrm{Fin}_* \end{array} \longmapsto (\mathcal{D}_{\langle 1 \rangle}, ?, \mathcal{D}_{\langle 0 \rangle} \to \mathcal{D}_{\langle 1 \rangle})$$

STRAIGHTENING – UNSTRAIGHTENING

Thm (Grothendieck): There is an equivalence of categories

$$\{\text{(Special) coCartesian fibrations over } \mathrm{Fin}_*\}$$
$$\cong$$
$$\{\text{(Special) pseudofunctors } \mathrm{Fin}_* \to \mathrm{Cat}\}$$

Idea:

$$\begin{array}{c} \mathcal{C} \\ \downarrow \\ \mathrm{Fin}_* \end{array} \longmapsto \begin{array}{c} \mathrm{Fin}_* \longrightarrow \mathrm{Cat} \\ \langle m \rangle \longmapsto \mathcal{C}_{\langle m \rangle} \end{array}$$

$$F : \mathrm{Fin}_* \longrightarrow \mathrm{Cat} \longmapsto \begin{array}{c} \int F \\ \downarrow \\ \mathrm{Fin}_* \end{array} \qquad \begin{array}{c} (\langle m \rangle, X \in F\langle m \rangle) \\ \downarrow \\ \langle m \rangle \end{array}$$

coCARTESIAN – LEFT DOUBLE

Thm [D]: There is an equivalence of categories

$$\{\text{(Special) left double fibrations over } \mathbb{H}_1(\mathrm{Fin}_*)\}$$
$$\cong$$
$$\{\text{(Special) coCartesian fibrations over } \mathrm{Fin}_*\}$$

Idea:

$$\begin{array}{c} \mathbb{E} \\ p \downarrow \\ \mathbb{H}_1(\mathrm{Fin}_*) \end{array} \longmapsto \begin{array}{c} \mathcal{G}(\mathbb{E}) \\ \downarrow \\ \mathrm{Fin}_* \end{array} \qquad \begin{array}{c} X \overset{h}{\rightsquigarrow} U \\ \downarrow \\ Y \\ \downarrow \\ p_0(X) \xrightarrow[p_1(h)]{} p_0(Y) \end{array}$$

$$\begin{array}{c} \mathcal{C} \\ p \downarrow \\ \mathrm{Fin}_* \end{array} \longmapsto \begin{array}{c} \mathbb{I}(\mathcal{C}) \\ \downarrow \\ \mathbb{H}_1(\mathrm{Fin}_*) \end{array} \qquad \begin{array}{c} X \overset{f}{\rightsquigarrow} Y \\ \downarrow \\ p(X) \xrightarrow[p(f)]{} p(Y) \end{array}$$

LEFT DOUBLE (∞,n)-FIBRATIONS

HORIZONTAL SATURATION

Suppose X is a presentation of $V \otimes W$, i.e

$$V \otimes W \xrightarrow[\cong]{f} X$$

we should be able to complete the square

$$\begin{array}{ccc} (V,W) & \dashrightarrow & V \otimes W \\ \| & & \cong \downarrow f \\ (V,W) & & X \end{array}$$

$$\begin{array}{ccc} \langle 1 \rangle & \xrightarrow{M_2} & \langle 1 \rangle \\ \| & & \| \\ [0] & \longrightarrow & \mathbb{V}\mathrm{ect}^{\otimes}_1 \end{array}$$

walking isomorphism

$$\begin{array}{ccc} [0] & \longrightarrow & \mathbb{V}\mathrm{ect}^{\otimes}_1 \\ \downarrow & \nearrow & \downarrow \\ \mathbb{I} & \longrightarrow & (\mathbb{V}\mathrm{ect}^{\otimes}_0 \times \mathbb{V}\mathrm{ect}^{\otimes}_0) \times_{\mathrm{H}_1(\mathrm{Fin}_*)_0 \times \mathrm{H}_1(\mathrm{Fin}_*)_0} \mathrm{H}_1(\mathrm{Fin}_*)_1 \end{array}$$

⚠ This is an instance of Reedy fibrancy.

FORMAL DEFINITION

Def: A left double (∞,n)-fibration over $N(\mathrm{Fin}_*)$ ~~$\mathrm{H}_1(\mathrm{Fin}_*)$~~ is a map of double (∞,n) categories $P: \mathbb{E} \longrightarrow \mathrm{H}_1(\mathrm{Fin}_*)$ ~~$N(\mathrm{Fin}_*)$~~ s.t

① $$\begin{array}{ccc} \mathbb{E}_1 & \longrightarrow & \mathrm{H}_1(\mathrm{Fin}_*)_1 \\ \downarrow & \lrcorner & \downarrow \\ \mathbb{E}_0 & \longrightarrow & \mathrm{H}_1(\mathrm{Fin}_*)_0 \end{array}$$ is a

~~pseudo pullback in Cat, equivalently~~

homotopy pullback in $(\mathrm{Cat})_{\mathrm{folk}}$ $M_{(\infty,n)}$.

