A CATEGORICAL MODEL STRUCTURE FOR GENERALIZED ALGEBRAIC THEORIES

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The goal is to:

- Give a brief introduction to Cartmell's **generalized algebraic theories** (**gats**) and their syntactic categories (**contextual categories**).
- Discuss the problem of comparing a "syntactic" and a "categorical" ways of defining a model of a gat, and some issues related to the strictness of the structure of contextual categories.
- Describe the categorical Quillen model structure on GAT and some of its features, and explain how it helps in understanding the strictification problems discussed before.

- The concept of a **generalized algebraic theory** was introduced by John Cartmell in his thesis in 1978.
- Generalized algebraic theories (gats for short) extend the setting of algebraic theories (or Lawvere theories) by allowing **dependent sorts/types**.
- Any (multi-sorted) algebraic theory is a gat, e.g. the theory of groups, rings, modules...

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- Any (multi-sorted) algebraic theory is a gat, e.g. the theory of groups, rings, modules...
- A classical example with sort dependency is:

The theory of categories, $\mathbb{T}_{cat}$

- We have a *sort* O that is "constant" (doesn't depend on other sorts), introduced by the axiom

$$\vdash O \text{ sort.}$$

- Arrows are encoded by the axiom

$$x, y : O \vdash A(x, y) \text{ sort}$$

(at this point we have the theory of directed graphs)

The theory of categories, $\mathbb{T}_{cat}$

- We have a sort O that is "constant" (doesn't depend on any parameters), introduced by the axiom

$$\vdash O \text{ sort.}$$

- Arrows are encoded by the dependent sort

$$x, y : O \vdash A(x, y) \text{ sort}$$

- We have *term symbols* $\circ$ and i introduced by the axioms

$$x, y, z : O,\ f : A(x, y),\ g : A(y, z) \vdash \circ(f, g) : A(x, z)$$

$$x : O \vdash i(x) : A(x, x)$$

$$x, y : O,\ f : A(x, y) \vdash i(x) \circ f \equiv f : A(x, y)$$

$$x, y : O,\ f : A(x, y) \vdash f \circ i(y) \equiv f : A(x, y)$$

$$w, x, y, z : O,\ f : A(w, x), g : A(x, y), h : A(y, z)$$
$$\vdash (f \circ g) \circ h \equiv f \circ (g \circ h) : A(w, z)$$

Generalized algebraic theories are presented via **judgments**

$$x_1 : X_1, ..., x_n : X_n \vdash \blacksquare$$

where $x_1 : X_1, ..., x_n : X_n$ (a "pre-context") is a list of variables of specified *sorts* and $\blacksquare$ is a statement being derived with respect to the parameters x_1, ..., x_n.

$$x_1 : X_1, ..., x_n : X_n \quad \text{ctx} \qquad \text{(context)}$$

$$x_1 : X_1, ..., x_n : X_n \vdash U \text{ sort} \qquad \text{(sort)}$$

$$x_1 : X_1, ..., x_n : X_n \vdash u : U \qquad \text{(term)}$$

$$x_1 : X_1, ..., x_n : X_n \vdash U \equiv V \text{ sort} \qquad \text{(sort equality)}$$

$$x_1 : X_1, ..., x_n : X_n \vdash u \equiv v : U \qquad \text{(term equality)}$$

A gat specifies sort/term symbols and certain judgments – its *axioms* – from which we obtain a set of *derivable* judgments.

(Inference rules omitted!)

Semantics for gats?

- A model of an (single-sorted) equational theory $\mathbb{T}$ (e.g. of groups) is a set equipped with functions interpreting the term symbols, subject to certain equalities.
- On the other hand, $\mathbb{T}$ has a *syntactic category* (which is a Lawvere theory) $\mathcal{C}(\mathbb{T})$, opposite to a skeleton of its category of finitely generated free models. An arbitrary model corresponds to a finite-product-preserving functor

$$\mathcal{C}(\mathbb{T}) \to \mathsf{Set}$$

- Similarly, we have two flavours of "set-valued" model of a gat.

