Double Orthogonal Factorization Systems

In fibrant double categories

Ruben Maldonado

Posgrado Conjunto en Ciencias Matemáticas
Universidad Nacional Autónoma de México

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Joint work with: Dorette Pronk, Luca Mesiti, Elena Caviglia, Matthew Kukla, C. B. Aberlé, Tanjona Ralaivaosaona

Outline

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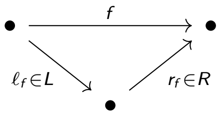

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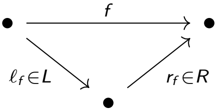
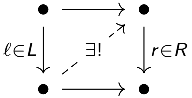


- The lifting property:



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- If $(C, (L_C, R_C))$ to $(D, (L_D, R_D))$ have a choice of factorizations each, then the functor F is a **strict morphism** if it preserves the chosen factorizations on the nose.

Double Categories

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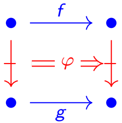

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Definition (DOFS with chosen factorizations)

A **DOFS with chosen factorizations** is a DOFS where the OFS in D_0 and D_1 have chosen factorizations, and the source, target, and unit functors are **strict morphisms**.

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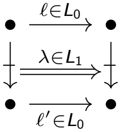

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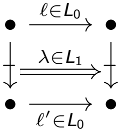

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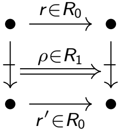

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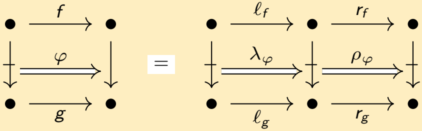

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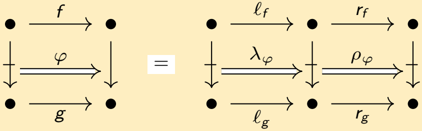

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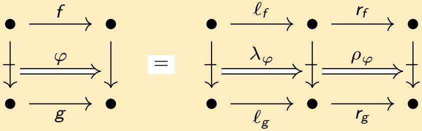

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- In a colax DOFS only the left class $\mathbb{L}$ is closed under vertical composition.

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- The double category of quartets $\mathbb{S}\mathrm{q}(C)$ over a category C with an OFS (L, R). The class $\mathbb{L}$ consists of cells whose horizontal arrows are in L and the class $\mathbb{R}$ consists of cells whose horizontal arrows are in R.

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- The double category of paths $\mathbb{P}\mathrm{ath}(\mathbb{Q}(\mathcal{B}))$ over a 2-category $\mathcal{B}$ with DOFS in $\mathbb{Q}(\mathcal{B})$. The class $\mathbb{L}$ and $\mathbb{R}$ are defined from the DOFS in $\mathbb{Q}(\mathcal{B})$.

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- The double category of relations $\mathbb{R}\mathrm{el}(C)$ over a regular category C. The left class $\mathbb{L}$ consists of cells with regular epimorphisms as horizontal arrows and a regular epimorphism for the arrow between the relations. The right class $\mathbb{R}$ consists of cells with monomorphisms as horizontal arrows and a monomorphism for the arrow between the relations.

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- The double category of profunctors $\mathbb{P}\mathrm{rof}$. The class $\mathbb{L}$ consists of surjective natural transformations with compositionally surjective functors as horizontal arrows. The class $\mathbb{R}$ consists of injective natural transformations with faithful functors that are injective on objects as horizontal arrows.

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Notice that in an OFS it is straight forward to choose factorizations. However, in a DOFS it is not always immediate, because the source, target, and unit functors must be strict respect to the chosen factorizations in (L_0, R_0) and (L_1, R_1).

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Then the factorizations in (L_1, R_1) can be adjusted to obtain a DOFS with chosen factorizations for $\mathbb{D}$

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In a fibrant double category with a DOFS we can always choose factorizations so that source, target, and unit functors are strict, and therefore, to use the monadicity.

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Future Work

Understanding the DOFS in a fibrant double category from the DOFS in their vertical bicategory and the OFS in their object category.

Thank You!

Questions?

Institutional: rubasmh@matmor.unam.mx

Personal: rub.maler.22@gmail.com