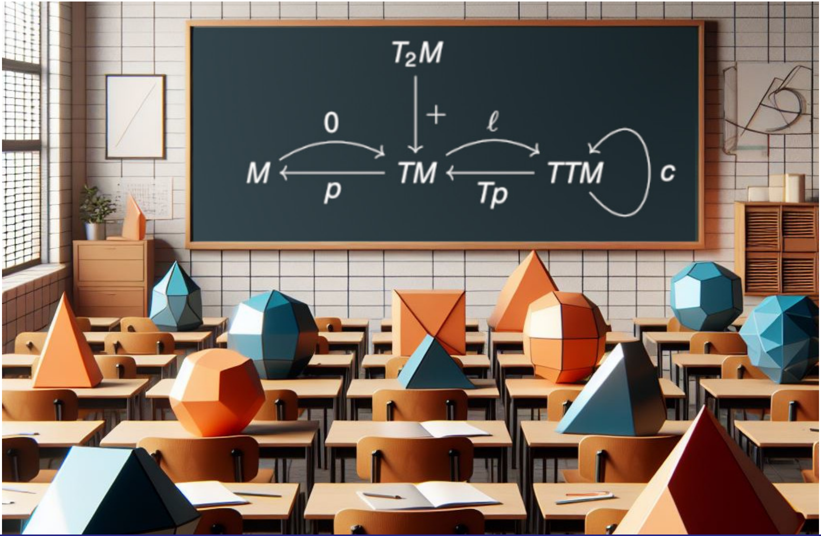
The dimension of the tangent bundle and the universality of the vertical lift

Florian Schwarz

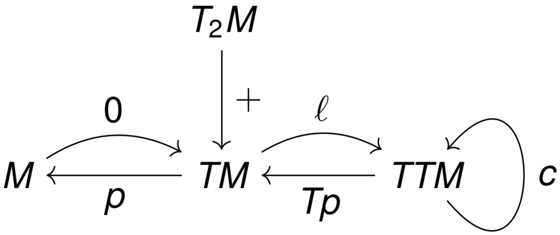
Category theory Octoberfest 2025

Tangent categories



A **tangent category** is a category $\mathbb{X}$ with:

- An endofunctor $T : \mathbb{X} \to \mathbb{X}$
- A projection natural transformation $p : T \to 1$ with pullback powers $T_n M$ preserved by T.
- A addition natural transformation $+ : T_2 \to T$
- A zero natural transformation $0 : 1 \to T$
- A vertical lift natural transformation $\ell : T \to T^2$
- A canonical flip natural transformation $c : T^2 \to T^2$



such that $(p, +, 0)$ is an additive bundle and $(\ell, 0)$ and $(c, 1_{TM})$ are additive bundle morphisms,

$$\begin{CD} T_3M @>{+\times_M 1}>> T_2M \\ @V{1\times_M +}VV @VV{+}V \\ T_2M @>>{+}> TM \end{CD} \qquad \begin{array}{ccc} T_2M & & \\ \downarrow{\scriptstyle \langle \pi_1, \pi_0\rangle} & \overset{+}{\searrow} & \\ T_2M & \xrightarrow{+} & TM \end{array} \qquad \begin{array}{ccc} TM & & \\ {\scriptstyle \langle 0\circ p, 1\rangle}\downarrow & \overset{1}{\searrow} & \\ T_2M & \xrightarrow[+]{} & TM \end{array}$$

$$\begin{CD} TM @>{\ell}>> T^2M \\ @V{p}VV @VV{T(p)}V \\ M @>>{0}> TM \end{CD} \qquad \begin{CD} T_2M @>{\langle \ell\circ\pi_0, \ell\circ\pi_1\rangle}>> T(T_2M) \\ @V{+}VV @VV{T(+)}V \\ TM @>>{\ell}> T^2M \end{CD} \qquad \begin{CD} M @>{0}>> TM \\ @V{0}VV @VV{T(0)}V \\ TM @>>{\ell}> T^2M \end{CD}$$

$$\begin{CD} T^2M @>{c}>> T^2M \\ @V{T(p_M)}VV @VV{p_{TM}}V \\ M @>>{1}> M \end{CD} \qquad \begin{CD} T(T_2M) @>{\langle c\circ\pi_0, c\circ\pi_1\rangle}>> T(T_2M) \\ @V{T(+_M)}VV @VV{+_{TM}}V \\ T^2M @>>{c}> T^2M \end{CD} \qquad \begin{CD} TM @>{1}>> TM \\ @V{T(0_M)}VV @VV{0_{TM}}V \\ T^2M @>>{c}> T^2M \end{CD}$$

