Higher dagger structures

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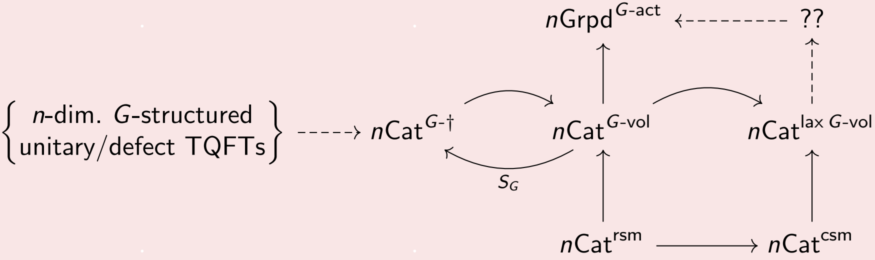
Motivation

The big picture (very conjectural)



Dagger categories

Definition (Dagger category, evil version)

Let $\mathcal{C}$ be a category. A *dagger structure* on $\mathcal{C}$ is a functor $\dagger\colon \mathcal{C} \to \mathcal{C}^{\mathrm{op}}$ satisfying $\dagger^2 = \mathrm{id}$ and $\dagger(c) = c$ for all $c \in \mathcal{C}$. The pair $(\mathcal{C}, \dagger)$ is called a *dagger category*.

Example (Hilbert spaces)

The category of Hilbert spaces Hilb with dagger structure defined by

$$\langle Tx, y\rangle = \langle x, T^\dagger y\rangle$$

Example (Spans)

Let $\mathcal{C}$ be a category with pullbacks. Then $\mathrm{Span}(\mathcal{C})$ is a dagger category with

$$\left(x \xleftarrow{f} z \xrightarrow{g} y \right)^\dagger = y \xleftarrow{g} z \xrightarrow{f} x$$

O(1)-volutive categories

Definition (O(1)-volutive categories)

Let $\mathcal{C}$ be a category. An O(1)-*volution* on $\mathcal{C}$ consists of a functor $d\colon \mathcal{C} \to \mathcal{C}^{\mathrm{op}}$ and a natural isomorphism $\eta\colon d^{\mathrm{op}} \circ d \Rightarrow \mathrm{id}_{\mathcal{C}}$ such that $d(\eta_a) = \eta^{-1}_{d(a)}$ for all $a \in \mathcal{C}$.

Example (Dagger categories)

Any dagger category $(\mathcal{C}, \dagger)$ defines an O(1)-volutive category $(\mathcal{C}, \dagger, \mathrm{id})$.

Lemma

Any rigid symmetric monoidal category admits an O(1)*-volutive structure.*

Sketch of Proof.

Define a functor $\mathcal{C} \to \mathcal{C}^{\mathrm{op}}$ by assigning $a \mapsto a^*$ and $X\colon a \to b$ to

$$X^* = \left(b^* \xrightarrow{\mathrm{coev}_a \otimes \mathrm{id}} a^* \otimes a \otimes b^* \xrightarrow{\mathrm{id} \otimes X \otimes \mathrm{id}} a^* \otimes b \otimes b^* \xrightarrow{\mathrm{id} \otimes \mathrm{ev}_b} a^* \right).$$

The natural isomorphism is induced by the symmetric braiding. □

O(1)-volutive categories and dagger categories

Lemma (Stehouwer-Steinebrunner, 23')

The forgetful functor $\mathrm{Cat}^{\dagger} \to \mathrm{Cat}^{\mathrm{O}(1)\text{-}vol}$ *admits a right adjoint* $S_{\mathrm{O}(1)}$.

Sketch of Proof.

Let $(\mathcal{C}, d, \eta)$ be an O(1)-volutive category. Consider the category

$$\hat{\mathcal{C}} := \begin{cases} (a, \theta_a \colon a \xrightarrow{\cong} d(a)) & \text{s.t.} \quad \theta_a^{-1} d(\theta_a) = \eta_a \\ X \colon a \to b \end{cases}$$

together with the dagger structure assigning $X \colon (a, \theta_a) \to (b, \theta_b)$ to

$$\dagger(X) = \left(b \xrightarrow{\theta_b} d(b) \xrightarrow{d(X)} d(a) \xrightarrow{\theta_a^{-1}} a \right)$$

□

Coherent dagger categories

Remark

$$\mathrm{Cat}^{\mathrm{O}(1)\text{-vol}} = \mathrm{Cat}_{(2,1)}^{\mathrm{O}(1)\text{-hfp}} \qquad \text{where} \qquad \mathrm{O}(1) \curvearrowright \mathrm{Cat}_{(2,1)},\ \mathcal{C} \mapsto \mathcal{C}^{\mathrm{op}}$$

Remark

O(1)-volutive structure (d, η) on $\mathcal{C}$ $\Rightarrow$ O(1)-action $((-)^{-1} \circ d, \eta)$ on $\mathcal{C}^\times \cong (\mathcal{C}^\times)^{\mathrm{op}}$

Definition (Dagger category, non-evil version)

A *coherent dagger category* is an O(1)-involutive category $(\mathcal{C}, d, \eta)$ together with a fully faithful subgroupoid $\mathcal{C}_0 \hookrightarrow (\mathcal{C}^\times)^{\mathrm{O}(1)\text{-hfp}}$ such that the induced functor $\mathcal{C}_0 \hookrightarrow (\mathcal{C}^\times)^{\mathrm{O}(1)} \to \mathcal{C}^\times$ is essentially surjective.

