Marcello Lanfranchi presents:

Local categories

A new approach to partiality

Joint work with JS Lemay

Chapter 0

Partiality

Different flavours of partiality

Motivation

Partial Functions

$f(x) = \frac{1}{x}$

$\ln(x)$

$\mathrm{arctg}(x)$

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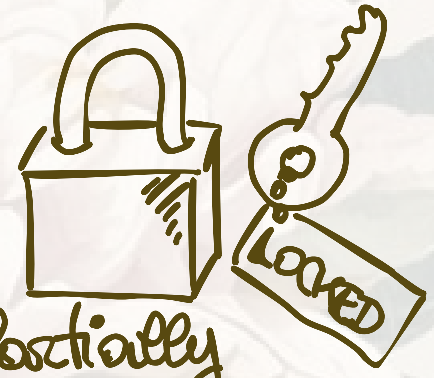
Partial Resources

open subsets of $\mathbb{R}^n$



Partially accessible resources

Different flavours of partiality

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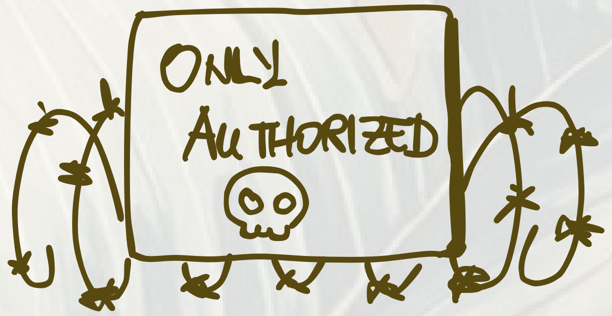
Control on Access of Resources



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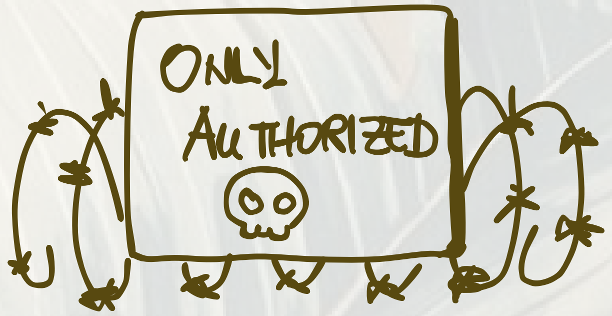

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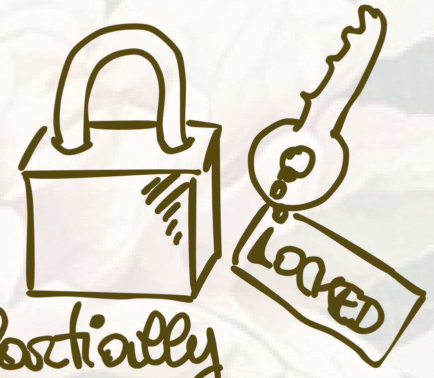
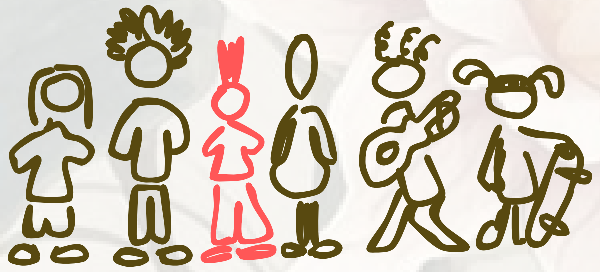
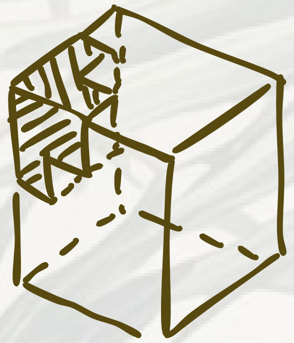


Partially accessible resources

Being part of something Bigger...



$A \subseteq B$



Different flavours of partiality

Motivation

Restriction categories capture well partiality on functions

Different flavours of partiality

Motivation

Restriction categories capture well partiality on functions

What about the other flavours?

