A 2-monadic approach to pseudo symmetric multicategories

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Outline

1. Multicategories in Homotopy Theory and Algebraic K-theory
2. Pseudo symmetry
3. 2-monadic approach

Multiplicative K-theory

If **Perm** is the category of small permutative categories and **Wald** is the category of Waldhausen categories, K-theory can be implemented as functors

$$K : \mathbf{Perm} \to \mathbf{Spectra} \text{ [Segal,May]},$$

or

$$K : \mathbf{Wald} \to \mathbf{Spectra} \text{ [Waldhausen]}.$$

There is a smash product in **Spectra** which allows us to define multiplicative structures such as E_∞-ring spectra. What is the relation between these structures and K-theory?
Obstruction: **Perm** and **Wald** are not symmetric monoidal.
Solution [Mandell,Elmendorf]: Use multicategories!
Thm[Elmendorf, Mandell, May]: The K-theory of a bipermutative category is an E_∞-ring spectrum.

Multicategories

Let $(C, \otimes, 1)$ be a symmetric monoidal category. A **C-multicategory** $\mathcal{M}$ consists of:

1. A collection of objects $\mathrm{Ob}(\mathcal{M})$.
2. For $\langle a \rangle$ and b objects of $\mathcal{M}$, $\mathcal{M}(a_1, \ldots, a_n; b) \in \mathrm{Ob}(C)$.
3. For $b \in \mathrm{Ob}(\mathcal{M})$, a morphism $1 \xrightarrow{1_b} \mathcal{M}(b;b)$ in C.
4. Multicomposition morphisms in C of the form
$$\mathcal{M}(\langle b \rangle; c) \otimes \bigotimes_{j=1}^{n} \mathcal{M}(\langle a_j \rangle; b_j) \xrightarrow{\gamma} \mathcal{M}(\langle a \rangle; c).$$
5. For $\sigma \in \Sigma_n$ and objects of $\mathcal{M}$,
$$\mathcal{M}(\langle a \rangle; b) \xrightarrow[\cong]{\sigma} \mathcal{M}(a_{\sigma(1)}, \ldots, a_{\sigma(n)}; b)$$

- These data are subject to a number of coherence axioms.

Examples of Multicategories

1. A symmetric monoidal category C is a **Set**-multicategory with the same set of objects as C and $C(a_1,\dots,a_n;b) = C(a_1 \otimes \cdots \otimes a_n, b)$.
2. A C-operad $\mathcal{O}$ is a C-multicategory with one object $\{*\}$, with $\mathcal{O}(n) = \mathcal{O}(*,\dots,*;*)$.
3. The associative operad Σ_*, with $\Sigma_*(n) = \Sigma_n$ is a **Set**-operad.
4. The right adjoint to Ob : **Cat** $\to$ **Set**, E : **Set** $\to$ **Cat** preserves products. Thus, $E\Sigma_*$ is a **Cat**-operad called the Barrat-Eccles operad.
5. **Perm** is a **Cat**-multicategory with $\mathbf{Perm}(C_1,\dots,C_n;D)$ having as objects functors $F : C_1 \times \cdots \times C_n \to D$ with natural isomorphisms

$$f(c_1,\dots,c_i,\dots,c_n) \otimes f(c_1,\dots,c_i',\dots,c_n) \xrightarrow{\delta_i} f(c_1,\dots,c_i \otimes c_i',\dots,c_n)$$

satisfying $f(c_1,\dots,c_n) = 1$ if $c_i = 1$ and a number of coherence axioms.

Multifunctors

- A C-multifunctor between C-multicategories $F : \mathcal{M} \to \mathcal{N}$ is an assignment on objects $F : \mathrm{Ob}(\mathcal{M}) \to \mathrm{Ob}(\mathcal{N})$ together with C-morphisms $\mathcal{M}(\langle a \rangle; b) \to \mathcal{N}(\langle Fa \rangle; Fb)$ preserving the identity, composition and the action of the symmetric group.
- An algebra in a C-multicategory $\mathcal{M}$ over a C-operad $\mathcal{O}$ is a C-multifunctor $\mathcal{O} \to \mathcal{M}$.