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Cartmell's family-valued ("strict") models

- Let's illustrate using a model of $\mathbb{T}_{cat}$.
- The sort O is interpreted as a set $\mathbf{O}$.
- The sort $x, y : O \vdash A(x, y)$ sort is interpreted as a family of (small) sets

$$(\mathbf{A}(\mathbf{x},\mathbf{y}))_{\mathbf{x},\mathbf{y}\in\mathbf{O}} = \mathbf{A} : \{(\mathbf{x},\mathbf{y}) \mid \mathbf{x},\mathbf{y} \in \mathbf{O}\} \to \mathscr{U} \quad \rightsquigarrow \textit{universe set}$$

- The composition symbol $\circ$ (similarly for i) is interpreted as a function

$$\{(\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{f},\mathbf{g}) \mid \mathbf{x},\mathbf{y},\mathbf{z} \in \mathbf{O},\ \ \mathbf{f} \in \mathbf{A}(\mathbf{x},\mathbf{y}),\ \ \mathbf{g} \in \mathbf{A}(\mathbf{y},\mathbf{z})\} \xrightarrow{\circ} \{(\mathbf{x},\mathbf{z},\mathbf{h}) \mid \mathbf{x},\mathbf{y} \in \mathbf{O},\ \mathbf{h} \in \mathbf{A}(\mathbf{x},\mathbf{z})\}$$

- Equalities, such as $(\mathbf{f} \circ \mathbf{g}) \circ \mathbf{h} = \mathbf{f} \circ (\mathbf{g} \circ \mathbf{h})$ whenever the expression is well-formed. In general, need to add equalities between sets!

Set-valued ("weak"/"categorical") models

- A gat $\mathbb{T}$ has a syntactic category $\mathcal{C}(\mathbb{T})$.
- Objects of $\mathcal{C}(\mathbb{T})$ are equivalence classes of contexts

$$[x_1 : X_1, ..., x_n : X_n]$$

- Morphisms are of the form

$$[f_1, ..., f_m] : [x_1 : X_1, ..., x_n : X_n] \to [y_1 : Y_1, ..., y_m : Y_m]$$

  for suitable terms f_1, ..., f_m. Composition encodes substitution (of variables by terms) inside terms.
- For $x : X_1, ..., x_n : X_n \vdash Y$ sort, we have a special arrow - a *display map* (Taylor?) -

$$[x_1, ..., x_n] : [x_1 : X_1, ..., x_n : X_n, y : Y] \twoheadrightarrow [x_1 : X_1, ..., x_n : X_n]$$

- **Display maps have pullbacks along arbitrary arrows**: encodes substitution inside sorts. Defined up to equality and strictly functorial!

$$\begin{array}{ccc} [x_1 : X_1, ..., x_n : X_n, z : Z[\overline{f} \mid \overline{y}]] & \xrightarrow{(f_1,...,f_m,z)} & [y_1 : Y_1, ..., y_m : Y_m, z : Z] \\ \downarrow & & \downarrow \\ [x_1 : X_1, ..., x_n : X_n] & \xrightarrow[\overline{f}=(f_1,...,f_m)]{} & [y_1 : Y_1, ..., y_m : Y_m] \end{array}$$

What is a model of a gat?

- $\mathcal{C}(\mathbb{T})$ is a **contextual category**.
- A *contextual category* (or *C-system*, following Voevodsky) is a category $\mathcal{A}$ where:
  - each object has a "length" $n \geq 0$;
  - there is a unique length-0 object, which is terminal;
  - a length-n object Γ is equipped with a *display map*

    $$\mathbf{p}_\Gamma : \Gamma \twoheadrightarrow \partial\Gamma$$

    where $\partial\Gamma$ has length $n - 1$;
  - display maps have strictly functorial pullbacks.
- Can't transfer contextual category structures along equivalences $A \simeq B$ in Cat!
- A *set-valued* (or "*weak/categorical*") *model* of a gat $\mathbb{T}$ is a functor

  $$F : \mathcal{C}(\mathbb{T}) \longrightarrow \mathrm{Set}$$

  that sends pullbacks of display maps to pullback squares, and preserves terminal objects (empty context is terminal!). **These are the models of $\mathcal{C}(\mathbb{T})$ as a clan.**
- Are these two concepts of model essentially the same? As it turns out, no – an example involving a sort equality axiom is given in Bardomiano-Henry's paper Homotopy Languages, Remark B.54.
- What is a natural setting to study this kind of comparison? What is the exact role of sort equality?

