Stability and safety for control-coalgebras

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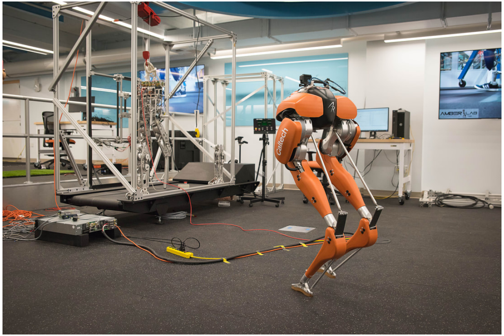
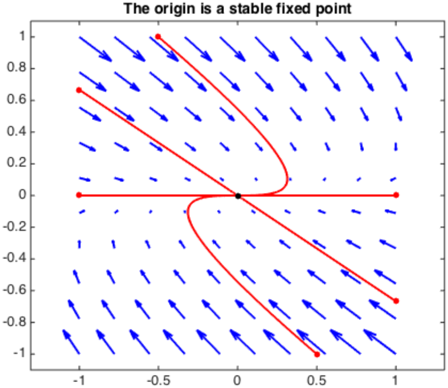


Dynamical Systems



- vector field: $\sigma : E \to \mathrm{T}(E)$

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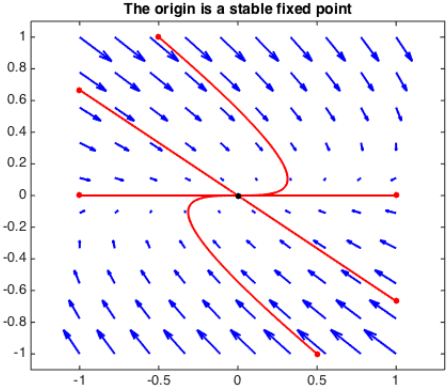

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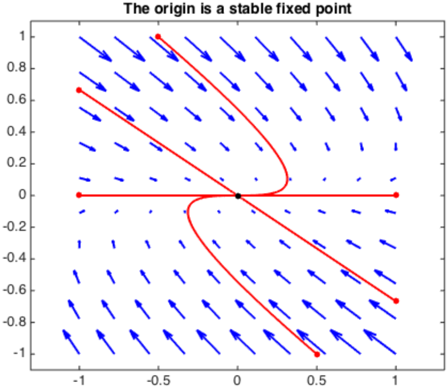

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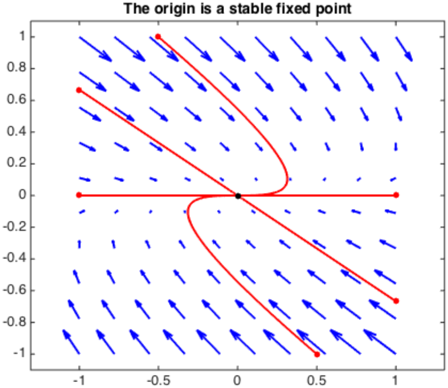

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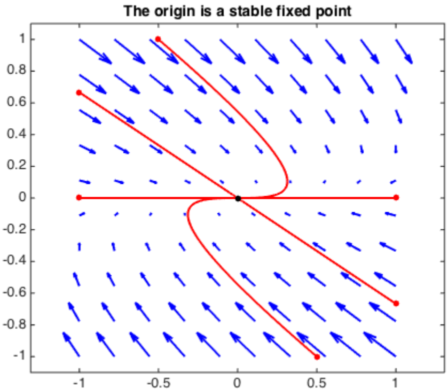

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- point: $x^* \colon 1 \to X$
- equilibrium:
  $\sigma(x^*) = \vec{0}$ or $\phi(t, x^*) = x^*$
- stable: $\forall \varepsilon > 0 \, \exists \delta > 0, ||x - x^*|| < \delta$ implies $||\phi(t, x) - x^*|| < \epsilon$

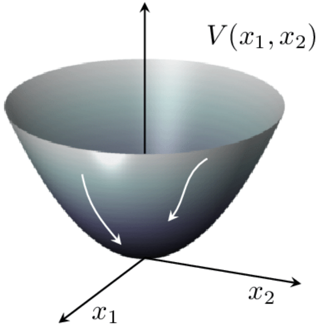
Classical Lyapunov's theorem

Theorem (Lyapunov, 1892)

Let $x = 0$ be an equilibrium point for an autonomous system $\dot{x} = f(x)$, and $M \subset \mathbb{R}^n$ be a domain containing $x = 0$. Let $V : M \to \mathbb{R}$ be a continuously differentiable function such that

- $V(0) = 0$
- $V(x) \geq 0$
- $\frac{\partial V}{\partial x} \cdot f(x) \leq 0$

Then $x = 0$ is a stable equilibrium point.



