On nondeterministic behaviors in double categorical systems theory

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Octoberfest 2025
October 25th, 2025

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- One framework I am interested in: Double Categorical Systems Theory. Quite an active area of research: Libkind and Myers 2025, Baez 2025, etc. For this talk, I mostly use ideas from the notes Myers 2023.
- Key feature: can handle both composition of systems and representations between systems.

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- Lots of work on probabilistic systems, e.g. monoidal streams (Di Lavore, Felice, and Román 2022); I won't delve into it.

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All the systems above have an interface, written $\begin{pmatrix} A \\ B \end{pmatrix}$ or $\begin{pmatrix} I \\ O \end{pmatrix}$.

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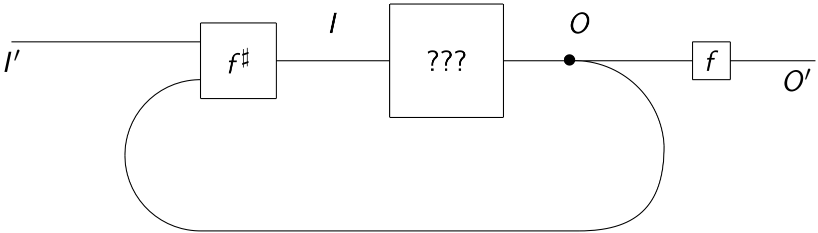
For Mealy machines, one can use combs (Chiribella, D'Ariano, and Perinotti 2008), which define a multicategory of interfaces.

Composing systems

Picture of a lens:



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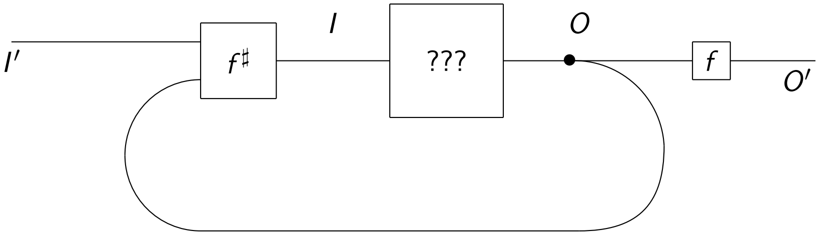

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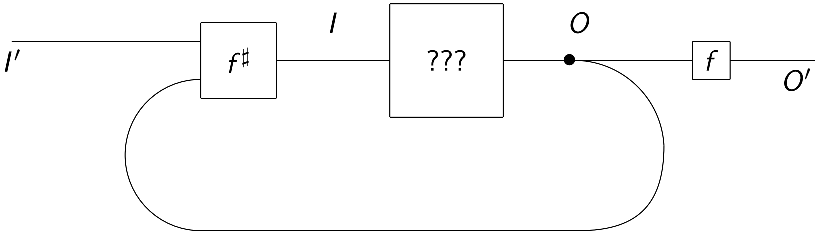

- Plugging a system of interface $\begin{pmatrix} I \\ O \end{pmatrix}$ in such a pattern then yields a system of interface $\begin{pmatrix} I' \\ O' \end{pmatrix}$.
- We compose the patterns together by nesting.

Composing systems, continued

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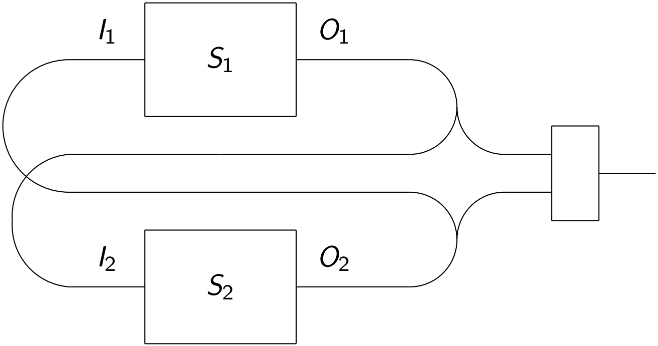
- There is also a tensor product operation on systems, that acts as tensor on the interfaces.
- Combining both, one can construct composites:



Outline

1. Simple systems theories
2. Comparing systems: (bi)simulations
3. Going double

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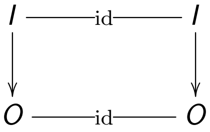
In Example 1, there can only be identity morphisms:



Simulations

In Example 2 (Mealy machines), such a morphism is given by a map $\phi : S \to S'$ such that the following diagram commutes:

$$
\begin{array}{ccc}
S \otimes I & \xrightarrow{\phi \otimes I} & S' \otimes I \\
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In Example 3 (Moore machines), a morphism is again given by a map $\phi : S \to S'$, with the condition that the following commute:

$$\begin{array}{ccc} S \otimes I & \xrightarrow{\phi \otimes I} & S' \otimes I \\ \downarrow & & \downarrow \\ S & \xrightarrow[\phi]{} & S' \end{array} \qquad \begin{array}{ccc} S & \xrightarrow{\phi} & S' \\ \downarrow & & \downarrow \\ O & \overset{}{\text{——id——}} & O \end{array}$$

Categories of systems and simulations

- This way, for each $\begin{pmatrix} I \\ O \end{pmatrix}$, we get a category $\mathrm{Sys}\begin{pmatrix} I \\ O \end{pmatrix}$ of "systems with interface $\begin{pmatrix} I \\ O \end{pmatrix}$ and simulations".