- We have a category GAT of (small) generalized algebraic theories and equivalence classes of interpretations.
- It is equivalent (Cartmell) to the category Cont of (small) contextual categories and *contextual functors* (or *strict morphisms*) – all available structure is preserved strictly, including the pullbacks encoding substitution of dependent sorts.
- Taking all natural transformations as 2-cells, we have a strict 2-category **Cont**.

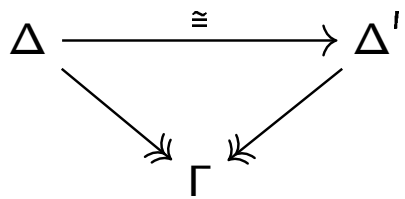
- For a contextual category $\mathcal{A}$ and $\Gamma \in \mathrm{Ob}(\mathcal{A})$, let

$$\mathrm{sort}_{\mathcal{A}}(\Gamma) = \{\text{display maps with codomain } \Gamma\}$$

- On the other hand, can consider the *groupoid* of sorts of each context (similar to a structure considered by Clairambault-Dybjer for categories with families):

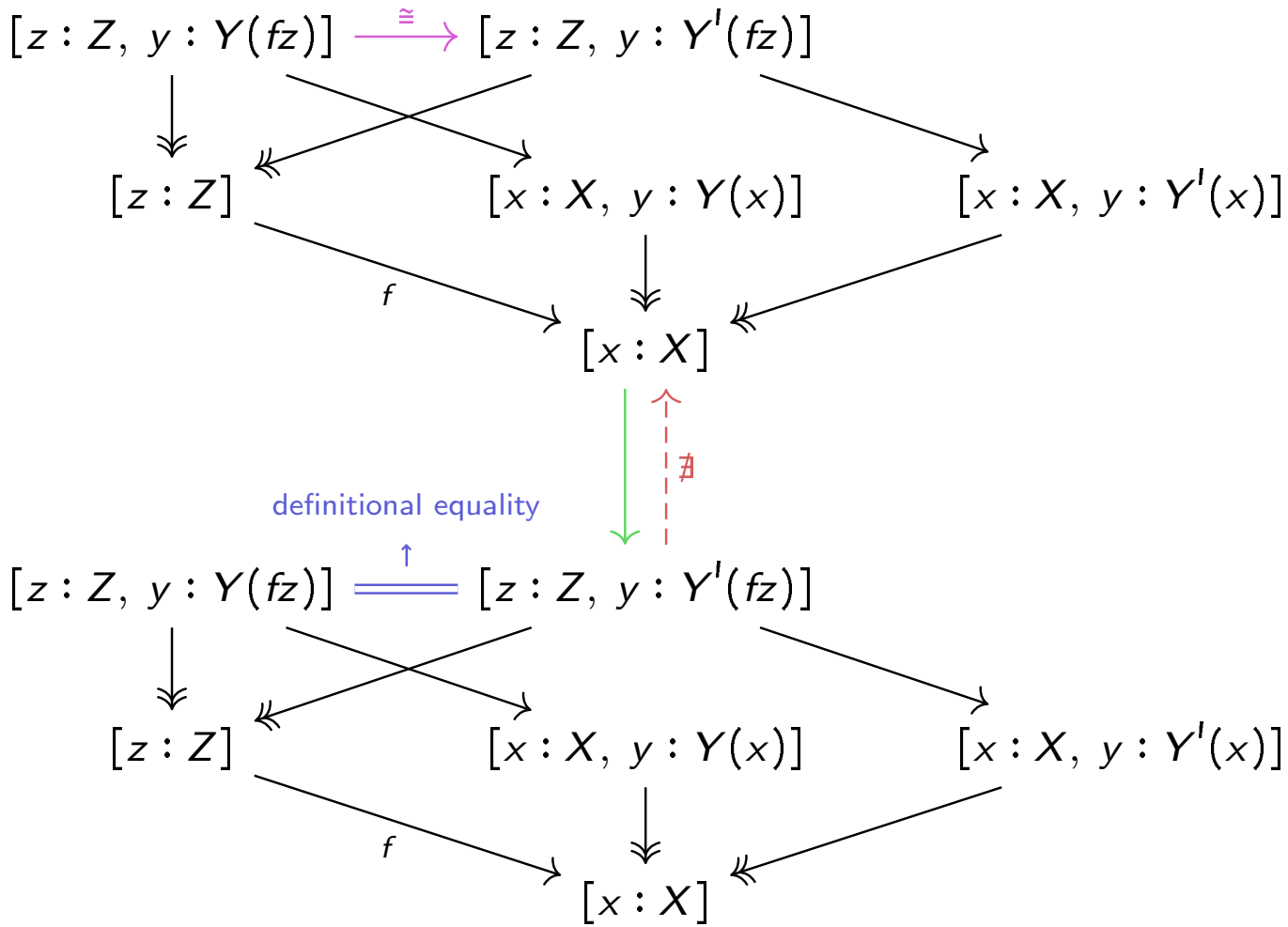
$$\mathbf{sort}_{\mathcal{A}} : \mathcal{A}^{op} \to \mathbf{Gpd}$$

