A compositional definition of automata

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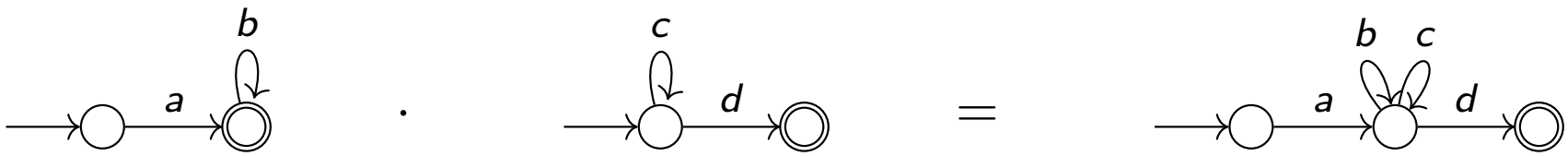
Plan

1. Motivation
2. Universal property of automata
3. Idea of the proof
4. Conclusion

Motivation

Naive composition of automata:



ab^* $\quad$ c^*d $\quad$ $a(b+c)^*d$

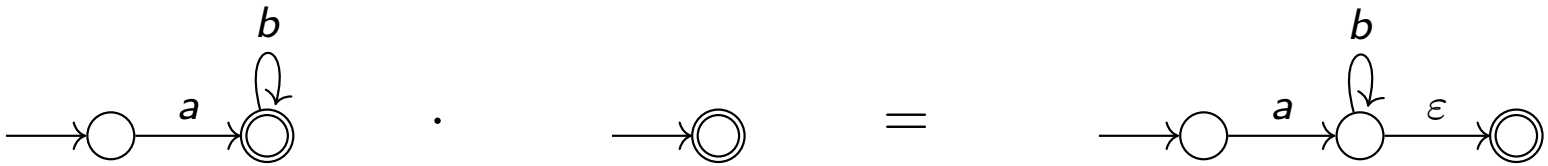
Standard solution

Use ε-transitions:



A category of automata

- Question: can we build a category whose objects are finite sets, and whose morphisms $I \to O$ are automata with input states I and output states O, together with a functor that computes the recognized language?
- Problem: we need to allow some ε-transitions to be contracted if we want composition to be unital:



- Solution: Define the category of automata by a universal property!

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Formal definition

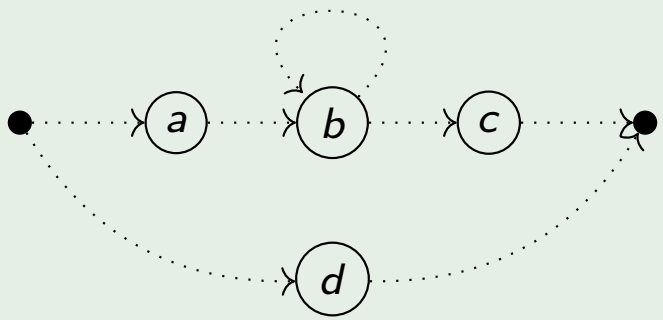
Definition

An *automaton* A is a tuple $A = (I, T, \ell, O, R)$ consisting of a set I of *inputs*, a set O of *outputs*, a set T of *transitions*, a *labeling* function $\ell : T \to \Sigma$ and a relation $R \subseteq (I + T) \times (T + O)$.

We write **Aut** for the category of finite sets and automata.

Example

An automaton recognizing the language $abb^*c + d$ is

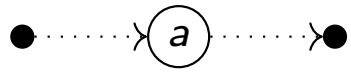


If you find this definition weird...

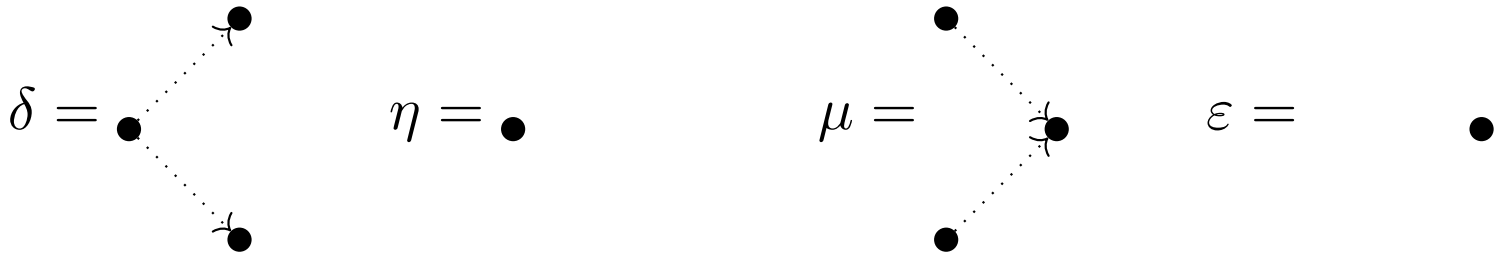
... we can prove that this is just an automaton with ε-transition, considered up to contraction of "useless" ε-transitions!