the equations $c \circ c = 1$, $c \circ \ell = \ell$ hold, the diagrams

$$\begin{array}{ccc} T & \xrightarrow{\ell} & T^2 \\ {\scriptstyle \ell}\downarrow & & \downarrow{\scriptstyle T(\ell)} \\ T^2 & \xrightarrow[\ell_T]{} & T^3 \end{array} \qquad \begin{array}{ccccc} T^3 & \xrightarrow{T(c)} & T^3 & \xrightarrow{c_T} & T^3 \\ {\scriptstyle c_T}\downarrow & & & & \downarrow{\scriptstyle T(c)} \\ T^3 & \xrightarrow[T(c)]{} & T^3 & \xrightarrow[c_T]{} & T^3 \end{array} \qquad \begin{array}{ccccc} T^2 & \xrightarrow{\ell_T} & T^3 & \xrightarrow{T(c)} & T^3 \\ {\scriptstyle c}\downarrow & & & & \downarrow{\scriptstyle c_T} \\ T^2 & & \xrightarrow[T(\ell)]{} & & T^3 \end{array}$$

commute and the following diagram is a pullback:

$$\begin{array}{ccc} T_2(M) & \xrightarrow{T(+) \circ \langle \ell \circ \pi_0, 0 \circ \pi_1 \rangle} & T^2(M) \\ {\scriptstyle p \circ \pi_0 = p \circ \pi_1}\downarrow & & \downarrow{\scriptstyle T(p)} \\ M & \xrightarrow{\qquad\qquad} & T(M) \end{array}$$

This last condition is the **universality of the vertical lift**.

Examples

The usual tangent bundle on **smooth manifolds** forms a tangent structure.

- $T(M) = TM$
- $p : TM \to M, v_p \mapsto p$
- $+ : T_2M \to TM, (v_p, w_p) \mapsto v_p + w_p$
- $0 : M \to TM, p \mapsto 0_p$
- $\ell : TM \to TTM, v_p \mapsto (v_{0_p})_{0_p}$
- $c : TTM \to TTM, (a_{v_p})_{w_p} \mapsto (a_{w_p})_{v_p}$

Trivial tangent structure
Any category $\mathbb{X}$ with the identity functor $1_{\mathbb{X}} : \mathbb{X} \to \mathbb{X}$ and the identity natural transformations $p = 0 = + = \ell = c = 1$ is a tangent category.

Non-example (Vooys)

Take Mfld, the category of smooth manifolds and smooth maps with

- $\hat{T}(M) = M \times \mathbb{R}$
- $\hat{p} = \pi_0 : M \times \mathbb{R} \to M, x \mapsto (x, 0)$
- $\hat{0} : M \to M \times \mathbb{R}, x \mapsto (x, 0)$
- $\hat{+} : M \times \mathbb{R} \times \mathbb{R} \to M \times \mathbb{R}, (x, a, b) \mapsto (x, a + b)$
- $\hat{\ell} : M \times \mathbb{R} \to M \times \mathbb{R} \times \mathbb{R}, (x, r) \mapsto (x, 0, 0)$
- $\hat{c} : M \times \mathbb{R} \times \mathbb{R} \to M \times \mathbb{R} \times \mathbb{R},$

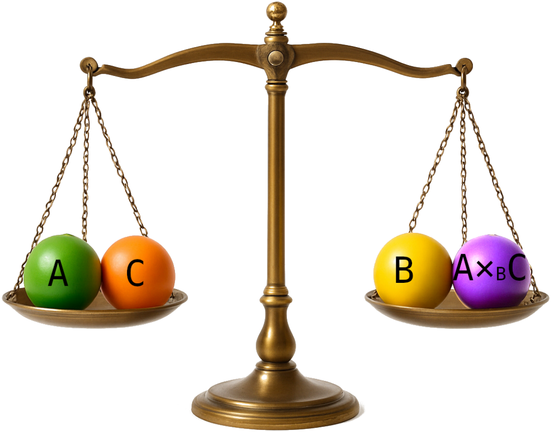
Universality of the vertical lift

$$\begin{array}{ccc} M \times \mathbb{R} \times \mathbb{R} & \longrightarrow & M \times \mathbb{R} \times \mathbb{R} \\ \downarrow & & \downarrow \\ M & \longrightarrow & M \times \mathbb{R} \end{array}$$