Examples:

1. A permutative category is the same as a multifunctor $E\Sigma_* \to \mathbf{Cat}$ [May].
2. A bipermutative category is the same as a multifunctor $E\Sigma_* \to \mathbf{Perm}$. [Mandell, Elmendorf]
3. $K : \mathbf{Perm} \to \mathbf{Spectra}$ and $K : \mathbf{Wald} \to \mathbf{Spectra}$ can be extended to multifunctors [Blumberg, Elmendorf, Mandell, Zakharevich].

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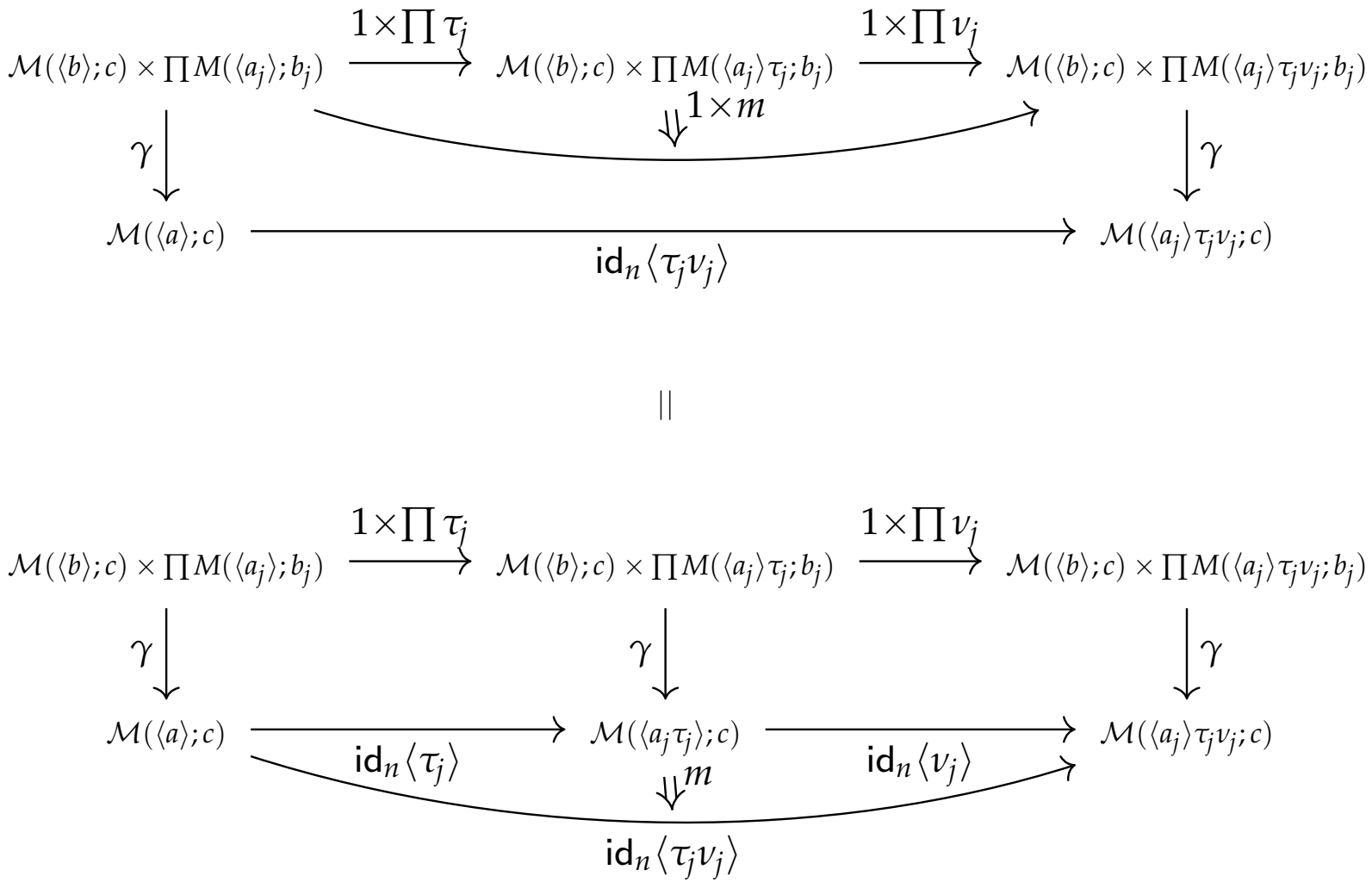
What should be pseudo symmetric multicategories?

