(Trick or) Trees associated with unitary partition complexes

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Outline

Partition complexes and trees

Unitary partition complexes and topological trees

Future directions

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Definition and examples of partition complexes

Let $\underline{n} = \{1, \ldots, n\}$ denote the set with n elements.

Definition

The *n-th partition complex* is the classifying space of the poset $\mathcal{P}_n$ of proper, nontrivial, unordered partitions of $\underline{n}$ under refinement.

- *Proper* means that we exclude the partition $\{\underline{n}\}$.
- *Nontrivial* means that we exclude the partition $\{\{1\}, \ldots, \{n\}\}$.

Otherwise, the resulting classifying space would be contractible!

Examples

- For $n = 1$ and $n = 2$, the n-th partition complex is empty.
- The third partition complex is

$$B(\mathcal{P}_3) = \{\{1\}, \{2,3\}\} \coprod \{\{2\}, \{1,3\}\} \coprod \{\{3\}, \{1,2\}\}.$$

Computations of partition complexes

- (Exercise?) The fourth partition complex is

$$B(\mathcal{P}_4) \simeq \bigvee_{i=1}^{6} S^1.$$

- (Robinson–Whitehouse, 1996) The n-th partition complex is

$$B(\mathcal{P}_n) \simeq \bigvee_{i=1}^{(n-1)!} S^{n-3}.$$

- (Arone–Dwyer–Lesh, 2016) Fixed-point spaces under the Σ_n-action and Bredon homology and cohomology.

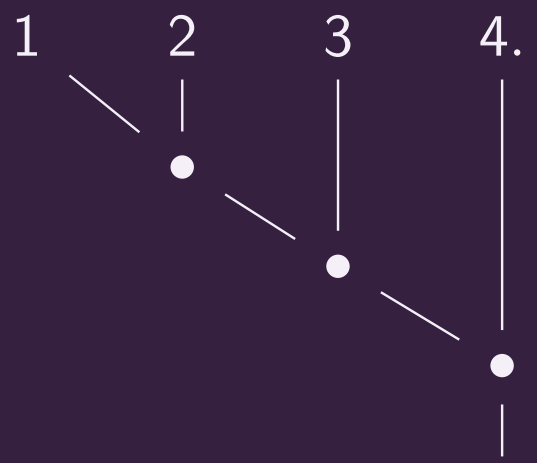
Trees associated with partition complexes

In a 2023 paper, Heuts and Moerdijk develop a poset of trees $\mathcal{A}_n$. The underlying set of $\mathcal{A}_n$ comprises isomorphism classes of

- finite, rooted trees T with set of leaves $\underline{n} = \{1, \ldots, n\}$
- such that every vertex of T has at least two incoming edges.

Example

An example of a tree T in $\mathcal{A}_4$ is



The partial order on trees

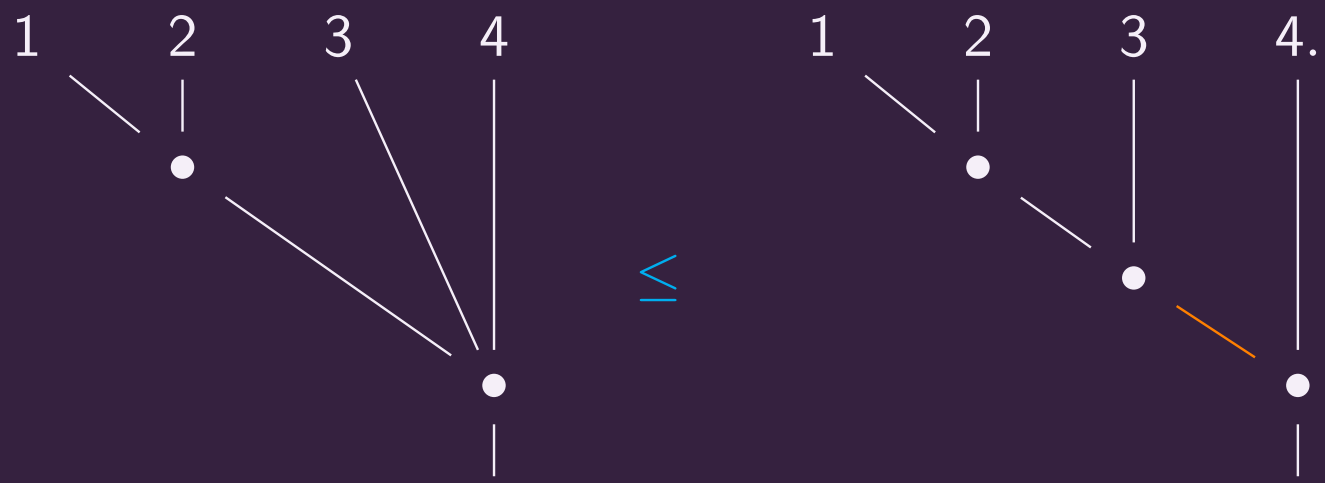
The partial order on $\mathcal{A}_n$ is defined by decreeing that