However, in Mfld, pullbacks along submersions fulfill

$$\dim(A \times_B C) = \dim(A) + \dim(C) - \dim(B).$$

Dimensions



$$\dim(A) + \dim(C) = \dim(A \times_B C) + \dim(B)$$

Definition

Let R be a commutative monoid and $\mathbb{X}$ be a category. An R-**valued dimension on** $\mathbb{X}$ is an assignment $\dim : \mathrm{Ob}(\mathbb{X}) \to R$ such that for pullbacks

$$\begin{array}{ccc} A \times_B C & \longrightarrow & A \\ \downarrow & & \downarrow f \\ C & \xrightarrow[g]{} & B \end{array}$$

- if f is a retraction and g is a section, and
- if f and g are both retractions,

the equation $\dim(A) + \dim(C) = \dim(A \times_B C) + \dim(B)$ holds.

Examples

- $\forall X, \dim(X) = c$ is a dimension for any constant c
- the classical dimension of manifolds

Theorem

Let R be a commutative monoid. Let $(\mathbb{X}, T)$ be a tangent category with an R-valued dimension dim.

a. *Then* $\dim(T^2(X)) + 2 \cdot \dim(X) = 3 \cdot \dim(T(X))$.
b. *If R is an integral domain and*

$$\exists a \in R \text{ such that } \forall X, \ \dim(T(X)) = a \cdot \dim(X),$$

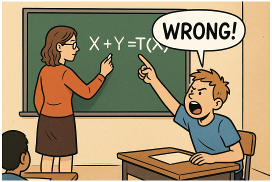
then $a = 1$ or $a = 2$ works.

Proof idea

What does this buy us?

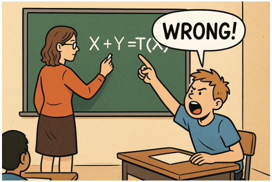
What does this buy us?

- Easy check to show something is **not** a tangent structure

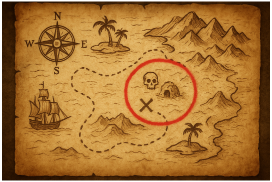


What does this buy us?

- Easy check to show something is **not** a tangent structure



- Learn what tangent structures to search for



Finite Sets

Example

Let FinSet be the category of finite sets and functions. The cardinality is an $\mathbb{N}$-valued dimension $\dim(X) = \#X$ on $\mathrm{FinSet}^{\mathrm{op}}$.

Consequences

- Any tangent structure on $\mathrm{FinSet}^{\mathrm{op}}$ fulfills

$$\#T^2(X) + 2 \cdot \#X = 3 \cdot \#T(X).$$

- Any Cartesian tangent structure on $\mathrm{FinSet}^{\mathrm{op}}$ fulfills

$$T(X) \cong X \qquad \text{or} \qquad T(X) \cong X \sqcup X.$$

CW complexes

Example
Let CW be the category of finite CW complexes and cellular maps. Then the Betti numbers

$$(B_n)_{n\in\mathbb{N}} : x \mapsto (\dim(H_n(X,\mathbb{Q})))_{n\in\mathbb{N}}$$

form an $\mathbb{N}^{\mathbb{N}}$-valued dimension on $\mathrm{CW}^{\mathrm{op}}$.

Consequence
Any tangent structure on $\mathrm{CW}^{\mathrm{op}}$ fulfills

$$B_n(T^2(X)) + 2 \cdot B_n(X) = 3 \cdot B_n(T(X)).$$

References

1. J. R. B. Cockett and G. S. H. Cruttwell, *Differential structure, tangent structure, and sdg*, Applied Categorical Structures 22 (2014), 331–417.
2. J. Rosický, *Abstract tangent functors*. Diagrammes 12 (1984), JR1–JR11.
3. P.W. Michor, *Topics in differential geometry*, Graduate studies in mathematics, American Mathematical Society (2008).
4. J. P. May, *A Concise Course in Algebraic Topology*, Chicago Lectures in Mathematics, University of Chicago Press (1999).

All figures were self-made using generative AI, Quiver and Gimp