Categorical Tiling Theory

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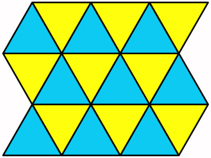
2024 UVA Topology REU

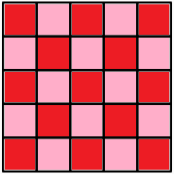
2025 Category Theory Octoberfest

Regular tilings of the plane

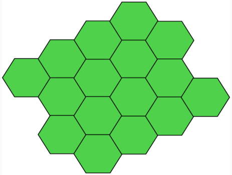
Euclidean tilings:



{3,6}

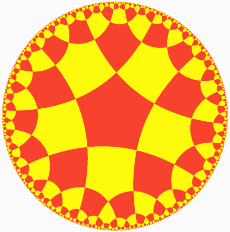


{4,4}

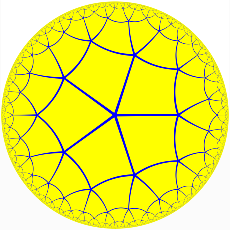


{6,3}

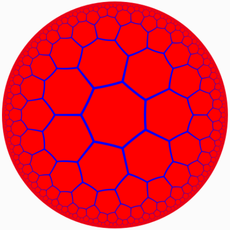
Hyperbolic Tilings:



{5,4}



{4,5}



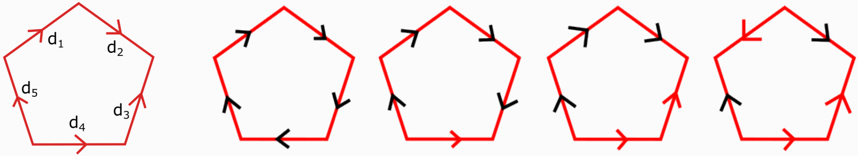
{7,3}

An *m-gon category* has objects and non-identity mophisms

$$\mathcal{C}_m := \begin{tikzcd} 0 \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] \arrow[rr, bend right=60, "v^1"'] \arrow[rr, bend right=90, "\vdots"', "v^m"'] & 1 \arrow[r, shift left=2, "d^1"] \arrow[r, phantom, "\vdots"] \arrow[r, shift right=2, "d^m"'] & 2 \end{tikzcd}$$

such that $\{d^i \circ s, d^i \circ t\} = \{v^i, v^{i+1 \pmod m}\}$ for $i = 1, \ldots, m$.

This corresponds to an m-gon with directed and labeled edges

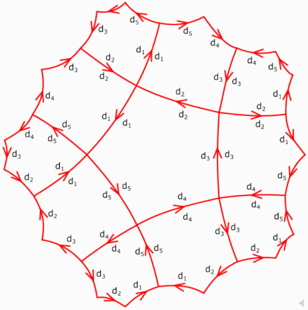


A presheaf on an m-gon category contains vertices, edges, and tiles

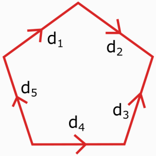
$$\mathcal{C}_m := \begin{tikzcd} 0 \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] \arrow[rr, bend right=50, "v^1"] \arrow[rr, bend right=80, "\vdots", "v^m"'] & 1 \arrow[r, shift left=2, "d^1"] \arrow[r, phantom, "\vdots"] \arrow[r, shift right=2, "d^m"'] & 2 \end{tikzcd}$$

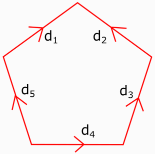
$$\begin{tikzcd} \{\text{vertices}\} & \{\text{edges}\} \arrow[l, shift right, "s"'] \arrow[l, shift left, "t"] & \{\text{tiles}\} \arrow[l, shift right=4, "d_1"'] \arrow[l, shift right=2] \arrow[l] \arrow[l, shift left=2] \arrow[l, shift left=4, "d_5"] \end{tikzcd}$$

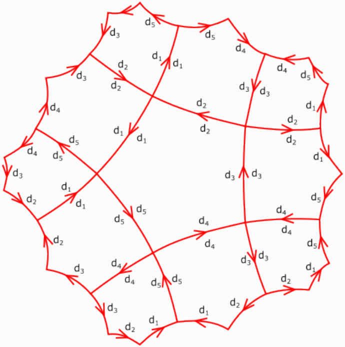
such that $\{s(d_i(x)), t(d_i(x))\} = \{v_i, v_{i+1 \pmod m}\}$ for each tile x.