$$T' \leq T$$

if and only if T' is obtained from T by contracting inner edges.

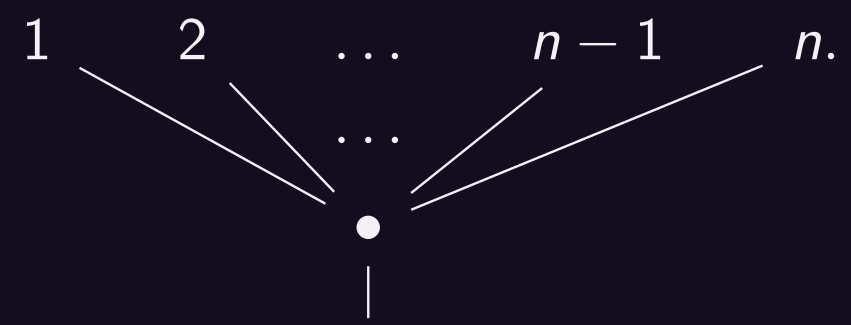
Example

In the poset $\mathcal{A}_4$, we find that



The curious case of the corolla

As it stands, the poset $\mathcal{A}_n$ has a least element, namely the *corolla*



To avoid a contractible classifying space, we remove the corolla, and we work with the poset $\mathcal{A}_n^+$.

The association with partition complexes

Theorem (Heuts–Moerdijk, 2023)

The n-th partition complex $B(\mathcal{P}_n)$ is homotopy equivalent to the classifying space $B(\mathcal{A}_n^+)$ of the poset of trees $\mathcal{A}_n^+$.

Proof sketch

Applying Quillen's Theorem A to the explicit zig-zag of functors

$$\mathcal{P}_n \xleftarrow{\zeta_n} (\Delta \downarrow N(\mathcal{P}_n)) \xrightarrow{\varphi_n} \mathcal{A}_n^+$$

yields the zig-zag of homotopy equivalences of classifying spaces

$$B(\mathcal{P}_n) \xleftarrow{\simeq} B(\Delta \downarrow N(\mathcal{P}_n)) \xrightarrow{\simeq} B(\mathcal{A}_n^+).$$

□

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Definition of unitary partition complexes

Partition complexes find a unitary analog due to Arone and Lesh.

Definition

The *n-th unitary partition complex* is the classifying space of the poset $\mathcal{L}_n$ of proper and unordered partitions of $\mathbb{C}^n$ as a direct sum of nonzero and pairwise orthogonal subspaces under refinement.

- *Proper* means that we exclude the partition $\{\mathbb{C}^n\}$.
- There are uncountably many partitions of $\mathbb{C}^n$ into n lines!

The poset $\mathcal{L}_n$ is *topological*, that is, internal to topological spaces, for it inherits a topology from the Grassmannians of $\mathbb{C}^n$.

Example

An example of an element of $\mathcal{L}_4$, that is, a partition of $\mathbb{C}^4$, is

$$\{\langle e_1, e_2\rangle, \langle e_3\rangle, \langle e_4\rangle\},$$

where e_i denotes the i-th standard basis vector of $\mathbb{C}^4$.

