Another look at the Vietoris Locale construction

Work in progress!

Dr Christopher F. Townsend

www.christophertownsend.org

25th October 2025

Overview

1. Background: locale theory
2. Define $V(X)$, the Vietoris locale of X
3. Vietoris Locale Construction - Examples
4. Vietoris Locale Construction - Key Facts
5. Vietoris Locale Construction - via the power locales
6. Extending the domain of definition
7. Application: $B\mathbb{G}$ is a topos for any localic groupoid $\mathbb{G}$

Background: locale theory

We shall be focusing on the category of locales **Loc** ($\equiv \mathbf{Fr}^{op}$ where **Fr** is the category of frames; $\mathbf{Fr} = \mathbf{Sup} \cap \mathbf{PreFr} \cap \mathbf{DLat}$).

- There is a Sierpński locale $\mathbb{S}$ - an internal distributive lattice in the category of locales.

The locale $\mathbb{S}$ can be used to describe lower (P_L) and upper (P_U) power locales:

Fact

$\mathbf{Loc}(Y, P_L(X)) \cong \mathbf{Sup}(\mathcal{O}X, \mathcal{O}Y) \cong \sqcup\text{-}Slat[\mathbb{S}^X, \mathbb{S}^Y].$
$\mathbf{Loc}(Y, P_U(X)) \cong \mathbf{PreFr}(\mathcal{O}X, \mathcal{O}Y) \cong \sqcap\text{-}Slat[\mathbb{S}^X, \mathbb{S}^Y].$

- Here $\mathbb{S}^X$ is a *presheaf* $\mathbf{Loc}^{op} \longrightarrow \mathbf{Set}$ defined by $W \mapsto \mathbf{Loc}(W \times X, \mathbb{S})$.
- The points $P_L(X)$ are the closed sublocales of X. The points of $P_U(X)$ are the fitted (intersection of open) sublocales of X with compact domain.

The Vietoris Locale construction

Definition

For every locale X we can associated the Vietoris locale $V(X)$. Its opens are generated by $\Box a$ and $\diamond a$ for each open a of X.

Subject to:

- $\diamond$ preserves arbitrary joins ('suplattice structure', **Sup**)
- $\Box$ preserves directed joins and finite meets ('preframe structure', **PreFr**)
- $\diamond a \wedge \Box b \leq \diamond(a \wedge b)$; and, $\Box(a \vee b) \leq \Box a \vee \diamond b$

- Introduced by Johnstone in 1985 ([J85]).
- Comes from the Vietoris hyperspace construction (1922), which was defined for only compact Hausdroff spaces.
- Johnstone saw that the construction could be done for all locales (not just compact Hausdorff locales).

Examples

Example

If X is a compact Hausdorff space, then we can define a Vietoris hyperspace by defining a topology on the set of of closed subsets of X. The topology is generated by

$$\diamond U = \{F|F \cap U \neq \phi\} \text{ and } \Box U = \{F|F \subseteq U\}$$

over all opens U of X. Purely formal/lattice theoretic matter to check that these opens satisfy the conditions just given in defining $V(X)$.

Example

If X is discrete then $V(X)$ is the set of finite subsets of X. *Proof:* A point of $V(X)$ will consist of (i) a suplattice hom. from PX; i.e. just a subset I of X, and (ii) and a preframe hom. from PX - i.e. a Scott open filter.

Then the 'modal' like conditions ensure that the this filter is $\uparrow I$ and so I is finite.

(If $\Box J = 1$ then $\forall i \in I$, $\diamond(\{i\}) = 1$; so, $\diamond(\{i\} \cap J) = 1$; i.e. $i \in J$. In the other direction $1 = \Box(I \cup I^c) \leq \Box(I) \vee \diamond(I^c) = \Box(I) \vee 0 = \Box I$; so $\Box J = 1, \forall J$ with $I \subseteq J$).

Facts about $V(X)$

Johnstone 1985 proved:

- V is a monad.
- If X is compact Hausdorff then $V(X)$ is compact Hausdorff.
- If X is locally compact, then so is $V(X)$.
- The points of $V(X)$ are the compact semi-fitted (a meet of fitted and closed) sublocales of $V(X)$.
- $V(X)$ has a semilattice structure
- there is at most one V-algebra structure on any X for a given semilattice structure.

A couple of other key facts:

(I) Between compact Hausdorff algebras of V, semilattice homomorphisms are always V-algebra homomorphisms but *not all compact Hausdorff semilattices are V-algebras*.

(II) For compact Hausdorff X, $\mathit{Patch}(\mathbb{S}^X) \cong V(X)$.

Looking at V via the power locales

It is immediate to see we can construct $V(X)$ as an equalizer in the category of locales:

$$VX \rightarrowtail P_L(X) \times P_U(X) \rightrightarrows Z$$

with the Z and the arrows determined by the exponential structure, so as to capture the two 'modal like' conditions. I was revisiting the area, trying to just exploit exponentiation as the universal way of looking at power locales.

How much of Johnstone 1985 can we recover?

- V is a monad - because P_L and P_U are.
- $V(X)$ is a semilattice, because $P_L(X)$ and $P_U(X)$ are semilattices.
- The points of $V(X)$ are the compact semi-fitted sublocales because they are the meets of pairs of points of P_L and P_U.
- V preserves compact Hausdorffness.

This is really mostly originally covered in Vickers work [V97] using generators and relations. For example, using the coverage theorems for the last one.