Systems as Coalgebras

Definition

Let $\mathcal{C}$ be a category, $\mathcal{F}: \mathcal{C} \to \mathcal{C}$ a functor. An **$\mathcal{F}$-coalgebra** *is an object $X \in \mathcal{C}$ and a map $f: X \to \mathcal{F}(X)$.*

Examples

- $\mathcal{T}$-coalgebras are (continuous-time) dynamical systems (not necessarily sections).
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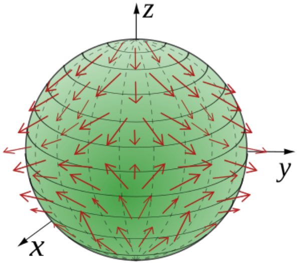

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- $\mathrm{id}_{\mathsf{Man}} \colon \mathsf{Man} \to \mathsf{Man}$, $\mathrm{id}_{\mathsf{Man}}$-coalgebras are discrete-time dynamical systems $f \colon M \to \mathrm{id}_{\mathsf{Man}}(M) = M$.
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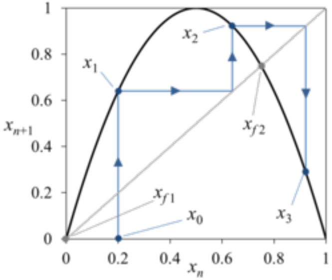

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- $\mathcal{P} \colon \mathsf{Set} \to \mathsf{Set}$ the power set, $\mathcal{P}$-coalgebras are transition systems.
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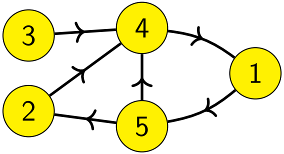
$$f \colon \{1,2,3,4,5\} \to \mathcal{P}(\{1,2,3,4,5\})$$



$$f(5) = \{2,4\}$$

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- $\mathrm{id}_{\mathsf{Man}} \colon \mathsf{Man} \to \mathsf{Man}$, $\mathrm{id}_{\mathsf{Man}}$-coalgebras are discrete-time dynamical systems $f \colon M \to \mathrm{id}_{\mathsf{Man}}(M) = M$.
- $\mathcal{P} \colon \mathsf{Set} \to \mathsf{Set}$ the power set, $\mathcal{P}$-coalgebras are transition systems.
- Fix a set L of "labels", $\mathcal{P}(L \times -) \colon \mathsf{Set} \to \mathsf{Set}$, $\mathcal{P}(L \times -)$-coalgebras are L-labeled transition systems.

Definition

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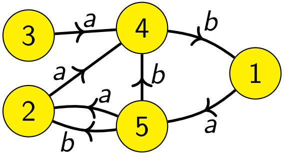
$$L = \{a, b\}$$

$$f \colon \{1,2,3,4,5\} \to \mathcal{P}(\{a,b\} \times \{1,2,3,4,5\})$$



$$f(5) = \{(a,2), (b,2), (b,4)\}$$

Measurement Object

Definition

An object $R \in \mathcal{C}$ is **posetal** *if each $\mathcal{C}(X, R)$ has a partial order such that for any $f : X \to Y$, if $g_1 \geq g_2$, then $g_1 \circ f \geq g_2 \circ f$.*

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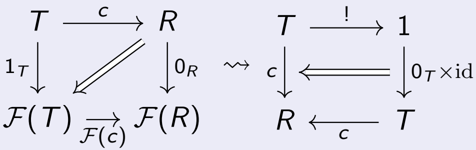

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$$
\begin{array}{ccc}
T & \xrightarrow{c} & R \\
{\scriptstyle 1_T}\downarrow & \swarrow\!\!\!\swarrow & \downarrow{\scriptstyle 0_R} \\
\mathcal{F}(T) & \xrightarrow[\mathcal{F}(c)]{} & \mathcal{F}(R)
\end{array}
\quad \rightsquigarrow \quad
\begin{array}{ccc}
T & \xrightarrow{!} & 1 \\
{\scriptstyle c}\downarrow & \Longleftarrow & \downarrow{\scriptstyle 0_T \times \mathrm{id}} \\
R & \xleftarrow[c]{} & T
\end{array}
$$