Computations of unitary partition complexes

- The first unitary partition complex is empty.
- (Bergner–Joachimi–Lesh–Stojanoska–Wickelgren, 2015) The second unitary partition complex is

$$B(\mathcal{L}_2) \simeq \mathbb{R}P^2.$$

  They also compute fixed-point spaces under the $U(n)$-action.
- (Bergner–Joachimi–Lesh–Stojanoska–Wickelgren, 2019) For $n \geq 3$, the n-th unitary partition complex is simply connected.

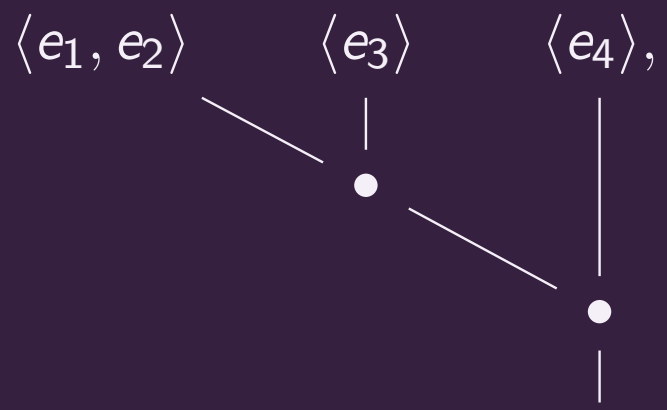
Trees associated with unitary partition complexes

In our forthcoming paper, we build a topological poset of trees $\mathcal{T}_n$. The underlying set of $\mathcal{T}_n$ comprises isomorphism classes of

- finite, rooted trees T whose set of leaves is an element of $\mathcal{L}_n$
- such that every vertex of T has at least two incoming edges.

Example

An example of a tree T in $\mathcal{T}_4$ is



where e_i denotes the i-th standard basis vector of $\mathbb{C}^4$.

The topological poset of trees

The partial order on $\mathcal{T}_n$ is again defined by decreeing that

$$T' \leq T$$

if and only if T' is obtained from T by contracting inner edges.
The poset $\mathcal{T}_n$ inherits a topology from the Grassmannians of $\mathbb{C}^n$.
We have a space of corollas $\mathrm{Cor}(\mathcal{T}_n)$, one for each partition in $\mathcal{L}_n$.

Proposition (Bergner–Brunialti–C.–Turner, 2026)

The bijection

$$\mathrm{Ob}(\mathcal{L}_n) \xrightarrow{\cong} \mathrm{Cor}(\mathcal{T}_n),$$

where $\mathrm{Ob}(\mathcal{L}_n)$ *is the object space of* $\mathcal{L}_n$*, is a homeomorphism.*

The association with unitary partition complexes

Theorem (Bergner–Brunialti–C.–Turner, 2026)

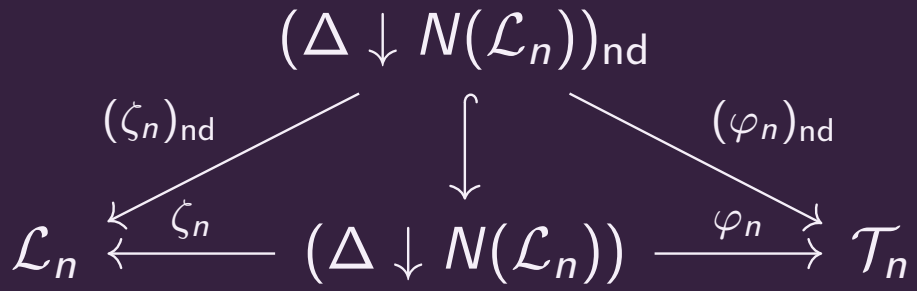
The n-th unitary partition complex $B(\mathcal{L}_n)$ is homotopy equivalent to the classifying space $B(\mathcal{T}_n)$ of the poset of trees $\mathcal{T}_n$.