Plan for today

The Menu

Restriction categories

Local categories

Partial categories

Inclusion systems

The fundamental theorem of partiality

Chapter 1

Restriction Categories

Partiality on morphisms

Restriction Categories

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Definition

A restriction structure on a category consists of:

$$\frac{f : A \to B}{\bar{f} : A \to A}$$

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[R.2] $B \xleftarrow{f} A \xrightarrow{g} C$, $\bar{f}\bar{g} = \bar{g}\bar{f}$

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[R.4] $A \xrightarrow{f} B \xrightarrow{g} C$, $f\overline{g} = \overline{fg}\, f$

Restriction Categories

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Examples

Category of SETS and PARTIAL functions:

Restriction Categories

Category of SETS and PARTIAL functions:

Objects: sets

Restriction Categories

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2000

Examples

Category of SETS and PARTIAL functions:

Objects: sets

Morphisms: partially defined functions

Restriction Categories

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Lack

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$$\frac{f : A \to B}{\bar{f}(a) := \begin{cases} a & \text{if } f(a) \text{ defined} \\ \text{undefined} & \text{else} \end{cases}}$$

Restriction Categories

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Restriction Categories

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Objects: Unital Rings

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Objects: Unital Rings

Morphisms: $\dfrac{R \to S}{f: S \to R \text{ s.t.}}$

$f(ab) = f(a)f(b)$

Restriction Categories

Examples

Category of SETS and PARTIAL functions:

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Opposite category of RINGS and NON-UNITAL morphisms:

Objects: Unital Rings

Morphisms:

$$\frac{R \to S}{f : S \to R \text{ s.t. } f(ab) = f(a)f(b)}$$

Restriction:

$\bar{f} : R \to R \quad \bar{f}(r) := f(1_S)r$

Restriction Categories

Concepts

In a restriction category:

- $f : A \to B$ total if $\overline{f} = id_A$

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Restriction Categories

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- $\overline{a} = a$ splits if

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[dr, "r"'] & A \\ & E \arrow[u, "s"'] \end{tikzcd} \qquad \begin{tikzcd} & A \arrow[dr, "r"] & \\ E \arrow[ur, "s"] \arrow[rr, equal] & & E \end{tikzcd}$$

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In a restriction category:

- $f: A \to B$ total if $\bar{f} = id_A$
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- partial isomorphism:

  $f: A \rightleftarrows B : f^\circ$, $f f^\circ = \bar{f}$

  $f^\circ f = \overline{f^\circ}$

The splitting completion

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Theorem

Every restriction category embeds in a split restriction category.

$\mathrm{Split}_R(\mathbb{X})$:

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Objects: (A, α) $A \in \mathbb{X}$

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Identities:

$\mathrm{id}_{(A,a)} : (A, a) \xrightarrow{a} (A, a)$

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$$\begin{tikzcd} (A, a) \arrow[rr, "\bar{f}"] \arrow[dr, "\bar{f}"'] & & (A, a) \\ & (A, \bar{f}) \arrow[ur, "\bar{f}"'] & \end{tikzcd}$$

Chapter 2

Local Categories

Partiality on objects

The L construction

Idea

Take total maps of Split_R:

$L[\mathbb{X}] := \mathrm{Tot}[\mathrm{Split}_R[\mathbb{X}]]$

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Morphisms: $f : (A, a) \to (B, b)$

$f : A \to B$

$$\boxed{\begin{array}{c} \bar{f} = a \\ fb = f \end{array}}$$

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$f : A \to B$

$\boxed{\begin{matrix} \bar{f} = a \\ fb = f \end{matrix}}$

The local structure:

$\boxed{(A, a) \rightsquigarrow L(A, a) := (A, id_A)}$

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IMPORTANT STUFF!