Needed structure

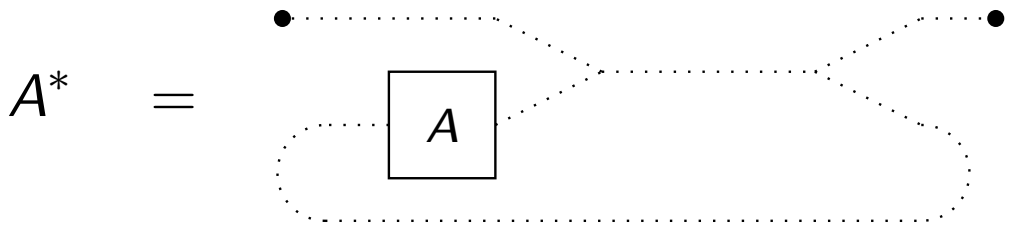
- (symmetric) monoidal category: $A \oplus A' : m \oplus m' \to n \oplus n'$;
- with endomorphisms $a : 1 \to 1$ for every letter $a \in \Sigma$:



- such that the object 1 has a bialgebra structure :



- with a trace structure, in order to define:



The universal property

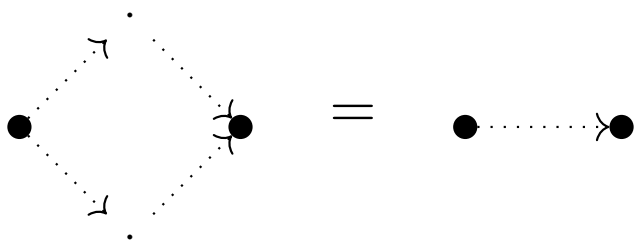
Theorem (baby version)

The category **Rel** *is the free loop-free traced symmetric monoidal category containing a qualitative bialgebra.*

Theorem

The category **Aut** *is the free loop-free traced symmetric monoidal category containing a qualitative bialgebra and Σ-endomorphisms.*

Here qualitive means that



and loop-free means that $(\mathrm{id}_1)^* = \mathrm{id}_1$.

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The Circ construction

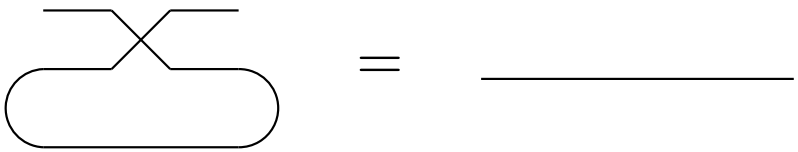
Definition

Given a symmetric monoidal category $\mathbb{C}$, we write $Circ(\mathbb{C})$ for the monoidal category with the same objects as $\mathbb{C}$, where a morphism $(u, f) : x \to y$ is a pair consisting of an object $u \in \mathbb{C}$ and a morphism $x \oplus u \to y \oplus u$ in the underlying category, up to isomorphism.

The feedback on v of a morphism $(u, f) : x \oplus v \to y \oplus v$ is the morphism $(v \oplus u, f)$.

Theorem

The category $Circ(\mathbb{C})$ is the free feedback category on $\mathbb{C}$; the free traced category on $\mathbb{C}$ can be obtained by quotienting it under (the congruence generated by) yanking.