In a pseudo symmetric **Cat**-multicategory $\mathcal{M}$ we should have an action of the symmetric group on multilinear maps up to natural isomorphisms of the form

$$\mathcal{M}(\langle a\rangle; b) \xrightarrow{\sigma} \mathcal{M}(\langle a\rangle\sigma; b) \xrightarrow{\tau} \mathcal{M}(\langle a\rangle\sigma\tau; b) \qquad \mathcal{M}(\langle a\rangle; b) \xrightarrow[\sigma\tau]{\Downarrow m_{\sigma,\tau}} \mathcal{M}(\langle a\rangle\sigma\tau; b)$$

satisfying some coherence. As an example, the pasting diagrams in the following slide should be equal.

An example of the coherence involved in the definition



Pseudo symmetric multifunctors

A pseudo symmetric **Cat**-multifunctor between pseudo symmetric **Cat**-multicategories $F\colon \mathcal{M} \to \mathcal{N}$ is an assignment on objects $F\colon \mathrm{Ob}(\mathcal{M}) \to \mathrm{Ob}(\mathcal{N})$ together with functors $F\colon \mathcal{M}(\langle a\rangle; b) \to \mathcal{N}(\langle Fa\rangle; Fb)$ that preserve the action of the symmetric group up to isomorphisms of the form

$$\begin{array}{ccc} \mathcal{M}(\langle a\rangle; b) & \xrightarrow{F} & \mathcal{N}(\langle Fa\rangle; Fb) \\ {\scriptstyle\sigma}\downarrow & \swarrow\!\!\swarrow_{\cong} & \downarrow{\scriptstyle\sigma} \\ \mathcal{M}(\langle a\rangle\sigma; b) & \xrightarrow[F]{} & \mathcal{N}(\langle Fa\rangle\sigma; b). \end{array}$$

subject to some coherence axioms. Pseudo symmetric multifunctors between symmetric **Cat**-multicategories were defined by Johnson and Yau in 2023.

Examples of pseudo symmetry, work in progress

- Inverse K-theory: $\mathcal{P}\colon \mathbf{Sp}_{\geq \mathbf{0}} \to \mathbf{Perm}$ is an example of a pseudo symmetric mulltifunctor [Johnson, Yau] .
  Thm[M., 2024]: Every E_n-algebra in connective spectra parametrized by a free **Cat**-operad can be realized as the K-theory of an E_n-algebra in **Perm**.
- The Grothendieck construction for a bipermutative category $(\mathcal{D}, \oplus, \otimes)$ is a non-symmetric multifunctor [Yau]

$$\mathcal{D}\text{-}\mathbf{Cat} \to (\mathbf{PermutativeOpfibrations}/\mathcal{D}).$$

  The objects of $\mathcal{D}$-**Cat** are symm. mon. functors from $(\mathcal{D}, \oplus)$ to $(\mathbf{Cat}, \times)$. Permutative opfibrations over $\mathbf{D}$ form a pseudo symmetric **Cat**-multicategory and the Grothendieck construction is pseudo symmetric [M.] It should be a pseudo symmetric equivalence (work in progress).

Examples of pseudo symmetry, work in progress

- The following example comes from equivariant K theory [Guillou, May, Merling, Osorno]. If $(\mathcal{C}, \otimes)$ is a symmetric monoidal category then, symmetric monoidal pseudo functors from $(\mathcal{C}, \otimes)$ to $(\mathbf{Cat}, \times)$ should form a pseudo symmetric $\mathbf{Cat}$-multicategory with multilinear maps from $F_1, \ldots, F_n \to G$ given by natural isos of the form

$$\begin{array}{ccc} \mathcal{C}^{\times n} & \xrightarrow{F_1 \times \cdots \times F_n} & \mathbf{Cat}^{\times n} \\ {\scriptstyle \otimes}\downarrow & \Downarrow \cong & \downarrow{\scriptstyle \times} \\ \mathcal{C} & \xrightarrow[G]{} & \mathbf{Cat}. \end{array}$$

Symmetric monoidal pseudo functors $F\colon \mathcal{C} \to \mathcal{D}$ should induce pseudo symmetric $\mathbf{Cat}$-multifunctors.