$L : (A, a) \mapsto (A, \mathrm{id}_A)$ NOT functorial

$\eta_{(A,a)}$ NOT natural

Local categories

Definition

A local structure on a category consists of:

$$A \in \mathbb{C} \rightsquigarrow LA \in \mathbb{C}$$

$$\eta_A : A \to LA$$

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[L1.] $LLA = LA, \eta_{LA} = id_{LA}$

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[L2.] η_A monic in $\mathbb{C}$

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[L2.] η_A monic in $\mathbb{C}$

[L3.] $f : A \to LB$ the pullback

$$\begin{tikzcd}
C \arrow[r] \arrow[d, "m_f"'] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "\eta_B"] \\
A \arrow[r, "f"'] & LB
\end{tikzcd}$$

exists and $LC = LA$,

$$\begin{tikzcd}
C \arrow[r, "m_f"] \arrow[d, "\eta_C"'] & A \arrow[d, "\eta_A"] \\
LC \arrow[r, equal] & LA
\end{tikzcd}$$

Local categories

Examples

The category of PARTIAL sets:

Objects: (U, A), U, A sets: $U \subseteq A$

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$L(U,A) := (A,A)$

$\dfrac{\eta:(U,A) \longrightarrow (A,A)}{U \hookrightarrow A}$

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The opposite category of Rings with idempotents:

Objects: (R, e_R), R unital ring $e_R \in R$, $e_R^2 = e_R$

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Local categories

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$f(ab) = f(a)f(b)$ $f(1_S) = e_R = f(e_S)$

$L(R, e_R) = (R, 1_R)$

$\eta:(R,e_R)\to(R,1_R): R \longrightarrow R$
$r \longmapsto e_R r$

The R construction

Idea

Start with a local category and construct the following restriction category: $R[\mathbb{C}]$:

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Morphisms: $\dfrac{A \nrightarrow B}{U \text{ s.t. } LU = A}$

$$\begin{tikzcd}
A & & B \\
& U \arrow[ul, "\eta_U"] \arrow[ur, "f"'] &
\end{tikzcd}$$

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Start with a local category and construct the following restriction category: $R[\mathbb{C}]$:

Objects: $A \in \mathbb{C},\ A = LA$

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$$\begin{tikzcd} A & & B \\ & U \arrow[ul, "\eta_U"] \arrow[ur, "f"'] & \end{tikzcd}$$

Identities: $\dfrac{A \nrightarrow A}{}$

$$\begin{tikzcd} A & & A \\ & A \arrow[ul, equal] \arrow[ur, equal] & \end{tikzcd}$$

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Start with a local category and construct the following restriction category: $R[\mathbb{C}]$:

Objects: $A \in \mathbb{C}$, $A = LA$

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\end{tikzcd}$$

Identities: $\dfrac{A \nrightarrow A}{\;}$

$$\begin{tikzcd}
A & & A \\
& A \arrow[ul, equal] \arrow[ur, equal] &
\end{tikzcd}$$

Composition:

$$\begin{tikzcd}
A \arrow[rr, "\shortmid" marking] & & B \arrow[rr, "\shortmid" marking] & & C \\
& U \arrow[ul, "\eta_U"] \arrow[ur, "f"'] & & V \arrow[ul, "\eta_V"] \arrow[ur, "g"'] & \\
& & W \arrow[ul] \arrow[ur] \arrow[uull, bend left, "\eta_W"] & &
\end{tikzcd}$$

The R construction

Idea

Start with a local category and construct the following restriction category: $R[\mathbb{C}]$:

Objects: $A \in \mathbb{C}$, $A = LA$

Morphisms: $\dfrac{A \nrightarrow B}{U \text{ s.t. } LU = A}$

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Identities: $\dfrac{A \nrightarrow A}{}$

$$\begin{tikzcd} A & & A \\ & A \arrow[ul, equal] \arrow[ur, equal] & \end{tikzcd}$$

Composition:

$$\begin{tikzcd} A \arrow[rr, "\shortmid" marking] & & B \arrow[rr, "\shortmid" marking] & & C \\ & U \arrow[ul, "\eta_U"] \arrow[ur, "f"'] & & V \arrow[ul, "\eta_V"] \arrow[ur, "g"'] & \\ & & \hat{W} \arrow[ul] \arrow[ur] \arrow[uull, bend left, "\eta_W"] & & \end{tikzcd}$$

