Finite presentability of $\pi_k(S^n)$ in Cubical Agda

Talk discusses work by: Reid Barton, Tim Campion, Axel Ljungström, Owen Milner, Anders Mörtberg, Loïc Pujet, and others

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Outline

Overview of the project

The Serre finiteness theorem (SFT) implies that for every n, and every k, $\pi_k(S^n)$ is finitely presentable (FP) [5].

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The formal proof provides a *verified* algorithm which can, in principle, compute presentations of these groups.

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The first proposition was proved in HoTT more than a decade ago by Licata and Shulman [2].

The bulk of the work in the Barton-Campion proof, and also in the formalization, goes towards proving the second proposition.

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and a handful individual files containing proofs about finitely presented abelian groups, finite CW complexes, and stably almost finite spaces.

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- A FP abelian group is (merely) equivalent to a direct sum of free groups and finite cyclic groups. This appears in LastMinuteLemmas
- (Following from the above) an induction principle for FP abelian groups: to prove that a proposition P, defined on all groups, holds for all FP abelian groups, it suffices to show it holds for cyclic groups, and that if it holds of two groups H and G, then it holds of $H \oplus G$.

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Some major contributions:

- Formalization of the CW structure on pushouts of maps between finite CW complexes.
- Direct computation of (the abelianization of) the lowest non-trivial homotopy group of a connected finite CW complex as a FP abelian group.

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Some major contributions:

- Formalization of the CW structure on pushouts of maps between finite CW complexes.
- Direct computation of (the abelianization of) the lowest non-trivial homotopy group of a connected finite CW complex as a FP abelian group.
- (Using the above) a proof of the Hurewicz theorem for finite CW complexes. Ljungström and Pujet observed that, contrary to expectations, this is not necessary for the proof of the SFT, all that's needed is the previous result.

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Definition

A space X is called **stably almost finite** if for every n there (merely) exists an m, and an $(m+n)$-connected map

$$C \to \mathbf{Susp}^m(X)$$

from a finite CW complex C.

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The proof of the SFT rests on an induction where, at each step, we apply this closure property.

The proof of the closure property itself uses the join construction/Ganea's theorem [4].

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$\pi_1(X) = 0$, and $\pi_2(X)$ must be FP using Freudenthal suspension theorem and SAF condition.

This is the base case for the induction.

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And so the induction proceeds...

Code

The final induction:

```
mutual
  saf→isFPn : (X : Pointed ℓ) (safX : saf X) (scX : isConnected 3 (typ X)) (n : ℕ)
    → isFP (πAb n X)
  saf→isFPn X safX scX zero = saf→isFPBottomn X safX 0 scX
  saf→isFPn X safX scX (suc n) =
    transport (isFPId X (suc n)) (saf→isFPBottomn (X < (2 + n) >) (saf→saf<-> X safX scX (suc n))

  saf→saf<-> : (X : Pointed ℓ) (safX : saf X) (scX : isConnected 3 (typ X)) (n : ℕ)
    → saf (X < (suc n) >)
  saf→saf<-> X safX scX 0 = transport (λ i → saf (1ConnCovEq X scX i)) safX
  saf→saf<-> X safX scX (suc n) =
    safTotal (<->EMFibSeq X n) (Conn→ConnConnCov X 3 (suc n) scX)
      (saf→saf<-> X safX scX n) (isFP→safEM' (πAb n X) (saf→isFPn X safX scX n) n)

-- Universe polymorphic for demonstration purposes,
-- but should get specialized back to spheres in ℓ-zero
isFPπAbSn : {ℓ : Level} (n m : ℕ) → isFP (πAb n (S {ℓ = ℓ} (2 + m)))
isFPπAbSn n m = saf→isFPn (S (2 + m)) (saf-Sn (2 + m)) rem2 n
  where
```

Code

Closure property for SAF spaces:

```
safTotal : {F E B : Pointed ℓ} (S : FiberSeq F E B) (scB : isConnected 3 (typ B))
    → saf B → saf F → saf E
safTotal {ℓ} {F'} {E'} {B} S scB hB hF' k = saf-E k (suc k) (saf-En k)
  where
    open Ganea^ (FiberSeqProj S)

    ΩB-connected : isConnected 2 (typ (Ω B))
    ΩB-connected = isConnectedPath 2 scB (pt B) (pt B)


    F0-≡ : F 0 ≡ F'
    F0-≡ = fst (PathPΣ (FibsEqOfFibSeq (FiberFiberSeq (FiberSeqProj S)) S refl))

    saf-ΩB : saf (Ω B)
    saf-ΩB = saf→safΩ scB hB

    saf-Fn : (n : ℕ) → saf (join-F n)
    saf-Fn n = join·^-saf (F 0) (Ω B) n
               (transport (λ i → saf (F0-≡ (~ i))) hF')
               saf-ΩB (isConnectedSubtr 1 1 ΩB-connected)

    connected-join-F' : (k : ℕ) → isConnected (suc k)
                                               (typ (join-F (suc k)))
    connected-join-F' k = join·^-connected (F 0) (Ω B) (suc k)
                            (isConnectedSubtr 1 1 ΩB-connected)
```

Code

Induction principle for FP abelian groups:

```
indFP : (P : AbGroup₀ → Type ℓ) → (∀ A → isProp (P A))
  → (∀ n → P (ℤMod n))
  → (∀ H K → P H → P K → P (AbDirProd H K))
  → (∀ A → isFP A → (P A))
indFP P pr bas prod A = PT.rec (pr A)
  λ FP → subst P (AbGroupPath _ _ .fst
        (GroupIso→GroupEquiv
          (invGroupIso (isFP→isFPGroupNormalForm FP .snd .snd))))
            (main _ _)
 where
 lem : AbGroupEquiv (ℤAbGroup/ 1) (trivialAbGroup {ℓ-zero})
 fst lem = isContr→Equiv FinAlt.isContrFin1 (tt* , (λ {tt* → refl}))
 snd lem = makeIsGroupHom λ _ _ → refl

 main : (l : List ℤ) (n : ℕ) → P (FPGroupNormalForm' (pos n) l)
 main [] zero = subst P (AbGroupPath _ _ .fst lem) (bas 1)
 main [] (suc n) = prod _ _ (bas 0) (main [] n)
 main (x ∷ l) n = prod _ _ (bas (abs x)) (main l n)
```

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Ljungström and Pujet's formalization of the theory of CW complexes led to a significant and unexpected simplification.

Thank you for your attention.

References I

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