Derived Functors in HoTT

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Outline:

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2. Derived Functors
3. F-Projectives & Tor_*^R
4. Exact Structures

Let R be a ring, $\mathcal{B}$ an abelian category, and $F : \mathrm{Mod}_R \to \mathcal{B}$ a right-exact functor.
The **n-th left derived functor** is

$$\mathrm{L}_n FA := H_n(F\mathrm{p}A)$$

where $\mathrm{p}A$ is a projective resolution of A.

The choice of a projective resolution only exists if the category Mod_R has enough projective objects.
Mod_R has enough projectives by the *axiom of choice* (AoC).

1. Introduction: Homological algebra in HoTT

Developing homological methods **internal to Homotopy Type Theory (HoTT)**.
Earlier work done by Flaten 2023, and Christensen and Flaten 2024.

HoTT is a language to reason about ∞-toposes, e.g. ∞-category of spaces and sheaves on sites of spaces.

We restrict to the 1-topos of sets in an ∞-topos:

- Sets are 0-truncated objects
- Abelian groups are abelian group objects of sets
- Rings are ring objects of sets
- Etc...

HoTT has to be *constructive*. We **cannot** assume the Axiom of Choice (AoC).

Question: How do we compute derived functors without resolutions?

2. Derived Functors: Ext

Fix abelian category $\mathcal{A}$.
$\text{Ext}^n_{\mathcal{A}}$ are the right-derived functors of $\text{Hom}_{\mathcal{A}}$.

Proposition (Yoneda 1954, Christensen and Flaten 2024)

$\text{Ext}^n_{\mathcal{A}}$ *is isomorphic to the sets:*

- $n = 0$*:* $\text{Ext}^0_{\mathcal{A}}(B, A) := Hom_{\mathcal{A}}(B, A)$*.*
- $n = 1$*:* $\text{Ext}^1_{\mathcal{A}}(B, A)$ *is the set of SES* $A \hookrightarrow E \twoheadrightarrow B$ *modulo isomorphism of SES.*
- $n > 0$*:* $\text{Ext}^n_{R}(B, A)$ *is the set of of n-fold exact sequences*

$$0 \longrightarrow A \longrightarrow \cdots \longrightarrow B \longrightarrow 0$$

modulo an equivalence relation.

$\text{Ext}^1_{\mathcal{A}}$ is an abelian groups under the "Baer sum" $\oplus$.
The SES $A \hookrightarrow A \oplus B \twoheadrightarrow B$ is the 0-element.

A **homological δ-functor** (F, δ^F) is $\{F_n : \mathcal{A} \to \mathcal{B}\}$ such that for any SES $A \overset{i}{\hookrightarrow} E \overset{p}{\twoheadrightarrow} B$ there are connecting morphisms $\delta_n^F(i,p) : F_{n+1}B \to F_nA$, natural in (i,p), which yields a chain complex

$$\cdots \longrightarrow F_nA \xrightarrow{F_n(i)} F_nE \xrightarrow{F_n(p)} F_nB \xrightarrow{\delta^F_{n-1}(i,p)} \cdots \longrightarrow F_0A \xrightarrow{F_0(i)} F_0E \xrightarrow{F_0(p)} F_0B$$

A δ-functor is **exact** if the induced chain complexes are exact.

A δ-functor (F, δ^F) is **left-universal** if for any other δ-functor (T, δ^T), there is a natural isomorphism

$$res_0 : Fun_\delta((T,\delta^T),(F,\delta^F)) \to Nat(T_0, F_0)$$

2. Derived Functors: Cohomological δ-Functors

A **cohomological δ-functor** (F, δ^F) is $\{F^n : \mathcal{A} \to \mathcal{B}\}$ such that for any SES $A \overset{i}{\hookrightarrow} E \overset{p}{\twoheadrightarrow} B$ there are connecting morphisms $\delta^n_F(i,p) : F^n B \to F^{n+1} A$, natural in (i,p), which yields a chain complex

$$\begin{array}{ccccccc} \cdots \longleftarrow & F^n A & \xleftarrow{F^n(p)} & F^n E & \xleftarrow{F^n(i)} & F^n B & \\ & & & \delta_F^{n-1}(i,p) & & & \\ \cdots \longleftarrow & F^0 A & \xleftarrow{F^0(p)} & F^0 E & \xleftarrow{F^0(i)} & F^0 B & \end{array}$$

A δ-functor is **exact** if the induced chain complexes are exact.