Restriction:

$$\dfrac{\bar{f} : A \nrightarrow A}{}$$

$$\begin{tikzcd} A & & A \\ & U \arrow[ul, "\eta_U"] \arrow[ur, "\eta_U"'] & \end{tikzcd}$$

Local = Restriction

Lanfranchi
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Theorem

There is a 2-equivalence

$$\mathrm{RCAT} \simeq \mathrm{LCAT}$$

between the 2-category of restriction categories and the 2-category of local categories.

Theorem

Local = Restriction

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Restriction categories	Local categories

Local = Restriction

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Theorem

Restriction categories	Local categories
Total maps f s.t. $\overline{f} = id$	Total objects $A = LA$

Theorem

Local = Restriction

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2025

Restriction categories	**Local categories**
Total maps f s.t. $\bar{f} = id$	Total objects $A = LA$
$f \smile g$, $\bar{f}g = \bar{g}f$	$A \smile B$, $LA = LB$

Local = Restriction

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Theorem

Restriction categories

Total maps f s.t. $\bar{f} = id$

$f \smile g$, $\bar{f}g = \bar{g}f$

$f \leq g$, $\bar{f}g = f$

Local categories

Total objects $A = LA$

$A \smile B$, $LA = LB$

$A \leq B$, $A \smile B$, $m : A \to B$ s.t.

$$\begin{tikzcd} A \arrow[r, "m"] \arrow[d, "\eta_A"'] & B \arrow[d, "\eta_B"] \\ LA \arrow[r, equal] & LB \end{tikzcd}$$

Local = Restriction

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Theorem

Restriction categories	**Local categories**
$e = \bar{e}$ splits: $e = rs$, $sr = id$	A splits: $A \cong E$, E total

Local = Restriction

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Theorem

Restriction categories

$e = \bar{e}$ splits: $e = rs$, $sr = \mathrm{id}$

$f : A \rightleftarrows B : f^\circ$ partial iso:

$$ff^\circ = \bar{f}, \quad f^\circ f = \overline{f^\circ}$$

Local categories

A splits: $A \cong E$, E total

$f : A \to B$ local iso: $\exists\, C \leq B$ s.t.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "\cong"'] & B \\ & C \arrow[u, "m"'] \end{tikzcd}$$

Local = Restriction

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Theorem

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$e = \bar{e}$ splits: $e = rs$, $sr = id$

$f : A \rightleftarrows B : f^{\circ}$ partial iso:

$$ff^{\circ} = \bar{f}, \; f^{\circ}f = \overline{f^{\circ}}$$

Restriction Terminal Object

Local categories

A splits: $A \cong E$, E total

$f : A \to B$ local iso: $\exists C \leq B$ s.t.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "\cong"'] & B \\ & C \arrow[u, "m"'] \end{tikzcd}$$

Terminal Object

Local = Restriction

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Theorem

Restriction categories	**Local categories**
$e=\bar{e}$ splits: $e = rs$, $sr = \mathrm{id}$	A splits: $A \cong E$, E total
$f : A \rightleftarrows B : f^\circ$ partial iso: $ff^\circ = \bar{f}$, $f^\circ f = \overline{f^\circ}$	$f : A \to B$ local iso: $\exists C \leq B$ s.t.
	$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "\cong"'] & B \\ & C \arrow[u, "m"'] \end{tikzcd}$$
Restriction Terminal Object	Terminal Object
Restriction Products	Products

Local = Restriction

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Theorem

Restriction categories	Local categories
$e=\bar{e}$ splits: $e=rs$, $sr=id$	A splits: $A \cong E$, E total
$f: A \rightleftarrows B : f^\circ$ partial iso: $ff^\circ = \bar{f}$, $f^\circ f = \overline{f^\circ}$	$f: A \to B$ local iso: $\exists C \leq B$ s.t. $\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "\cong"'] & B \\ & C \arrow[u, "m"'] \end{tikzcd}$
Restriction Terminal Object	Terminal Object
Restriction Products	Products
Restriction Limits	???