② ~~The horizontal nerve of P~~

$$N^h P : N^h \mathbb{E} \longrightarrow N^h \mathrm{H}_1(\mathrm{Fin}_*)$$

is a Reedy fibration in $(\mathrm{Cat}_{\mathrm{folk}})^{\Delta^{op}}_{M_{(\infty,n)}}$.

$(\infty,1)$ - COMPARISON RESULTS

coCARTESIAN DEFINITION

Def / Thm (Lurie) (~~classical~~): ~~There is an equivalence of categories~~

$$\{\text{Symmetric monoidal } (\infty,1)\text{-categories}\}$$

$$\text{"="}$$

$$\{(\text{Special}) \text{ coCartesian } (\infty,1)\text{-fibrations over } \frac{\text{Fin}_*}{N(\text{Fin}_*)}\}$$

Idea:

$$(\mathcal{C},\otimes,\mathbb{1}) \longmapsto \begin{array}{ccc} \mathcal{C}^{\otimes} & & (X_1,\dots,X_n) \\ \downarrow & & \downarrow \\ \text{Fin}_* & & \langle n\rangle \end{array}$$

$$\begin{array}{c} \mathcal{D} \\ \downarrow \\ \text{Fin}_* \end{array} \longmapsto \left(\mathcal{D}_{\langle 1\rangle},\ ?,\ \mathcal{D}_{\langle 0\rangle}\to\mathcal{D}_{\langle 1\rangle}\right)$$

$(\infty,1)$-STRAIGHTENING - UNSTRAIGHTENING

Thm (Lurie) (~~Grothendieck~~): There is an equivalence of $(\infty,1)$-categories

$$\{(\text{Special}) \text{ coCartesian } (\infty,1)\text{-fibrations over } \frac{\text{Fin}_*}{N(\text{Fin}_*)}\}$$

$$\text{is}$$

$$\{(\text{Special}) \text{ (~~pseudo~~)functors } N(\text{Fin}_*) \to (\infty,1)\text{-Cat}\}$$

Idea:

$$\begin{array}{c} \mathcal{C} \\ \downarrow \\ \text{Fin}_* \end{array} \longmapsto \begin{array}{ccc} \text{Fin}_* & \longrightarrow & \text{Cat} \\ \langle m\rangle & \longmapsto & \mathcal{C}_{\langle m\rangle} \end{array}$$

$$F:\text{Fin}_*\to\text{Cat} \longmapsto \begin{array}{ccc} \int F & & (\langle m\rangle, X\in F\langle m\rangle) \\ \downarrow & & \downarrow \\ \text{Fin}_* & & \langle m\rangle \end{array}$$

coCARTESIAN - LEFT DOUBLE

Thm [D]: There is an equivalence of $(\infty,1)$-categories

$$\{(\text{Special}) \text{ left double } (\infty,1)\text{-fibrations over } \frac{\text{Fin}_*}{N(\text{Fin}_*)}\}$$

$$\text{is}$$

$$\{(\text{Special}) \text{ coCartesian } (\infty,1)\text{-fibrations over } \frac{\text{Fin}_*}{N(\text{Fin}_*)}\}$$

Idea:

$$\begin{array}{ccc} \mathbb{E} & & \mathcal{G}(\mathbb{E}) \\ \downarrow p & \longmapsto & \downarrow \\ \Pi_1(\text{Fin}_*) & & \text{Fin}_* \end{array} \qquad \begin{array}{ccc} X & \xrightarrow{h} & U \\ & & \downarrow \\ & & Y \\ \downarrow & & \downarrow \\ p_0(X) & \xrightarrow[p_1(h)]{} & p_0(Y) \end{array}$$

$$\begin{array}{ccc} \mathcal{C} & & \mathbb{\Pi}(\mathcal{C}) \\ \downarrow p & \longmapsto & \downarrow \\ \text{Fin}_* & & \Pi_1(\text{Fin}_*) \end{array} \qquad \begin{array}{ccc} X & \xrightarrow{f} & Y \\ & & \downarrow \\ p(X) & \xrightarrow[p(f)]{} & p(Y) \end{array}$$

Definition [D]: Let $M_{(\infty,n)}$ be a suitable model category presenting (∞,n)-categories. A symmetric monoidal (∞,n)-category relative to $M_{(\infty,n)}$ is a special left double (∞,n)-fibration over $N(\mathrm{Fin}_*)$.

WHAT'S NEXT?

1. How does this presentation recover strict symmetric monoidal 2-categories? [Almost done!]
2. Construct explicit examples: Morita $(\infty,2)$-category / (∞,∞)-category of bordisms.
3. Investigate delooping hypothesis in this framework.

THANK YOU