A δ-functor (F, δ_F) is **right-universal** if for any other δ-functor (T, δ_T), there is a natural isomorphism

$$res^0 : Fun^\delta((F,\delta_F),(T,\delta_T)) \to Nat(F^0, T^0)$$

2. Derived Functors: δ-Pair

A δ-**pair** (F_1, F, δ^F) consists of two functors $F, F^1 : \mathcal{A} \to \mathcal{B}$ and for every SES $A \overset{i}{\hookrightarrow} E \overset{p}{\twoheadrightarrow} B$ there is a connecting morphism $\delta^F(i,p) : F_1 B \to FA$, natural in (i,p), which yields a 6-term chain complex

$$F_1 A \xrightarrow{F_1(i)} F_1 E \xrightarrow{F_1(p)} F_1 B \xrightarrow{\delta^F(i,p)} FA \xrightarrow{F(i)} FE \xrightarrow{F(p)} FB$$

A δ-pair is **exact** if the induced chain complexes are exact.

A δ-pair (F_1, F, δ^F) is **left-universal** if for any other δ-pair (T_1, T, δ^T), there is a natural isomorphism

$$res^0 : Fun_\delta((T_1, T, \delta^T), (F_1, F, \delta^F)) \to Nat(T, F)$$

Example

Fix $M \in \mathcal{A}$.
$(\mathrm{Ext}^n_{\mathcal{A}}(M, \cdot\,), \delta_{\mathrm{Hom}_R})$ is an exact right-universal cohomological δ-functor.
Every SES $A \hookrightarrow E \twoheadrightarrow B$ induces an exact sequence

$$\mathrm{Hom}_{\mathcal{A}}(M, A) \xrightarrow{i_*} \mathrm{Hom}_{\mathcal{A}}(M, E) \xrightarrow{p_*} \mathrm{Hom}_{\mathcal{A}}(M, B) \xrightarrow{\delta^0_{\mathrm{Hom}_{\mathcal{A}}}}$$

$$\mathrm{Ext}^1_{\mathcal{A}}(M, A) \xrightarrow{i_*} \mathrm{Ext}^1_{\mathcal{A}}(M, E) \xrightarrow{p_*} \mathrm{Ext}^1_{\mathcal{A}}(M, B) \xrightarrow{\delta^1_{\mathrm{Hom}_{\mathcal{A}}}}$$

$$\mathrm{Ext}^2_{\mathcal{A}}(M, A) \xrightarrow{i_*} \mathrm{Ext}^2_{\mathcal{A}}(M, E) \xrightarrow{p_*} \mathrm{Ext}^2_{\mathcal{A}}(M, B) \longrightarrow \cdots$$

Let $F : \mathcal{A} \to \mathcal{B}$ and $W : \mathcal{A} \to \text{Ab}$ be additive functors.
If there exists a representing object for the functor

$$T \mapsto Nat(W, \mathcal{B}(T, F))$$

we call it the **functor-hom exponential**, and denote it by $\{W, F\}^{\mathcal{A}}$.
Here $\mathcal{B}(T, F)(A) := \mathcal{B}(T, FA)$ for any $T \in \mathcal{A}$.

The **first left-satellite functor** $S_1 FA$ is

$$S_1 FA := \{\text{Ext}^1(A, \cdot), F\}^{\mathcal{A}}$$

If $\text{S}_1 FA$ exists for every $A \in \mathcal{A}$, then we obtain a functor $\text{S}_1 F$.
Higher satellites are defined iteratively: $S_{n+1} F := S_1 S_n F$.

Theorem (Yoneda 1960, Buchsbaum 1960)

If $S_1 F$ exists, then $(S_1 F, F, \delta_F)$ is a left-universal homological δ-pair.