Chapter 3

Partial Categories

Operational partiality

Partial categories

Idea

In a local category we can define two operations:

RESTRICTION:

$$\frac{f : A \to B,\ U \leq A}{f{\restriction}U : U \xrightarrow{m} A \xrightarrow{f} B}$$

Idea

Partial categories

In a local category we can define two operations:

RESTRICTION:

$$\frac{f : A \to B,\ U \leq A}{f\restriction U : U \xrightarrow{m} A \xrightarrow{f} B}$$

CONTRACTION:

$$\frac{f : A \to LB}{A \bullet_f B \leq A,\ f \bullet B : A \bullet_f B \to B :}$$

$$\begin{tikzcd}
A \bullet_f B \arrow[r, "f \bullet B"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "\eta_B"] \\
A \arrow[r, "f"'] & LB
\end{tikzcd}$$

Partial categories

Definition

A partial structure on a category consists of a partial order on the objects, and two operators:

Partial categories

Definition

A partial structure on a category consists of a partial order on the objects, and two operators:

RESTRICTION:

$$\downharpoonright \; \frac{f : A \to B, \; U \leq A}{f\downharpoonright U : U \to B}$$

Partial categories

Definition

A partial structure on a category consists of a partial order on the objects, and two operators:

RESTRICTION:

$$\frac{f : A \to B,\ U \leq A}{f \rfloor U : U \to B}$$

CONTRACTION:

$$\frac{f : A \to B,\ V \leq B}{A \,!_f\, V \leq A,\ f \,!\, V : A \,!_f\, V \to V}$$

Partial categories

Definition

A partial structure on a category consists of a partial order on the objects, and two operators:

RESTRICTION:

$$\text{⫰} \quad \frac{f : A \to B,\ U \leq A}{f \text{⫰} U : U \to B}$$

⫰ Restricts the access of f at $U \leq A$

CONTRACTION:

$$\text{⫯} \quad \frac{f : A \to B,\ V \leq B}{A \text{⫯}_f V \leq A,\ f \text{⫯} V : A \text{⫯}_f V \to V}$$

⫯ contracts f s.t. its image lands in $V \leq B$

Partial categories

Definition

A partial structure on a category consists of a partial order on the objects, and two operators:

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CONTRACTION:

⫯ $\dfrac{f : A \to B,\ V \leq B}{A ⫯_f V \leq A,\ f ⫯ V : A ⫯_f V \to V}$

⫯ contracts f s.t. its image lands in $V \leq B$

+Axioms

Examples

Partial categories

Every local category:

$\leqslant$: $U \leqslant A \Leftrightarrow LU = LA$ and

$\exists m : U \to A$ s.t. $m\eta_A = \eta_U$

Partial categories

Examples

Every local category:

$\leq$: $U \leq A \Leftrightarrow LU = LA$ and
$\exists m: U \to A$ s.t. $m\eta_A = \eta_U$

δ: $\dfrac{f: A \to B,\ U \leq A}{U \xrightarrow{m} A \xrightarrow{f} B}$

Partial categories

Examples

Every local category:

$\leq$: $U \leq A \iff LU = LA$ and $\exists m : U \to A$ s.t. $m\eta_A = \eta_U$

δ : $\dfrac{f : A \to B,\ U \leq A}{U \xrightarrow{m} A \xrightarrow{f} B}$

ρ : $\dfrac{f : A \to B,\ V \leq B}{}$

$$\begin{tikzcd} A\rho_f V \arrow[r, "f\rho V"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & V \arrow[d, "m"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

Partial categories

Examples

Every local category:	**The category of SETS:**

Every local category:

$\leq$: $U \leq A \iff LU = LA$ and $\exists m: U \to A$ s.t. $m\eta_A = \eta_U$

$\downarrow$: $\dfrac{f: A \to B,\ U \leq A}{U \xrightarrow{m} A \xrightarrow{f} B}$