V via the power locales - newer perspectives

Some initial added perspective:

- V is formally self-dual.
- $V(X)$ is discrete if X is (and this is formally the same result that compact Hausdorffness is preserved).
- For discrete X, $P_L X \cong Idl(V(X))$ (cf 'most basic result' $PA = idl(FA)$; the power set is the ideal completion of the set of finite subsets of A for any set A).
- By duality $Patch(\mathbb{S}^X) = V(X)$ for compact Hausdorff X (as patch reverses ideal completion and $P_U X \cong \mathbb{S}^X$)
- for discrete/compact Hausdroff V-algebras, semilattice homomorphisms are V-algebra homomorphisms.

(These 'categorical' proofs are at various degrees of maturity.) But what really interested me was ...

Extending the domain of definition

Let $\mathbb{G}$ be a localic groupoid. Then we can naturally extend the definition of P_L and P_U to the category $[\mathbb{G}, \mathbf{Loc}]$ of $\mathbb{G}$-objects. For example (lower case, $\mathbb{G}$ just a group, G)):

$$P_L^{\mathbb{G}}(X, a : G \times X \longrightarrow X) \equiv ((P_L(X), G \times P_L(X) \longrightarrow P_L(G \times X) \xrightarrow{P_L(a)} P_L(X))$$

where the first un-labelled arrow is the strength that arises through the definition of P_L via exponentiation.

All of the key 'categorical' constructions work relative to $[\mathbb{G}, \mathbf{Loc}]$; for example we can trivially lift the definition of V:

$$V^{\mathbb{G}} \rightarrowtail P_L^{\mathbb{G}} \times P_U^{\mathbb{G}} \rightrightarrows \ldots$$

and the results under discussion lift to this broader context.

This generality is important because:

- It 'covers all bounded toposes' since all such toposes are of the form $B(\mathbb{G}) =$ $\mathbb{G}$-equivariant sheaves. These embed in $[\mathbb{G}, \mathbf{Loc}]$ as the discrete $\mathbb{G}$-objects.
- It is an even more general context, as for connected localic groups G, $B(G)$ is trivial but $[\mathbb{G}, \mathbf{Loc}]$ is not.

Novel proof that $B(\mathbb{G})$ is a topos (avoiding Girard's theorem)

Proposition:
If $Idl^{\mathbb{G}}(\_) : V^{\mathbb{G}}\text{-}Alg_{\mathbf{Dis}} \longrightarrow [\mathbb{G}, \mathbf{Loc}]^{op}$ has a right adjoint then $B(\mathbb{G})$ is a topos.

Proof.

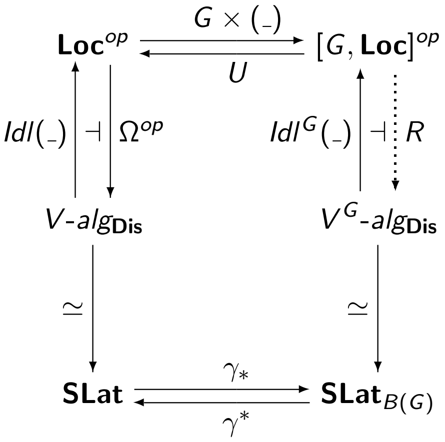
Let R be the right adjoint $[\mathbb{G}, \mathbf{Loc}]^{op} \longrightarrow V^{\mathbb{G}}\text{-}alg_{\mathbf{Dis}}$ be the right adjoint. Then, for any two discrete $\mathbb{G}$-objects, A, B:

$$
\begin{aligned}
\mathbf{Dis}_{[\mathbb{G},\mathbf{Loc}]}(B, RA) &\cong V^{\mathbb{G}}\text{-}_{\mathbf{Dis}}(V^{\mathbb{G}}B, (RA, \alpha_{RA})) \\
&\cong [\mathbb{G}, \mathbf{Loc}](A, Idl^{\mathbb{G}}V^{\mathbb{G}}B) \cong [\mathbb{G}, \mathbf{Loc}](A, P_L^{\mathbb{G}}(B)) \\
&\cong [\mathbb{G}, \mathbf{Loc}](A, \mathbb{S}_{\mathbb{G}}^{B}) \text{ (Vickers)} \\
&\cong [\mathbb{G}, \mathbf{Loc}](A \times B, \mathbb{S}_{\mathbb{G}}) \\
&= \text{“ relations on } A \times B\text{''}
\end{aligned}
$$

So RA must be the powerset on A. □

Outline of how the right adjoint R exists:

Hinges on the fact that discrete semilattices are the same thing as discrete V-algebras in any topos. Use forgetful functor from $B\mathbb{G}$ to $Sh(G_0)$ (not using that $B(\mathbb{G})$ is a topos!).



where γ is derived from the 'geometric morphism' $\gamma : \mathbf{Set} \longrightarrow B(G)$. NB: U is monadic.

Conclusions

- V can be defined relative to $[\mathbb{G}, \mathbf{Loc}]$ - i.e. even more general than previously known.
- Some known 'hard' results are duals of 'easy' results (e.g. $V(X)$ is the patch of $\mathbb{S}^X$ for compact Hausdorff X).
- There is a novel way of looking at constructing power objects in $B(\mathbb{G})$.
- *Still to do:* preservation of local compactness and uniqueness of V-algebra structure (given a fixed semilattice structure).
- Approach could help to describe étale completion of localic groupoids.

References

Johnstone, P. *Vietoris Locales and Localic Semilattices* Continuous Lattices and Their Applications. 1985 Imprint CRC Press Pages 26

Vickers, S. *Constructive points of powerlocales* Mathematical Proceedings of the Cambridge Philosophical Society, vol. 122, Issue 2, p.207-222