Fixed Point Properties in Synthetic homotopy Theory

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Fixed points in classical topology

Definition

A topological space X has the **strict fixed point property** (SFPP) if every continuous map $f : X \to X$ has a fixed point, i.e. a point $x \in X$ such that $f(x) = x$.

Brouwer's fixed point theorem: Bounded convex subspaces of $\mathbb{R}^n$ have the SFPP. Many more results like this in topology!

- These proofs usually rely on the law of excluded middle.
  No control over the *choice* of fixed point. In particular, it is not continuous with respect to the map.
  (exception: Banach's fixed point theorem, which uses metric spaces)
- Not homotopy-invariant: $\{*\}$ has it, but $\mathbb{R}$ does not (e.g. $x \mapsto x + 1$).

Fixed points in classical topology

Szymik [Szy15] considers continuous choices of fixed points. Motivated by order theory.

Definition

A **selection map** for a space X is a continuous $f : C(X, X) \to X$ that sends maps to fixed points.

This is a very strong property:

Proposition (Szymik)

No D^n has selection maps for $n \geqslant 1$.

Remark

Having selection maps is closed under retracts.

From these, we can conclude:

Proposition

If X has a selection map, then it is weakly contractible.

We will consider a homotopy-invariant version of the SFPP that is also weaker than Szymik's in a model-independent setting [HoTT].

Very short introduction to HoTT

Homotopy type theory (HoTT) is a foundational system with which we can do (among other things) model-independent/synthetic homotopy theory.

Everything is a space, every map we can define is continuous. Everything holds up to homotopy.

Everything true in HoTT is true in any ∞-topos, which is one motivation for using HoTT as a foundation.

All statements are of the form $a : A$ (a has type A).

Type theory	**Homotopy theory**	**Language/logic**
Dependent sum type $\sum_{(a:A)} C(a)$	fibre bundle $\mathrm{pr}_1 \colon \sum_{(a:A)} C(a) \to A$	all a satisfying $C(a)$
Dependent function type $\prod_{(a:A)} C(a)$	sections of $\mathrm{pr}_1 \colon \sum_{(a:A)} C(a) \to A$	for all a, $C(a)$
Intensional identity type $a =_A b$	path space	identifications of a and b

In particular, identity types are also types, so there are types of paths between paths, etc.

- For $f, g : A \to B$, we write $f \sim g :\equiv \prod_{(a:A)} (f(a) = g(a))$.
- We assume a **univalent universe** $\mathcal{U}$, i.e. $(A =_{\mathcal{U}} B) \simeq (A \simeq B)$ for all $A, B : \mathcal{U}$.
- We form more complicated spaces, **higher inductive types**, similarly to CW-complexes, by adding paths.

Note: the type of **pointed** maps from (A, a_0) to (B, b_0) is $\sum_{f:A\to B} (f(a_0) = b_0)$.

Very short introduction to HoTT

Definition

Define the property of a type A being n-**truncated** inductively as follows:

- A is (-2)-truncated if it is contractible.
- A is $(n+1)$-truncated if the path space $x = y$ is n-truncated for all $x, y \in A$.

In particular, a type is a **proposition** if it is (-1)-truncated and a **set** if it is 0-truncated.

For any type A there is a universal n-**truncation** $\|A\|_n$ together with a map $|-|_n : A \to \|A\|_n$.

Definition

Let (A, a_0) be a pointed type and let $n \in \mathbb{N}$. The (pointed) **loop space** $\Omega^n(A, a_0)$ is defined inductively as follows:

- $\Omega^0(A, x_0) :\equiv (A, a_0)$;
- $\Omega^1(A, a_0) :\equiv ((a_0 = a_0), \mathsf{refl}_{a_0})$;
- $\Omega^{n+1}(A, a_0) :\equiv \Omega^n(\Omega^1(A, a_0))$.

We also define $\pi_n(A, a_0) :\equiv \|\Omega^n(A, a_0)\|_0$.

In particular, $\pi_0 A = \|A\|_0$.

Fixed points in HoTT

Definition

For a type A, say $\mathsf{FP}(A) :\equiv \prod_{f:A\to A} \sum_{(a:A)} (f(a) = a)$.

Interpretation in spaces:

A space X has the **fixed point property** (FPP) if there is a continuous $(s, H) : C(X, X) \to X \times \mathrm{Path}(X)$ such that $H(f) \in \mathrm{Path}(f(s(f)), s(f))$ for all $f : X \to X$.

This is now homotopy-invariant!

Remark

Keeping track of the path is important: if A is pointed and connected, then $\prod_{f:A\to A} \sum_{(a:A)} \|f(a) = a\|_{-1}$.

Partial converse, constructively:

Proposition (Bauer)

If $\prod_{f:A\to A} \| \sum_{(a:A)} (f(a) = a)\|_{-1}$, then for all $f : A \to \{0, 1\}$ and $x, y : A$, $f(x) = f(y)$.

Fixed points in HoTT

Definition

For a type A, say $\mathrm{FP}(A) :\equiv \prod_{f:A\to A} \sum_{(a:A)} (f(a) = a)$.

