# An algebra modality admitting countably many deriving transformations[1]

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[1] If you want to see the GIFs, you will need to use a compatible PDF reader such as Acrobat Reader. However, textual descriptions of the GIFs are provided on the slides.

*Is differentiation unique in differential categories?*

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(Cat making no with its head)

# Additive symmetric monoidal categories

## Definition

An *additive symmetric monoidal category* is a symmetric monoidal category $(\mathsf{C}, \otimes, I)$ enriched in commutative monoids such that:

$$0 \otimes f = f \otimes 0 = 0,$$

$$(f + g) \otimes h = (f \otimes h) + g \otimes h,$$

$$f \otimes (g + h) = (f \otimes g) + (f \otimes h)$$

whenever it makes sense.

## Example

$(\mathsf{CMon}, \otimes, \mathbb{N})$

# Algebra modality

Let $(\mathsf{C}, \otimes, I)$ be an additive symmetric monoidal category.

## Definition

An *algebra modality* on $(\mathsf{C}, \otimes, I)$ is given by:

- a monad $(S, m, u)$ on $\mathsf{C}$;
- natural transformations $\nabla_A \colon SA \otimes SA \to SA$ and $\eta_A \colon A \to SA$;

such that for every $A \in \mathsf{C}$:

1. $(SA, \nabla_A, \eta_A)$ is a commutative monoid
2. the diagram

$$\begin{array}{ccc} SSA \otimes SSA & \xrightarrow{\nabla} & SSA \\ {\scriptstyle m \otimes m}\downarrow & & \downarrow{\scriptstyle m} \\ SA \otimes SA & \xrightarrow[\nabla]{} & SA \end{array}$$

   commutes.

## Example

The symmetric algebra monad = free rig monad is an algebra modality on $(\mathsf{CMon}, \otimes, \mathbb{N})$.

# Deriving transformation

Let $(S, m, u, \nabla, \eta)$ be an algebra modality on an additive symmetric monoidal category $(\mathsf{C}, \otimes, I)$.

## Definition

A *deriving transformation* (on this algebra modality) is a natural transformation $\mathsf{d}_A \colon SA \to SA \otimes A$, such that the four following rules are satisfied.

1. Product rule:
$$\mathsf{d}_A \circ \nabla_A = [(\nabla_A \otimes \mathrm{id}_A) \circ (\mathrm{id}_{SA} \otimes \mathsf{d}_A)] + [(\nabla_A \otimes \mathrm{id}_A) \circ (\mathrm{id}_{SA} \otimes \sigma_{A,SA}) \circ (\mathsf{d}_A \otimes \mathrm{id}_{SA})].$$
2. Linear rule:
$$\mathsf{d}_A \circ u_A = \eta_A \otimes \mathrm{id}_A.$$
3. Chain rule:
$$\mathsf{d}_A \circ m_A = (\nabla_A \otimes \mathrm{id}_A) \circ (m_A \otimes \mathsf{d}_A) \circ \mathsf{d}_{SA}.$$
4. Interchange rule:
$$(\mathsf{d}_A \otimes \mathrm{id}_A) \circ \mathsf{d}_A = (\mathrm{id}_{SA} \otimes \sigma_{A,A}) \circ (\mathsf{d}_A \otimes \mathrm{id}_A) \circ \mathsf{d}_A.$$

## Example

The following is a deriving transformation on the symmetric algebra monad $S$ on $(\mathrm{CMon}, \otimes, \mathbb{N})$:

$$\begin{aligned} d_A \colon SA &\longrightarrow SA \otimes A \\ a_1 \otimes_s \cdots \otimes_s a_n &\longmapsto \sum_{0 \leq k \leq n} (a_1 \otimes_s \cdots \otimes_s \hat{a_k} \otimes_s \cdots \otimes_s a_n) \otimes a_k \end{aligned}$$

If $A \simeq \mathbb{N}^n$, then $SA \simeq \mathbb{N}[x_1, \ldots, x_n]$, and $d_A$ is given by

$$f \longmapsto \sum_{0 \leq k \leq n} \frac{\partial f}{\partial x_i} \otimes x_i .$$

# An open problem

At Octoberfest 2022, JS Lemay presented the following result (obtained with Marie Kerjean):[2]

## Theorem

*If $d_1, d_2$ are two deriving transformations on a same* **comonoidal**[3] *algebra modality $(S, m, u, \nabla, \eta)$, then $d_1 = d_2$.*

That is:

*Differentiation is unique in models of differential linear logic.*

JS then asked:

*Does this theorem extend to* **arbitrary** *algebra modalities?*

That is:

*Is differentiation unique in arbitrary differential categories?*

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[2] later appeared in Lemay J.-S. P., *Additive Enrichment from Coderelictions* (2025)

[3] that is, $S$ is a symmetric comonoidal functor, $m, u, \nabla, \eta$ are comonoidal natural transformations + two other equations.

# The answer

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Should we trust our differential kitty? Proof?

# The idea

There are infinitely many derivations $\partial : \mathbb{R}[x] \to \mathbb{R}[x]$. They are of the form

$$\partial = \partial(x)\frac{\mathrm{d}}{\mathrm{d}x}$$

where $\partial(x)$ is any polynomial in $\mathbb{R}[x]$.