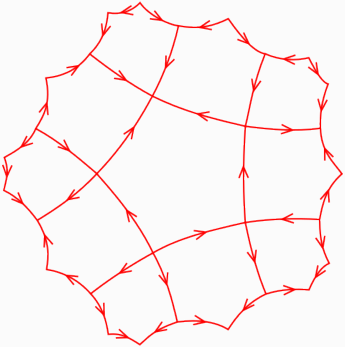


There are many different directed $\{m, n\}$ tilings:



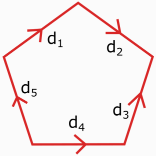


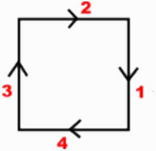


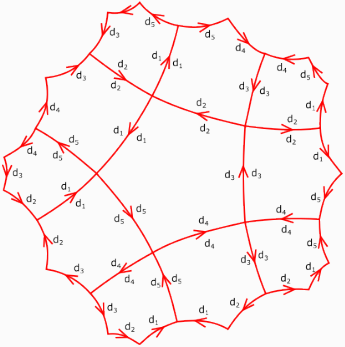


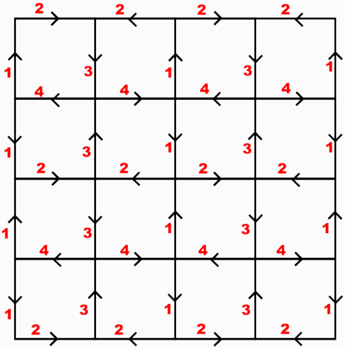
Reflective tilings

Whenever n is even, there is a reflective directed $\{m, n\}$ tiling:



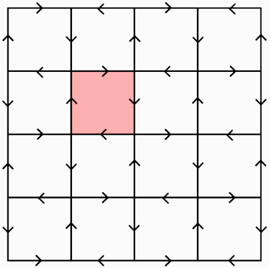


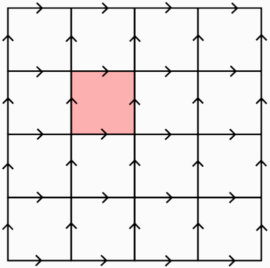




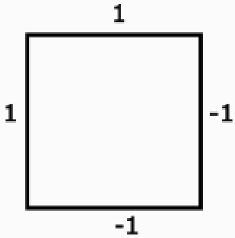
Reflection generated tilings

How to change the directions of a reflective tiling:





Reverse some edges of one tile:



Reflection generated tilings

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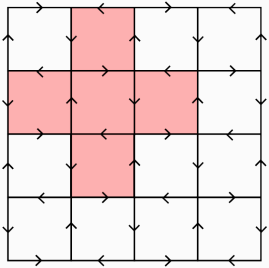
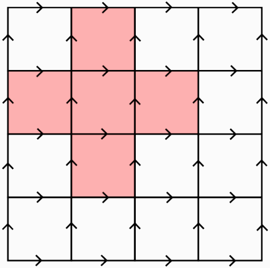
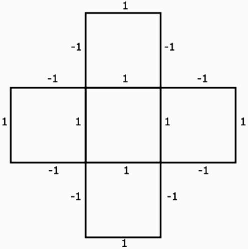
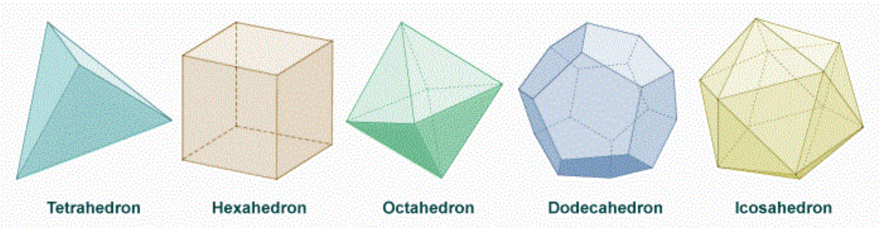