Remark

$$\text{coherent dagger categories} \iff \text{dagger categories}$$

Lax O(1)-volutive categories

Definition (Lax O(1)-volutive categories)

Let $\mathcal{C}$ be a category. A *lax* O(1)*-volution* on $\mathcal{C}$ consists of a functor $d \colon \mathcal{C} \to \mathcal{C}^{\mathrm{op}}$ and a natural transformation $\eta \colon \mathrm{id}_{\mathcal{C}} \Rightarrow d^{\mathrm{op}} \circ d$ such that $d(\eta_a) \circ \eta_{d(a)} = \mathrm{id}_{d(a)}$ for all $a \in \mathcal{C}$.

Example (O(1)-volutive categories)

Any O(1)-volutive category $(\mathcal{C}, d, \eta)$ defines a lax O(1)-volutive category.

Lemma (L, 25')

Any closed symmetric monoidal category admits a lax O(1)*-volutive structure.*

Remark

Any lax O(1)-volutive category $(\mathcal{C}, d, \eta)$ gives rise to an O(1)-volutive category $(\hat{\mathcal{C}}, \hat{d}, \hat{\eta})$ where $\hat{\mathcal{C}} \subseteq \mathcal{C}$ is the largest full subcategory s.t. η_a is invertible for all $a \in \hat{\mathcal{C}}$. This does not immediately extend to a 2-functor.

One-dimensional dagger category theory

Remark

We have a diagram of functors

$$
\begin{array}{ccccc}
\mathrm{Cat}^{\dagger} & \underset{S_{O(1)}}{\rightleftarrows} & \mathrm{Cat}^{O(1)\text{-vol}} & \longrightarrow & \mathrm{Cat}^{\text{lax } O(1)\text{-vol}} \\
 & & \uparrow & & \uparrow \\
 & & \mathrm{Cat}^{\mathrm{rsm}} & \longrightarrow & \mathrm{Cat}^{\mathrm{csm}}
\end{array}
$$

Conjectures on higher dagger categories

Conjecture

We have a diagram of functors

$$\begin{array}{ccccc} n\mathrm{Cat}^{G\text{-}\dagger} & \underset{S_G}{\rightleftarrows} & n\mathrm{Cat}^{G\text{-vol}} & \longrightarrow & n\mathrm{Cat}^{\mathrm{lax}\ G\text{-vol}} \\ & & \uparrow & & \uparrow \\ & & n\mathrm{Cat}^{\mathrm{rsm}} & \longrightarrow & n\mathrm{Cat}^{\mathrm{csm}} \end{array}$$

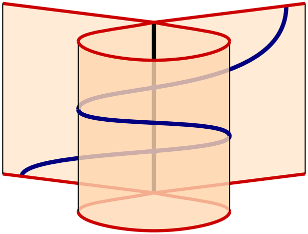
Conjecture/Definition

- $\mathrm{Aut}(\mathrm{Cat}_n^{\mathrm{adj}}) \cong \mathrm{O}(n)$, and $G \to \mathrm{O}(n)$ induces $G \to \mathrm{Aut}(\mathrm{Cat}_n^{\mathrm{adj}})$.
- $n\mathrm{Cat}^{G\text{-vol}} = n\mathrm{Cat}_{(n+1,1)}^{G\text{-hfp}}$, and have functor $n\mathrm{Cat}^{G\text{-vol}} \to n\mathrm{Grpd}^{G\text{-act}}$
- A (coherent) *G-dagger n-category* consists of a G-volutive n-category $\mathcal{C}$ together with a fully faithful subgroupoid $\mathcal{C}_0 \hookrightarrow (\mathcal{C}^\times)^{G\text{-hfp}}$ such that the induced functor $\mathcal{C}_0 \hookrightarrow (\mathcal{C}^\times)^{G\text{-hfp}} \to \mathcal{C}^\times$ is essentially surjective.

Remark

What is known for $n = 2$, $G = \mathrm{SO}(2)$?

- SO(2)-volutive structure: $S\colon \mathrm{id}_{\mathcal{B}} \overset{\cong}{\Longrightarrow} (-)^{RR}$ + coherence
- $\mathcal{B} \in 2\mathrm{Cat}^{\mathrm{rsm}} \Rightarrow$ Serre isomorphism $\mathcal{S}$ defines SO(2)-volutive structure
- $\mathcal{B} \in 2\mathrm{Cat}^{\mathrm{csm}} \Rightarrow$ lax Serre morphism $\mathcal{S}$ defines lax SO(2)-volutive structure



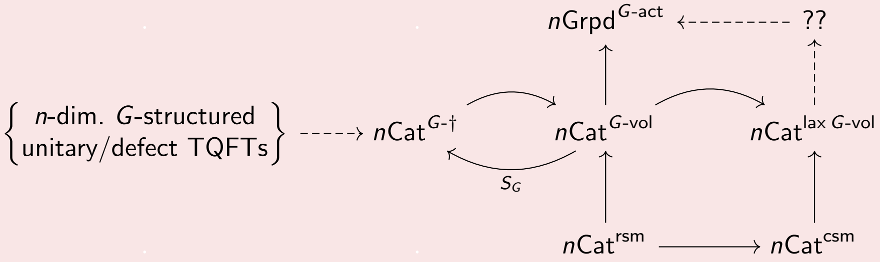
Remark

What about $\mathrm{O}(2) = \mathrm{O}(1) \rtimes \mathrm{SO}(2)$?

- O(2)-volution = O(1)-volution + SO(2)-volution + interaction
- $2\mathrm{Cat}^{\mathrm{rsm}} \Rightarrow$ Dualization + Serre
- Lax case: work in progress

Summary/Outlook

The big picture (very conjectural)



Thank you for your attention!