Definition

A **semi-metric** *is a map $d\colon X \times X \to R$ such that*

- $d \Rightarrow 0$
- $\ker(d) \cong \Delta\colon X \to X \times X$

For a fixed $x_*\colon 1 \to X$, let $\|\cdot\|_{x^*}$ denote the composite

$$X \xrightarrow{\mathrm{id}_X \times x^*} X \times X \xrightarrow{d} X$$

called the **semi-norm** *relative to x^*.*

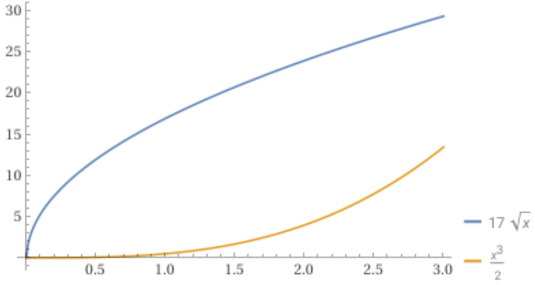
Class K Morphisms

Definition

A morphism $\alpha \colon R \to R$ *is* **class** $\mathcal{K}$ *if:*

- α *is an order-preserving map*
- α *has an order-preserving inverse* α^{-1}
- $\alpha \circ 0_R = 0_R$.

Replaces ε-δ stuff: $\alpha(\varepsilon) = \delta$.



Stable Equilibria

An equilibrium point $x^* \in M$ is **stable** if there is a class $\mathcal{K}$ function α such that for any solution curve c:

$$\|c(t) - x^*\| \leq \alpha(\|c(0) - x^*\|)$$

Definition

An equilibrium point $x^* : 1 \to X$ is **stable** *if there is a class $\mathcal{K}$ morphism α such that the following diagram lax commutes for any solution curve c:*

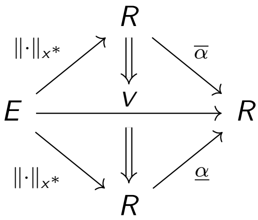
$$\begin{array}{ccccc} 1 & \xrightarrow{c_0} & E & \xrightarrow{\|\cdot\|_{x^*}} & R \\ {\scriptstyle !}\uparrow & & \rotatebox{90}{$\leq$} & & \downarrow{\scriptstyle \alpha} \\ I & \xrightarrow[c]{} & E & \xrightarrow[\|\cdot\|_{x^*}]{} & R \end{array}$$

$$\|c(t) - x^*\| \leq \alpha(\|c(0) - x^*\|), \ \forall t \in I$$

Lyapunov morphisms

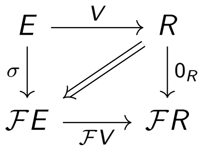
A **Lyapunov morphism** $V\colon M \to R$ for flow ϕ and equilibrium x^* if:

1. (positive definite) V is bounded by class $\mathcal{K}$ morphisms



$$V(x) \geq 0,\ V(x) = 0 \text{ iff } x = x^*.$$

2. (decrescent) the following diagram lax commutes.



$$\frac{\partial V}{\partial x}\sigma(x) \leq 0$$

Lyapunov's Theorem

Theorem

Let $\mathcal{F}\colon \mathcal{C} \to \mathcal{C}$ be a functor. Let $x^*\colon 1 \to X$ be an equilibrium point for a $\mathcal{F}$-coalgebra $f\colon X \to \mathcal{F}(X)$. Let $V\colon M \to R$ be a morphism of $\mathcal{C}$ such that

- *(positive definite)* $\underline{\alpha} \circ \|\cdot\|_{x^*} \leq V \leq \overline{\alpha} \circ \|\cdot\|_{x^*}$.
- *(decrescent)* $\mathcal{F}(V) \circ f \leq 0_R \circ V$.

Then x^* is a stable equilibrium point.