Examples of pseudo symmetry

- Let $T : \mathcal{C} \to \mathcal{C}$ be a symmetric, pseudo commutative 2-monad on a symmetric monoidal 2-category. Then, $T\text{-}alg$, the 2-category of algebras and pseudo morphisms, admits the structure of a **Cat**-multicategory, and $T\colon \mathcal{C} \to T\text{-}alg$ is a pseudo symmetric multifunctor [M. 2025]. This can be considered as a coherence theorem for this kinds of 2-monads.

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Multi-icons

Def[M.]: Let $\mathcal{M}$ and $\mathcal{N}$ be non-symmetric **Cat**-multicategories and $F, G\colon \mathcal{M} \to \mathcal{N}$ non-symmetric **Cat**-multifunctors s.t. $Fa = Ga$ for $a \in \mathrm{Ob}(\mathcal{M})$. A multi-icon $\alpha\colon F \to G$ is a collection of natural transformations $\mathcal{M}(\langle a\rangle; b) \overset{F}{\underset{G}{\Downarrow \alpha_{a;b}}} \mathcal{N}(\langle Fa\rangle; Fb)$.

$$\mathcal{M}(\langle b\rangle; c) \times \prod \mathcal{M}(\langle a_i\rangle; b_i) \xrightarrow{\ \gamma\ } M(\langle a\rangle; c) \overset{F}{\underset{G}{\Downarrow \alpha}} \mathcal{N}(\langle Fa\rangle; Fc)$$

must equal

$$\mathcal{M}(\langle b\rangle; c) \times \prod \mathcal{M}(\langle a_i\rangle; b_i) \overset{\prod F}{\underset{\prod G}{\Downarrow \prod \alpha}} \mathcal{N}(\langle a\rangle; c) \times \prod \mathcal{N}(\langle Fa_i\rangle; Fb_i) \overset{\gamma}{\to} \mathcal{N}(\langle Fa\rangle; Fc)$$

and another axiom for the units has to be satisfied.

The 2-monad for symmetric monoidal categories

Thm[M.]**:** There is a 2-category $\mathbf{nCat\text{-}Multicat}$ with non symetric $\mathbf{Cat}$-multicategories as 0-cells, non-symmetric multifunctors as 2-cells, and multi-icons as 2-cells. Similarly in the symmetric case.Furthermore, if $T : \mathbf{nCat\text{-}Multicat} \to \mathbf{nCat\text{-}Multicat}$ is the 2-monad for symmetric $\mathbf{Cat}$-multicategories, then:

- Algebras over T are symmetric $\mathbf{Cat}$-multicategories.
- Pseudo morphisms between algebras for T are pseudo symmetric $\mathbf{Cat}$-multifunctors (as defined by Johnson and Yau).

Def[M.]: Pseudo symmetric $\mathbf{Cat}$-multicategories are pseudo algebras over T and pseudo symmetric $\mathbf{Cat}$-multifunctors between them are pseudo morphisms between pseudo T-algebras.
We have extracted from this a definition suitable for examples with coherence axioms similar to the one in slide 9.

Partial coherence results

Thm[M.]**:** Every pseudo symmetric **Cat**-multicategory is equivalent to a symmetric one. Furthermore, the inclusion of the 2-category of symmetric **Cat**-multicategories, symmetric **Cat**-multifunctors, and some 2-cells into the category of pseudo symmetric **Cat**-multicategories, pseudo symmetric **Cat**-multifunctors and some 2-cells admits a left 2-adjoint

$$\mathbf{Cat\text{-}Multicat} \underset{i}{\overset{\psi}{\underset{\longrightarrow}{\overset{\longleftarrow}{\perp}}}} \mathbf{PsCat\text{-}Multicat}.$$

Work in progress: understand what ψ does. When $\mathcal{M}$ is symmetric, the adjoint ψ becomes $\psi(\mathcal{M}) = \mathcal{M} \times E\Sigma_*$ (product with the Barrat-Eccles operad).

Future work

- Finish establishing the examples.
- Compare the two current different multifunctorial constructions for equivariant K-theory.
- Develop the concept of multibicategory and establish relations with different versions of symmetric monoidal 2-categories (adjunctions and representability issues).
- Understand the parametrization of multiplicative structures in pseudo symmetric multicategories and its preservation through pseudo symmetric multifunctors.

Questions?