where $\mathbf{sort}_{\mathcal{A}}(\Gamma)$ is the subgroupoid of $\mathcal{A}/\Gamma$ consisting of the display maps $\Delta \twoheadrightarrow \Gamma$ and isomorphisms between them,



- A contextual functor $F : \mathcal{A} \to \mathcal{B}$ is a *weak equivalence* if it is an equivalence categories and induces an equivalence $\mathbf{sort}_{\mathcal{A}}(\Gamma) \longrightarrow \mathbf{sort}_{\mathcal{B}}(F(\Gamma))$ for every context $\Gamma \in \mathrm{Ob}(\mathcal{A})$.

Not every weak equivalence is an actual equivalence in **Cont**:



Quillen model structures

- Model categories, introduced by Quillen, give a framework for categorical homotopy theory.
- A **model category** is a co/complete category C equipped with classes of arrows, weak equivalences, cofibrations, fibrations, such that
  - the class of weak equivalences W contains all isos satisfies the 2-out-of-3 property: if two among f, g, $g \circ f$ are in W, then so is the third.
  - have weak factorization systems

$$(\text{cofibrations} \cap \text{weak equivalences},\ \ \text{fibrations})$$

$$(\text{cofibrations},\ \ \text{fibrations} \cap \text{weak equivalences})$$

- Main idea (in modern language): a Quillen model structure $(C, W, \cdots)$ allows us to study the $(\infty, 1)$-category $C[W^{-1}]_\infty$ (its homotopy $(\infty, 1)$-category) obtained by freely adding weak inverses to every arrow in W.

Theorem: the category GAT ($\simeq$ Cont) admits a combinatorial Quillen model structure where

- Weak equivalences are the ones described previously.
- We have three kinds of generating cofibrations:
  (1) freely adding a sort in a length-n context

$$\left(\Gamma_n \twoheadrightarrow \cdots \twoheadrightarrow \Gamma_1 \twoheadrightarrow \Gamma_0 = 1\right) \hookrightarrow \left(\Gamma_{n+1} \twoheadrightarrow \Gamma_n \twoheadrightarrow \cdots \twoheadrightarrow \Gamma_1 \twoheadrightarrow \Gamma_0\right)$$

  (2) freely adding a term in a length-n context

$$\left(\Gamma_{n+1} \twoheadrightarrow \Gamma_n \twoheadrightarrow \cdots \twoheadrightarrow \Gamma_0 = 1\right) \hookrightarrow \left(\Gamma_{n+1} \overset{\text{section}}{\twoheadrightarrow} \Gamma_n \twoheadrightarrow \cdots \twoheadrightarrow \Gamma_0\right)$$

  (3) identifying two terms of the same sort in a length-n context

$$\left(\Gamma_{n+1} \rightrightarrows \Gamma_n \twoheadrightarrow \cdots \twoheadrightarrow \Gamma_0 = 1\right) \longrightarrow \left(\Gamma_{n+1} \overset{\leftarrow}{\twoheadrightarrow} \Gamma_n \twoheadrightarrow \cdots \twoheadrightarrow \Gamma_0\right)$$

(Similar to the "folk" model structure on Cat!)