Fix $B \in \mathcal{A}$.
Let $\int_{A:\mathcal{A}} \operatorname{Ext}^1_{\mathcal{A}}(B, A)$ be the category of SES ending in B.
Morphisms are diagrams on the form:

$$\begin{array}{ccccc} A & \hookrightarrow & E & \twoheadrightarrow & B \\ \downarrow & & \downarrow & & \downarrow \text{id} \\ A' & \hookrightarrow & E' & \twoheadrightarrow & B \end{array}$$

Define the functor

$$\mathrm{K}^F_B : \int_{A:\mathcal{A}} \operatorname{Ext}^1_{\mathcal{A}}(B, A) \to \mathcal{B} \quad \text{by} \quad A \overset{i}{\hookrightarrow} E \overset{p}{\twoheadrightarrow} B \mapsto \operatorname{Ker}(F(i))$$

Theorem
If $\varprojlim \mathrm{K}^F_B$ exists, then it is a model for the left-satellite functor $\mathrm{S}_1 FB$.

Let $\mathcal{P} \subseteq \mathcal{A}$ and $\mathcal{P}_B := \{A \hookrightarrow P \twoheadrightarrow B \mid P \in \mathcal{P}\} \subseteq \int_{A:\mathcal{A}} \mathrm{Ext}^1_{\mathcal{A}}(B, A)$.

$\mathcal{A}$ has **enough $\mathcal{P}$-covers** if for any $A \in \mathcal{A}$, there is an object $P \in \mathcal{P}$ with an epi $P \twoheadrightarrow A$.

Lemma (Buchsbaum 1960)

If $\mathcal{A}$ has enough $\mathcal{P}$-covers, then $\mathcal{P}_B$ is final in $\int_{A:\mathcal{A}} \mathrm{Ext}^1_{\mathcal{A}}(B, A)$.

Example

Let $\mathrm{Free}_R \subseteq \mathrm{Mod}_R$ be the collection of free R-modules.
There are enough Free_R-covers as $R[M] \twoheadrightarrow M$.

Example

Let $\mathrm{Proj}_{\mathcal{A}} \subseteq \mathcal{A}$ be the collection of projectives.
$\mathcal{A}$ has enough $\mathcal{P}$-covers if it has enough projectives.

Let $\mathcal{P} \subseteq \mathcal{A}$.
$F : \mathcal{A} \to \mathcal{B}$ is $\boldsymbol{\mathcal{P}}$ **left-exact** if for every SES $A \overset{i}{\hookrightarrow} E \overset{p}{\twoheadrightarrow} P$ where $P \in \mathcal{P}$, $F(i)$ is the kernel of $F(p)$.

Lemma (Röhrl 1962)

If F is $\mathcal{P}$ left-exact, then the functor $\mathrm{K}_B^F|_{\mathcal{P}_B}$ is essentially constant.

Example

An R-module F is **flat** if the functor $F \otimes_R \cdot$ is exact.
Let $\mathrm{Flat}_R \subseteq \mathrm{Mod}_R$ be the flat R-modules.
For $M \in \mathrm{Mod}_R$, the functor $\cdot \otimes_R M$ is Flat_R left-exact.

Example

Any additive functor $F : \mathcal{A} \to B$ is $\mathrm{Proj}_{\mathcal{A}}$ left-exact.

Let $\mathcal{P} \subseteq \mathcal{A}$ and $F : \mathcal{A} \to \mathcal{B}$. $\mathcal{P}$ is a **collection of F-projectives** if:

- $\mathcal{A}$ has enough $\mathcal{P}$-covers,
- F is $\mathcal{P}$ left-exact.

Pick a SES $A \overset{i}{\hookrightarrow} P \twoheadrightarrow B$ such that P is F-projective, then
$S_1 FB := \varprojlim \mathrm{K}^F_B \simeq \varprojlim \mathrm{K}^F_B \mid_{\mathcal{P}_B} \simeq \mathrm{Ker}(F(i))$

A functor $F : \mathrm{Mod}_R \to \mathcal{B}$ is **half-exact** if every SES $A \overset{i}{\hookrightarrow} E \overset{p}{\twoheadrightarrow} B$ is mapped to an exact sequence $FA \to FE \to FB$.

Theorem (T.)

Suppose that F is a half-exact functor. If $\mathcal{P}$ is a collection of F-projectives, then $(\mathrm{S}_1 F, F, \delta^F)$ is an **exact** *left-universal δ-pair.*

To use theorem with $(\cdot) \otimes_R M$, we must show that any free R-module is flat

Lemma (Mines, Richman, and Ruitenburg 1988, Flaten 2023)

A free R-module N is flat.

Proof.

R is projective, therefore also flat.
Since N is free, there exists a set X such that $N \simeq R[X]$.
Arbitrary direct sums of flat is flat, and $N = \bigoplus_{x \in X} R$, so N is flat. $\square$

Theorem (T.)