The element $x \in \mathbb{R}[x]$ must be understood:

- algebraically, as a generic element $x \in \mathbb{R}[x]$,
- differentially, as a smooth function from $\mathbb{R}$ to $\mathbb{R}$ with a chosen derivative $\partial(x) \in \mathbb{R}[x]$.

Derivations don't care about composition so that $x \in \mathbb{R}[x]$ does not have to be interpreted by a derivation as the identity map on $\mathbb{R}$ !

We take inspiration from this to build an algebra modality $F$ on $(\mathrm{CMon}, \otimes, \mathbb{N})$ with countably many deriving transformations.

Given a commutative monoid $A$, $FA$ will be a commutative rig with a function $\mathbf{f} : FA \to FA$.

We will also build for every $n \in \mathbb{N}$ a deriving transformation

$$_n\mathrm{d} : FA \to FA \otimes A.$$

The function $\mathbf{f} : FA \to FA$ must be understood:

- algebraically, as a generic function $\mathbf{f} : FA \to FA$,
- differentially, as a smooth function such that $_n\mathrm{d}(\mathbf{f}(t)) = n \cdot {}_n\mathrm{d}(t)$.

# The proof

$F$ will be the free **commutative rig with a self map** monad on CMon.

## Definition

A *commutative rig with a self-map* is a couple $(R, \mathbf{f})$ where $R$ is a commutative rig and $\mathbf{f}\colon R \to R$ is a function.

A morphism $\phi\colon (R, \mathbf{f}) \to (S, \mathbf{g})$ is a rig homomorphism $\phi\colon R \to S$ such that the diagram

$$\begin{array}{ccc} R & \xrightarrow{\ \phi\ } & S \\ {\scriptstyle \mathbf{f}}\big\downarrow & & \big\downarrow{\scriptstyle \mathbf{g}} \\ R & \xrightarrow[\ \phi\ ]{} & S \end{array}$$

commutes.

The resulting category is denoted by $\mathrm{CRig}^{\circlearrowleft}$.

## Theorem

*The forgetful functor* $U\colon \mathsf{CRig}^{\circlearrowleft} \to \mathsf{CMon}$ *admits a left adjoint* $\mathcal{F}\colon \mathsf{CMon} \to \mathsf{CRig}^{\circlearrowleft}$.

*We thus obtain a monad* $(F, m, u)$ *on* $\mathsf{CMon}$ *where*

$$F = U \circ \mathcal{F}\colon \mathsf{CMon} \to \mathsf{CMon}.$$

*Moreover, we obtain an algebra modality* $(F, m, u, \nabla, \eta)$ *on* $(\mathsf{CMon}, \otimes, \mathbb{N})$.

## Proof.

For every commutative monoid $A$, we define by induction a set $F_0A$ of terms and an appropriate equivalence relation $\sim$ on $F_0A$. Then we set $FA = F_0A/\sim$.

Some other things are defined and proved by *structural* induction on $FA$ and on $\sim$.

In the paper: 12 pages. □

# More detail on $FA$

If $A$ is a commutative monoid, then $F_0 A$ is defined by induction in this way:

- we have symbols $0, 1 \in F_0 A$,
- for every $a \in A$, we have a symbol $x_a \in F_0 A$,
- for all terms $s, t \in F_0 A$, we have a term $(s + t) \in F_0 A$ and a term $(st) \in F_0 A$,
- for every term $s \in F_0 A$, we have a term $f(s) \in F_0 A$.

The equivalence relation $\sim$ on $F_0 A$ is defined by 16 induction clauses ensuring that $FA = F_0 A/\sim$ is a commutative rig with a self-map.

The self-map $\mathbf{f} : FA \to FA$ is defined by $\mathbf{f}([a]) = [f(a)]$.

If $\phi : A \to B$ is a commutative monoid homomorphism, then $F\phi : FA \to FB$ is the unique morphism in $\mathrm{CRig}^{\circlearrowleft}$ which sends $[x_a]$ to $[x_{\phi(a)}]$.

# The countable family of deriving transformations

For every $n \in \mathbb{N}$, we define a deriving transformation

$$_n\mathrm{d}_A : FA \to FA \otimes A.$$

This is the unique deriving transformation such that

$$_n\mathrm{d}_A([f(a)]) = n \cdot {}_n\mathrm{d}_A([a]).$$

That is, $_n\mathrm{d}$ acts as if we had $\mathbf{f} = \mathrm{n} \cdot \mathrm{id}_{FA} : FA \to FA$ !

But we have $\mathbf{f} \neq \mathrm{n} \cdot \mathrm{id}_{FA} : FA \to FA$ for every $n \in \mathbb{N}$. (proved in the paper)

Constructing (by induction) these deriving transformations takes 22 pages in the paper.

It is then quite easy to prove that

$$_n\mathrm{d} \neq {}_p\mathrm{d}$$

if $n \neq p$.

# Conclusion

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(A flower is placed on the head of a cat and it suddenly understands the meaning of the universe.)

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If you have a problem with differential categories,
ask your differential cat.

And look at my paper `https://arxiv.org/abs/2510.03953` if you want the full details on today's problem.