Remark

Lawvere's fixed point theorem has two forms in HoTT:

- If there is a surjective function $A \twoheadrightarrow B^A$, then $\prod_{f:B\to B} \| \sum_{(b:B)} (f(b) = b) \|_{-1}$.
- If there is a function $f : A \to B^A$ that has a section, then $\mathrm{FP}(B)$.

Proposition (C., following Szymik)

- If $\mathrm{FP}(A)$ and R is a retract of A, then $\mathrm{FP}(R)$.
- $\mathrm{FP}(A \times B)$ if and only if $\mathrm{FP}(A)$ and $\mathrm{FP}(B)$.

Fixed points in HoTT

Definition

An **H-space** is a pointed type (A, a_0) equipped with an operation $\mu : A \times A \to A$ and homotopies $\mu(a_0, -) \sim \mathrm{const}_{a_0}$ and $\mu(-, a_0) \sim \mathrm{const}_{a_0}$.

Proposition

Let (A, μ) be an H-space such that for all $a : A$, $\mu(-, a)$ is a homotopy equivalence. If A has the fixed point property, then it is contractible.

Therefore none of the following spaces have the fixed point property (unless they are contractible):

1. $\mathbb{S}^1$, $\mathbb{S}^3$ and $\mathbb{S}^{7*}$;
2. Eilenberg-Mac Lane spaces of Abelian groups;
3. loop spaces of any space;
4. Any products where one component is such a space, or any space that has such a space as a retract.

What about examples? We look at 1-truncated types, which have simpler mapping spaces.

Maps between classifying spaces

Let G be a group. We assume that groups are 0-truncated.

The **classifying space** BG is the higher inductive type consisting of

$$\begin{aligned} \mathsf{base} &: BG \\ \mathsf{loop} &: G \to (\mathsf{base} = \mathsf{base}) \\ \mathsf{loop\text{-}comp} &: \textstyle\prod_{x,y:G} (\mathsf{loop}(xy) = \mathsf{loop}(x) \cdot \mathsf{loop}(y)) \end{aligned}$$

+ 1-truncation.

There is an equivalence of categories

$$B : \mathrm{Grp} \simeq \mathcal{U}_*^{\leqslant 1}_{\geqslant 0} : \pi_1.$$

In particular, $(BG \to_* BH) \simeq \mathrm{Hom}(G, H)$. What about unpointed maps?

Theorem (Gottlieb, Christensen-C.)

1. *There is a bijection* $\pi_0(BG \to BH) \simeq \mathrm{Hom}(G, H)/{\sim_{\mathrm{conj}}}$.
2. *For any* $v : \mathrm{Hom}(G, H)$*, there is an iso* $\pi_1((BG \to BH), Bv) \simeq C_H(v(G))$.

Note: if X is pointed connected, $(X \to BH) \simeq (B(\pi_1 X) \to BH)$.

Maps between classifying spaces

By analysing the bijection for $\pi_0(BG \to BH)$, we can prove the following without using the axiom of choice:

Proposition

Let G and H be finite groups. Then we can choose a pointed map in every path component of $(BG \to BH)$.

More generally, this works if G, H and $\mathrm{Hom}(G, H)$ are **compact types** (as defined by Escardó) and H has decidable paths.

We now want to show:

Theorem

If G is a finite, simple, non-Abelian group, then $\mathrm{FP}(BG)$.

Proof Outline. Let $f : BG \to BG$. By the above proposition, there is a $v : \mathrm{Hom}(G, G)$ such that $\| f \sim Bv \|_{-1}$. This v is either constant or an isomorphism by the assumptions on G.

Case 1: If v is constant, then Bv is constant. Then f is also constant since the following result applies to BG:

Proposition

Let X be a pointed type and (Y, y_0) a pointed connected type. Then the function $\mathrm{const}_{(-)} : Y \to (X \to Y)_{(\mathrm{const}_{y_0})}$ is an equivalence if and only if the space $(X \to_* \Omega Y)$ is contractible.

Maps between classifying spaces

Theorem (Gottlieb, Christensen-C.)

1. *There is a bijection* $\pi_0(BG \to BH) \simeq \mathrm{Hom}(G,H)/{\sim_{\mathrm{conj}}}$.
2. *For any* $v : \mathrm{Hom}(G,H)$, *there is an iso* $\pi_1((BG \to BH), Bv) \simeq C_H(v(G))$.

Theorem

If G is a finite, simple, non-Abelian groups, then $\mathrm{FP}(BG)$.

Case 2: If v an isomorphism, then $C_G(v(G)) \simeq \{*\}$.

By Whitehead's theorem (which holds for truncated types in HoTT), the component of Bv is contractible and therefore $f \sim Bv$. In particular, f is pointed. □

Since Szymik's examples are interpreted as contractible spaces, this is the first non-contractible example with this fixed point property in the ∞-category of spaces.

Formalisation

We have formalised our work in Rocq (formerly Coq), available at

https://github.com/thchatzidiamantis/HoTT-FixedPoints.

The $\pi_k(BG \to BH)$ computations are now part of the Coq-HoTT library.

References

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