- Every object is fibrant.
- A gat is cofibrant iff it is a retract of some gat that doesn't have sort equality axioms.
- The model structure on GAT is Cat_{folk}-enriched (**Cont**), and monoidal with respect to a certain tensor product operation for gats

idea: $\mathrm{Mod}(\mathbb{B}, \mathrm{Mod}(\mathbb{A})) \simeq \mathrm{Mod}(\mathbb{A} \otimes \mathbb{B}) \simeq \mathrm{Mod}(\mathbb{A}, \mathrm{Mod}(\mathbb{B}))$

- **Theorem:** the homotopy $(\infty, 1)$-category of GAT is the $(2, 1)$-category $\mathbf{DMC}_r$ of *rooted display map categories* (first considered by P. Taylor).
- These are categories C equipped with a set of arrows P such that
  (1) P is closed under isomorphism and pullbacks along arbitrary arrows
  (2) C has a terminal object, and every object Γ can be connected (not necessarily uniquely) to 1 by a sequence of arrows in P:

$$\Gamma = \Gamma_n \twoheadrightarrow \Gamma_{n-1} \twoheadrightarrow \cdots \twoheadrightarrow \Gamma_0 = 1$$

- Every contextual category $\mathcal{A}$ can be "weakened" into a rooted display map category $\mathcal{A}_w$ by closing the set of display maps under isomorphism and forgetting the terminal object.
- This gives a 2-functor

$$(-)_w : \mathbf{Cont} \longrightarrow \mathbf{DMC}_r$$

  which sends weak equivalences to "true" equivalences.
- The above theorem states that (after discarding non-invertible 2-cells), $(-)_w$ is initial among $(\infty, 1)$-functors $\mathbf{Cont} \to K$ that send weak equivalences to equivalences.

For a generalized algebraic theory $\mathbb{A}$:

- Models of $\mathbb{A}$ valued in families of sets (Cartmell's semantics) correspond to contextual functors $\mathcal{C}(\mathbb{A}) \to \mathrm{Fam}$ for a contextual category Fam of "iterated families of sets". These form a category

$$\mathrm{Mod}_{strict}(\mathbb{A}) = \mathbf{Cont}^{+}(\mathcal{C}(\mathbb{A}), \mathrm{Fam})$$

- Models of the clan (or finite-limit sketch) $\mathcal{C}(\mathbb{A})$ form a category

$$\mathrm{Mod}_{clan}(\mathbb{A}) = \mathbf{DMC}_r^{+}(\mathcal{C}(\mathbb{A})_w, \mathrm{Set})$$

  where every arrow in Set is regarded as a display map.

- But $\mathrm{Fam}_w \simeq \mathrm{Set}$, so we have a full-and-faithful comparison functor

$$\mathrm{Mod}_{strict}(\mathbb{A}) = \mathbf{Cont}^{+}(\mathcal{C}(\mathbb{A}), \mathrm{Fam}) \xrightarrow{(-)_w} \mathbf{DMC}_r^{+}(\mathcal{C}(\mathbb{A})_w, \mathrm{Fam}_w) \simeq \mathrm{Mod}_{clan}(\mathbb{A})$$

- If $\mathbb{A}$ is cofibrant, this functor is an equivalence.
- For non-cofibrant $\mathbb{A}$, it might not be an equivalence. This was observed in Bardomiano-Henry's Homotopy Languages paper, Rem. B.54.

  A characterization of which Set-models M of $\mathbb{A}$ can be strictified into a Fam-model is given in arXiv:2608.27667 in terms of triviality of a certain groupoid cover associated with M.

Thank you! :)