Closed Geometric Logic

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26.10.2025

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- Topos theory provides categorical semantics for intuitionistic logic.
- But to both Hegel and Marx, contradiction was central.
- So where are the contradictions?

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- but in his œuvre these are barely footnotes.

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So what are they?

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- often at the boundary,
- unstable,
- a transition between stable qualities,
- a driver of development,
- part of a logic of qualities.

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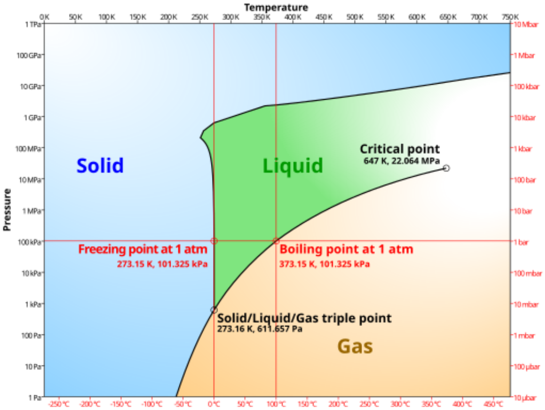
- So something closed.
- We can see how this applies in phase spaces.

Phase Diagram of Water[1]



[1]Courtesy of Cmglee on Wikipedia.

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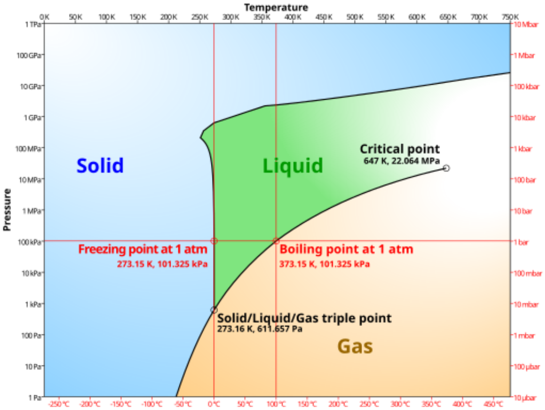

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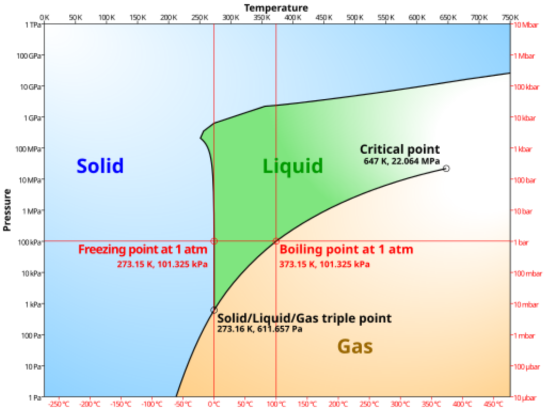

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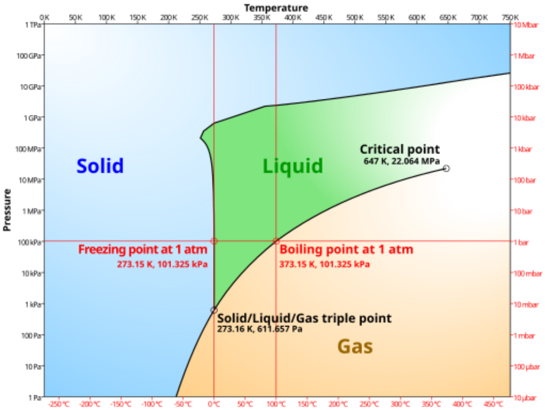

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- Most (not all) instability is at the boundary,
- Instability is small.

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When is something small?

If we can describe smallness, we can use it to define classifying morphisms that are internal functions outside a small domain and logically describe situations as before.

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So what does it mean to be small?

Leading Example

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- The join is the closure of the set-theoretic union,
- We have a notion of smallness given by nowhere dense sets.
- A closed subset is nowhere dense if and only if it is in the closure of disjoint open subset.

Smallness

Definition

In an upward-bounded preorder P, for $a \in P$ we call a *small* if whenever a family B satisfies

$$a \vee \bigvee B = \top,$$

then the join $\bigvee B$ exists and equals $\top$.

Context: Upward-Bounded Preorders

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- We can still talk about joins if we don't assume they always exist,
- Most developed notions have dual "open" analogues.

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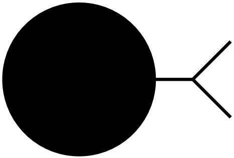

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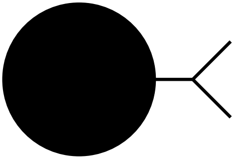

Example

Closed subsets are reduced if and only if they are regular.

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In $\mathbf{C}(X)$, $\neg C$ is the closure of the set-theoretic complement of C.

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In the lattice of closeds, this reproduces the usual topological boundary.

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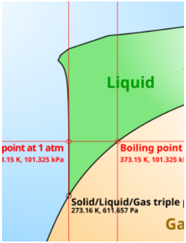

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- a is reduced if and only if $\neg a = a$.

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- Topological product projections are not closed maps,
- Generating colimits of closed/compact spaces will add "open" spaces.

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Every co-Heyting algebra is a co-Heyting structure.

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Remark

The subdivision coverage has no open equivalent.

Sheaves inherit co-Heyting structure

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Theorem

A co-Heyting structure on a site induces a co-Heyting structure on its sheaf topos (with respect to the subdivision coverage).

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This yields cohesive topoi; in particular compact analogues of $\mathbf{CartSp}$ (as in *Differential Cohomology in a Cohesive* ∞*-Topos*).

Closing Remarks

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- I think this largely validates Hegel's treatment of contradiction.

Thank you for your attention!