Examples

- cts-time: $\frac{\partial V}{\partial x} f(x) \leq 0$
- discrete-time: $\nabla V(X) = V(f(x)) - V(x) \leq 0$
- transition system: $\max_{s' \in f(s)} V(s') \leq V(s)$

Stabilizing Control Systems

A control system has an extra variable

$$\dot{x} = f(x, u)$$

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Control theory is all about finding a $k(x)$ so the closed system $\dot{x} = f(x, k(x))$ has some desired property

Stabilizing Control Systems

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Control theory is all about finding a $k(x)$ so the closed system $\dot{x} = f(x, k(x))$ has some desired property

Theorem (Artstein–Sontag Theorem)

If you have a **control Lyapunov function** $V: M \to \mathbb{R}$

- *V is positive definite w.r.t. x^**
- *for each $x \in M$, there exists $u \in U$ such that $\frac{\partial V}{\partial x} \cdot f(x, u) \leq 0$*

then there exists a stabilizing controller k.

Control Coalgebras

Definition

Let $\mathcal{C}$ have finite products, $\mathcal{F} \colon \mathcal{C} \to \mathcal{C}$ an endofunctor, $\pi \colon \mathcal{F} \Rightarrow \mathrm{id}_{\mathcal{C}}$ a natural transformation. A **control coalgebra***:*

$$\begin{array}{ccc} U & \xrightarrow{\quad f \quad} & \mathcal{F}M \\ & {\scriptstyle p}\searrow \qquad \swarrow{\scriptstyle \pi_M} & \\ & M & \end{array}$$

consists of

- *a* **state space** *object $M \in \mathcal{C}$*
- *a* **control bundle** *$p \colon U \to M$*
- *an M-bundle map $f \colon U \to \mathcal{F}M$*

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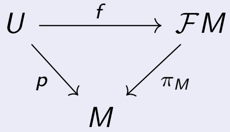

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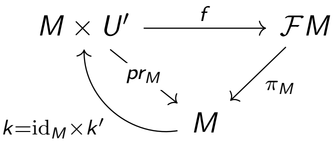
Definition

A **controller** *is a section of p.*

Think of:

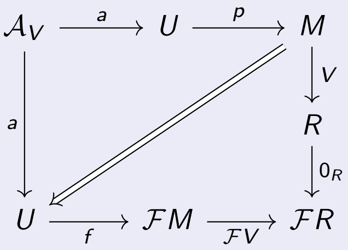


$\dot{x} = f(x, k'(x))$ is an autonomous system

The Admissible Locus

Definition

Given a control coalgebra $(p\colon U \to M, f)$ *and a map* $V\colon M \to R$, *the* **admissible locus** *of* V *is the lax equalizer of* $\mathcal{F}V \circ f$ *and* $0_R \circ V \circ p$:



This is basically the subspace of points that make this classical condition holds

$$\mathcal{A}_V = \{(x, u) \mid \frac{\partial V}{\partial x} \cdot f(x, u) \le 0\}$$

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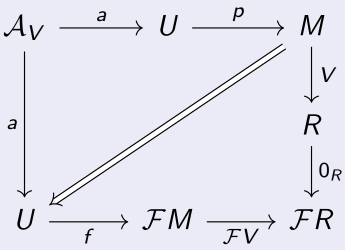

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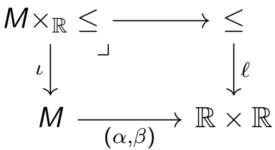
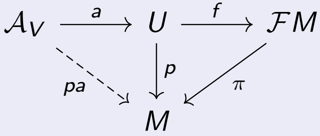
This lax equalizer can be realized as a pullback



Control Lyapunov Morphisms

Definition

A **control Lyapunov morphism** *is a map $V\colon M \to R$ such that pa is epi.*



Theorem (Moeller-Ames)

If $V\colon M \to R$ is a CLM, then any section $u\colon M \to \mathcal{A}_V$ of pa yields a stabilizing controller: $a \circ u\colon M \to U$.

Thanks!

A. Ames, J. Moeller, P. Tabuada, *Categorical Lyapunov Theory I: Stability of Flows*, arXiv:2502.15276

- systems are flows (monoid actions)
- notion of stability native to the setting
- Theorem: if a Lyapunov morphism exists for the flow, then the point is stable.

A. Ames, S. Mattenet, J. Moeller, *Categorical Lyapunov theory II: Stability of systems*, arXiv:2505.22968

- systems are $\mathcal{F}$-coalgebras
- notion of stability induced from flows
- Theorem: if a Lyapunov morphism exists for the coalgebra, it is a Lyapunov morphism for the solution flow.