The first left-satellite $\mathrm{Tor}_1^R(\cdot, M) := \mathrm{S}_1((\cdot) \otimes_R M)$ *exists and* $(\mathrm{Tor}_1^R(\cdot, M), (\cdot) \otimes_R M, \delta^{(\cdot)\otimes_R M})$ *is an exact left-universal* δ*-pair.*

A collection $\mathcal{P} \subseteq \mathcal{A}$ is **right-hereditary**
if any epi $p : P \twoheadrightarrow Q$ with $P, Q \in \mathcal{P}$ has $\mathrm{Ker}(f) \in P$.

Theorem (Röhrl 1962, T.)

Let $\mathcal{P}$ be a right-hereditary collection of F-projectives. Then there exists an exact left-universal homological δ-functor $(\mathrm{S}F, \delta_F)$ such that $\mathrm{S}_0 F \simeq F$.

Lemma (Lombardi and Quitté 2015)

Let $f : A \twoheadrightarrow B$ be a surjection between flat modules, then $\mathrm{Ker}(f)$ is flat.

Theorem (T.)

There is an exact left-universal homological δ-functor $\mathrm{Tor}_R^*((\cdot), M)$ such that $(\cdot) \otimes_R M = \mathrm{Tor}_R^0((\cdot), M)$.
Furthermore, these groups can be computed by free resolutions.

Let $\mathcal{A}$ be an additive category.
An **exact structure** on $\mathcal{A}$ is a collection $\mathcal{E}$ of SES, plus axioms.

Each exact structure yields an $\mathrm{Ext}^1_{\mathcal{E}}$ functor.
Can define δ-functor and satellite-functor relative to $\mathcal{E}$.

Exact substructures $\mathcal{E}' \subseteq \mathcal{E}$ biject with *biadditive half-exact subfunctors* $F \subseteq \mathrm{Ext}^1_{\mathcal{E}}$.

$\mathcal{E}_{\mathrm{min}}$ is the collection of split short exact sequences.

Let $\mathcal{A}$ be abelian.
$\mathcal{E}_{\mathrm{max}}$ is the collection of all short exact sequences.

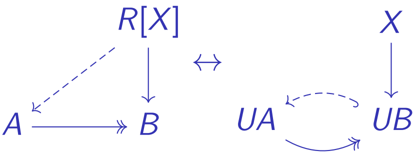
4. Relative Homological Algebra: Split-Epi Exact Structure

Let R be a ring.
An epi $f : A \twoheadrightarrow B$ is set-split if the morphism admits a section on the underlying sets.

Proposition (Christensen-T.)

The collection $\mathcal{E}_{\text{set-split}}$ of SES with set-split epi is an exact structure.

Every free module is $\mathcal{E}_{\text{set-split}}$-projective.



Tor^*_R and $\mathrm{Tor}^*_{R,\text{set-split}}$ coincide.

Assuming the axiom of choice, then $(\mathrm{Mod}_R, \mathcal{E}_{\max}) \simeq (\mathrm{Mod}_R, \mathcal{E}_{\text{set-split}})$.

$A \in \mathrm{Mod}_R$ is of type $\mathbf{FP}_\infty$ if it admits a projective resolution by finite rank free modules.

$I \in \mathrm{Mod}_R$ is $\mathbf{FP}_\infty$**-injective** if any mono $i : A \hookrightarrow B$ between FP_∞-modules is mapped to an epi $\mathrm{Hom}_R(i, I)$.

Proposition (Christensen-T.)

The collection $\mathcal{E}_{\mathrm{FP}_\infty}$ of SES with monos $i : A \to B$ being mapped to an epi $\mathrm{Hom}_R(i, I)$ for any FP_∞-injective I is an exact structure.

$\mathrm{Ext}^1_R(A, \cdot) \simeq \mathrm{Ext}^1_{\mathrm{FP}_\infty}(A, \cdot)$ whenever A is of type FP_∞.

There are enough FP_∞-injectives

FP_∞-injectives are divisible.

Assuming the axiom of choice, then $(\mathrm{Ab}, \mathcal{E}_{\max}) \simeq (\mathrm{Ab}, \mathcal{E}_{\mathrm{FP}_\infty})$.

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Thank you!