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For interpretability of AI systems, this notion of morphism gives a tentative definition of one of the goals:
"Given a system $\mathcal{S}$, find a simpler system $\mathcal{S}'$ with similar observable behavior and a morphism of systems from $\mathcal{S}$ to $\mathcal{S}'$."

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- Assuming I get that, what does it mean to *compare systems over a given morphism of interfaces*?

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- There are several candidates for the notion of "representation/chart" between interfaces $\begin{pmatrix} I \\ O \end{pmatrix} \rightrightarrows \begin{pmatrix} I' \\ O' \end{pmatrix}$:

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- Once such a notion is chosen, you can try to define a notion of "morphism/representation of systems over a given representation of interfaces".

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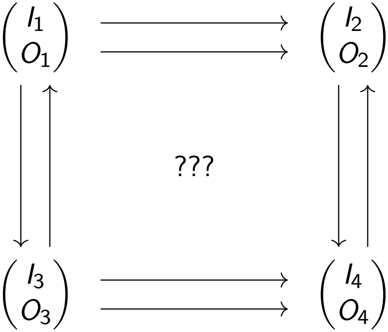

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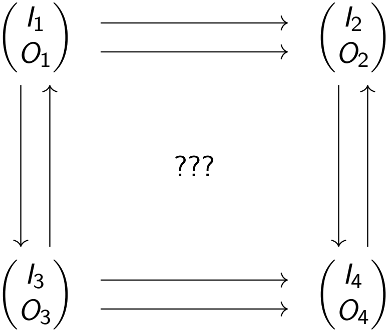

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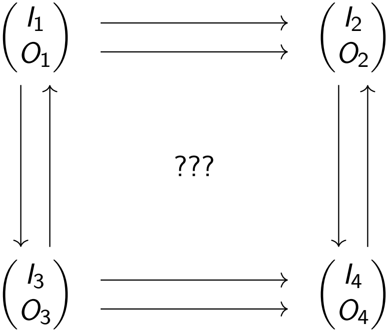

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Should it be given by compatibility conditions? Compatibility data?

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Issue: interchange remains elusive...

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References

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Bonus: charts between interfaces

Charts $\begin{pmatrix} I \\ O \end{pmatrix} \rightrightarrows \begin{pmatrix} I' \\ O' \end{pmatrix}$ are pairs of morphisms $O \to O'$ and $I \otimes O \to I' \otimes O'$ such that the following commutes:

$$\begin{array}{ccc} I \otimes O & \longrightarrow & I' \otimes O' \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ O & \longrightarrow & O' \end{array}$$

Bonus: squares of interfaces

$$\begin{array}{ccc} \begin{pmatrix} I_1 \\ O_1 \end{pmatrix} & \rightrightarrows & \begin{pmatrix} I_2 \\ O_2 \end{pmatrix} \\ \downarrow \uparrow & & \downarrow \uparrow \\ \begin{pmatrix} I_3 \\ O_3 \end{pmatrix} & \rightrightarrows & \begin{pmatrix} I_4 \\ O_4 \end{pmatrix} \end{array}$$

corresponds to $s : O_1 \otimes I_3 \to O_2 \otimes I_4$, such that:

$$\begin{array}{ccc} O_1 \otimes I_3 & \longrightarrow & O_2 \otimes I_4 \\ \downarrow & \circlearrowleft & \downarrow \\ O_3 \otimes I_3 & \longrightarrow & O_4 \otimes I_4 \end{array} \qquad \begin{array}{ccc} O_1 \otimes I_3 & \longrightarrow & O_2 \otimes I_4 \\ \downarrow & & \downarrow \\ O_1 \otimes O_1 \otimes I_3 & \circlearrowleft & O_2 \otimes O_2 \otimes I_4 \\ \downarrow & & \downarrow \\ O_1 \otimes I_1 & \longrightarrow & O_2 \otimes I_2 \end{array}$$

Bonus: Composing squares

- Horizontal composition is straightforward.
- Vertical composition uses *conditional products*; for associativity, you need some extra assumption, e.g. determinism of the lenses.
- Interchange fails in general... Is it a bug, or a feature/general obstruction?