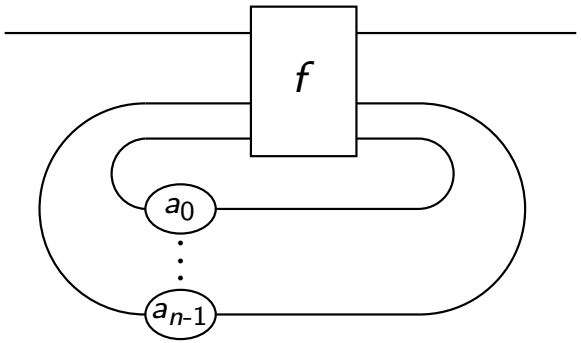
Here yanking means 

Colored Circ construction

Definition

Given a set Σ and a PROP $\mathbb{C}$ whose only isomorphisms are symmetries, we define $Circ_\Sigma(\mathbb{C})$ as the category with the same objects as $\mathbb{C}$ and where a morphism $(n, \ell, f) : x \to y$ is a triple consisting of an object $n \in \mathbb{C}$, a *labeling* function $\ell : [n] \to \Sigma \sqcup \{\varepsilon\}$, and a morphism $f : x \oplus n \to y \oplus n$ in $\mathbb{C}$, considered up to label-preserving isomorphism.

The feedback of a morphism $(n, \ell, f) : x \oplus m \to y \oplus m$ is $(m \oplus n, \ell', f)$ with $\ell'(i) = \ell(i)$ for $i \in [n]$ and $\ell'(i) = \varepsilon$ for $i \in [m]$. Graphically, writing $a_i = \ell(i)$, a morphism (n, ℓ, f) in $Circ_\Sigma(\mathbb{C})$ can be pictured as



where the circled $a_i := \ell(i)$ are not morphisms but only indicate the labels.

An abstract proof

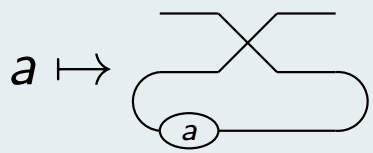
Definition

Given a PROP $\mathbb{C}$ and a set Σ, we write $\mathbb{C}_\Sigma$ for the free PROP with Σ-endomorphisms generated by $\mathbb{C}$.

Proposition

Let $\mathbb{C}$ be a PROP whose only isomorphisms are symmetries and Σ be a set. There is a feedback monoidal functor $F : Circ(\mathbb{C}_\Sigma) \to Circ_\Sigma(\mathbb{C})$ which becomes an equivalence after quotienting both sides by the yanking axiom.

Proof.



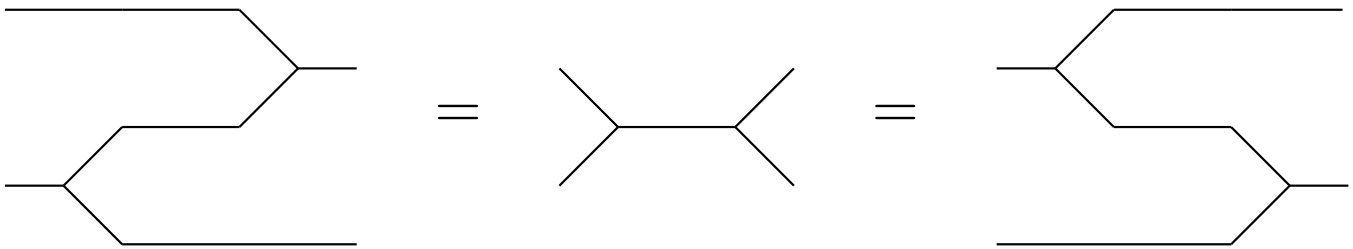
□

The theorem is then proved by taking $\mathbb{C} = \mathbf{Rel}$ and quotienting both sides by the loop-free equation.

Bonus

What happens if we replace "bialgebra" by "Frobenius algebra"?

- We have to replace **Rel** by **Cospan**...
- ... and we get the standard notion of automata with the "bad" composition!
- Indeed, a morphism $I \to O$ of $Circ_\Sigma(\mathbf{Cospan})$ is of the form (E, ℓ, f) with $\ell : E \to \Sigma \sqcup \{\varepsilon\}$ a labeling function, and f a cospan $E \sqcup I \to E \sqcup O$, i.e. a set V and maps $s : E \sqcup O \to V$ and $t : E \sqcup I \to V$.
- Composition indeed behaves according to the Frobenius laws



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Conclusion

- We can define a language (traced monoidal) functor

$$\mathrm{Lang} : \mathbf{Aut} \to \mathrm{Mat}(\mathbf{Reg})$$

  by the universal property, and show that it is full: this is essentially the content of Kleene's theorem.
- We can study the language of an automaton in the light of this functor, by noticing that it is a quotient.
- One should also study determinization, minimization... in the same way.
- One can also try to capture other variants of automata within this framework, e.g. stochastic automata.