We need to invoke Quillen's Theorem A for topological categories; such a topological analog is established in a 2024 paper of Roberts. We also develop new results in the theory of topological categories. Moreover, our arguments also work for the case of partitions of $\mathbb{R}^n$.

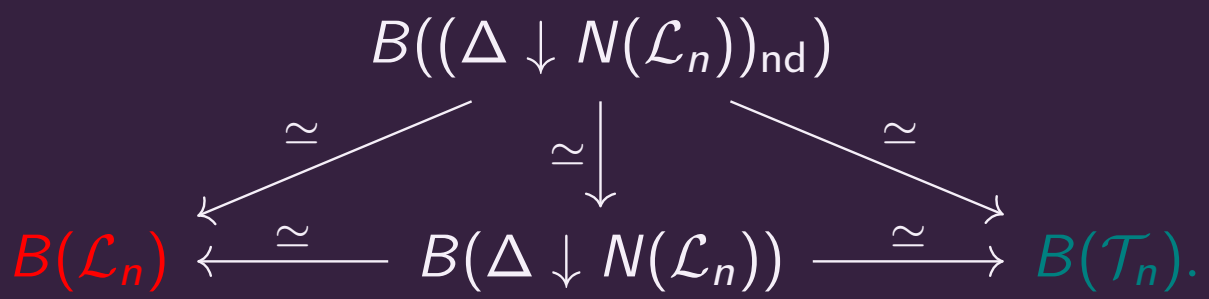
Schematic of our proof

Proof sketch

The commutative diagram of continuous functors



induces the commutative diagram of homotopy equivalences





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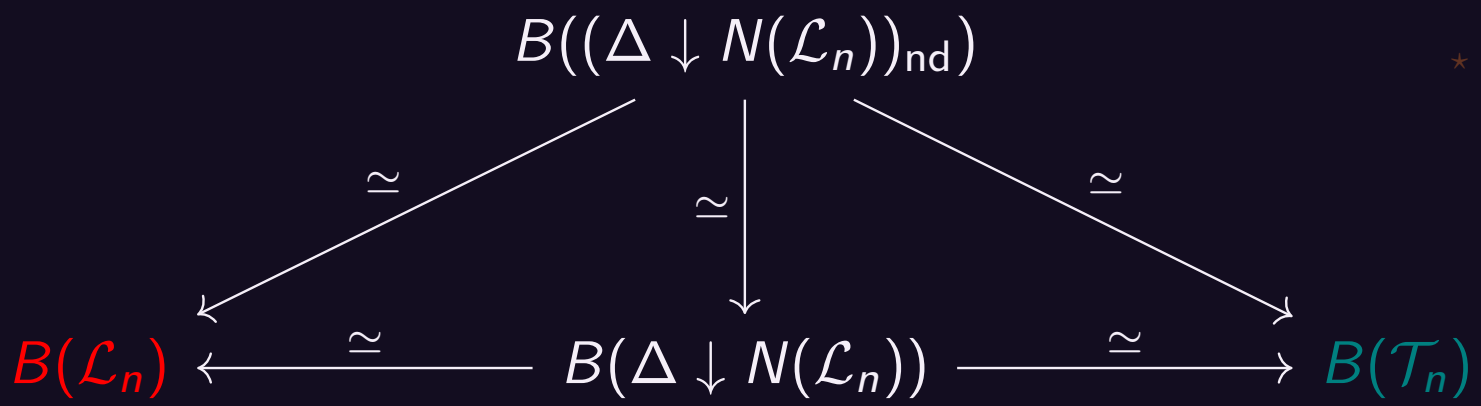
Open questions

We are interested in

- connections of our work with the study of operads;
- a (Robinson–Whitehouse) description of the homotopy type of unitary partition complexes; and
- a classification of problematic subgroups of $O(n)$ akin to that for $U(n)$ by Bergner–Joachimi–Lesh–Stojanoska–Wickelgren.

Thank you!

What part of



do you not understand? No, seriously - ask me your questions!