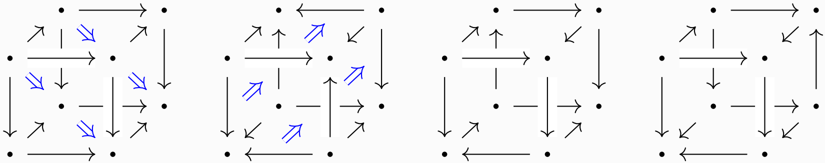
Reflect that tile outward and reverse more edges:



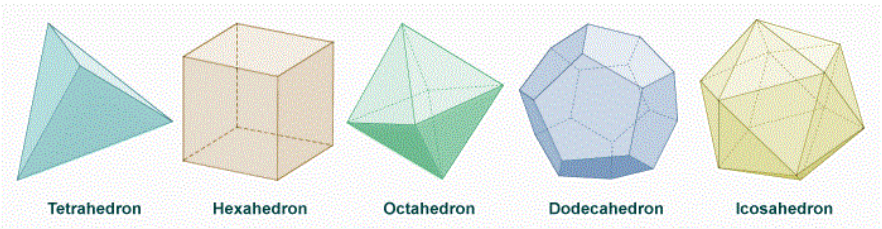
Higher Dimensions

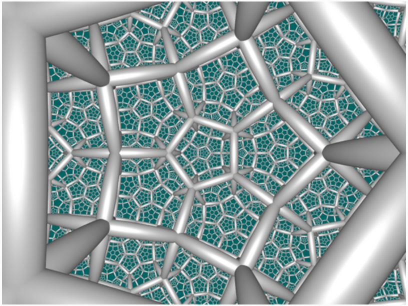


Cube categories: $$0 \underset{t}{\overset{s}{\rightrightarrows}} 1 \;\overset{d^1}{\underset{d^4}{\vdots}}\; 2 \;\overset{d^1}{\underset{d^6}{\vdots}}\; 3$$



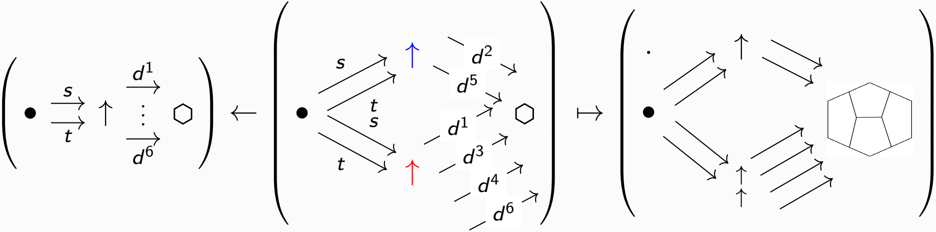
Higher Dimensions





Subdivisions and Cat$^{\#}$

$$\mathcal{C}_6 \xleftarrow{\text{discrete fibration}} \overline{\mathcal{C}_6} \longrightarrow \widehat{\mathcal{C}_5}$$



This is the data of a parametric right adjoint functor $\widehat{\mathcal{C}_6} \leftarrow \widehat{\mathcal{C}_5}$.
To subdivide a hexagon set X, choose $\overline{X}$ in the preimage of X in $\widehat{\overline{\mathcal{C}_6}}$ and apply the left adjoint $\widehat{\overline{\mathcal{C}_6}} \to \widehat{\mathcal{C}_5}$.

References

- "Categorical Tiling Theory: Constructing Directed Planar Tilings via Edge Reversal" - DiLeo, Sessoms, S. arXiv:2509.06363
- "Categorical Tiling Theory II: Schläfli Categories and Parametric Subdivisions" - Huffman, S. Work in progress

Thanks for coming!

https://en.wikipedia.org/wiki/Order-4_pentagonal_tiling
https://en.wikipedia.org/wiki/Order-5_square_tiling
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https://www.technologyuk.net/mathematics/geometry/platonic-solids.shtml
https://en.wikipedia.org/wiki/Order-4_dodecahedral_honeycomb