$\uparrow$: $\dfrac{f: A \to B,\ V \leq B}{}$

$$\begin{tikzcd}
A \uparrow_f V \arrow[r, "f \uparrow V"] \arrow[d, "f"'] \arrow[dr, phantom, "\lrcorner", very near start] & V \arrow[d, "m"] \\
A \arrow[r, "f"'] & B
\end{tikzcd}$$

The category of SETS:

$\leq$: $U \leq A \iff U \subseteq A$

Examples

Partial categories

Every local category:

$\leq$: $U \leq A \iff LU = LA$ and $\exists m : U \to A$ s.t. $m\eta_A = \eta_U$

$⫰$: $\dfrac{f : A \to B,\ U \leq A}{U \xrightarrow{m} A \xrightarrow{f} B}$

$⫯$: $\dfrac{f : A \to B,\ V \leq B}{\begin{tikzcd} A ⫯_f V \arrow[r, "f ⫯ V"] \arrow[d, "f"'] \arrow[dr, phantom, "\lrcorner", very near start] & V \arrow[d, "m"] \\ A \arrow[r, "f"'] & B \end{tikzcd}}$

The category of SETS:

$\leq$: $U \leq A \iff U \subseteq A$

$⫰$: $\dfrac{f : A \to B,\ U \subseteq A}{f|_U : U \to B}$

Examples

Partial categories

Every local category:

$\leq$: $U \leq A \iff LU = LA$ and $\exists m : U \to A$ s.t. $m\eta_A = \eta_U$

$⫰$: $\dfrac{f : A \to B,\ U \leq A}{U \xrightarrow{m} A \xrightarrow{f} B}$

$⫯$: $\dfrac{f : A \to B,\ V \leq B}{}$

$$\begin{tikzcd} A ⫯_f V \arrow[r, "f⫯V"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & V \arrow[d, "m"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

The category of SETS:

$\leq$: $U \leq A \iff U \subseteq A$

$⫰$: $\dfrac{f : A \to B,\ U \subseteq A}{f|_U : U \to B}$

$⫯$: $\dfrac{f : A \to B,\ V \leq B}{A ⫯_f V := \{a \in A \text{ s.t. } f(a) \in V\}}$

$f ⫯ V : A ⫯_f V \ni a \mapsto f(a) \in V$

Chapter 4

Inclusion Systems

Partiality via inclusions

Inclusion systems

Idea

In a partial category we can define a family of monics:

$$U \leq A \rightsquigarrow \mathrm{id}_A \downharpoonleft U : U \to A$$

Inclusion systems

Idea

In a partial category we can define a family of monics:

$$U \leq A \rightsquigarrow \mathrm{id}_A \circ U : U \to A$$

Stable under pullbacks:

$$\begin{tikzcd}
A \bullet_f V \arrow[r, "f \bullet V"] \arrow[d, "\mathrm{id}_A \circ (A \bullet_f V)"'] \arrow[dr, phantom, very near start, "\lrcorner"] & V \arrow[d, "\mathrm{id}_B \circ V"] \\
A \arrow[r, "f"'] & B
\end{tikzcd}$$

Inclusion systems

Definition

An inclusion system on a category consists of a family of monics s.t.:

Inclusion systems

Definition

An inclusion system on a category consists of a family of monics s.t.:

[IS1] Identities belong to $\mathcal{I}$

Inclusion systems

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An inclusion system on a category consists of a family of monics s.t.:

[IS1] Identities belong to $\mathcal{F}$

[IS2] $\mathcal{F}$ stable under composition

Inclusion systems

Definition

An inclusion system on a category consists of a family of monics s.t.:

[IS1] Identities belong to $\mathcal{F}$

[IS2] $\mathcal{F}$ stable under composition

[IS3] $\mathcal{F}$ stable under pullbacks

Inclusion systems

Definition

An inclusion system on a category consists of a family of monics s.t.:

[IS1] Identities belong to $\mathcal{I}$

[IS2] $\mathcal{I}$ stable under composition

[IS3] $\mathcal{I}$ stable under pullbacks

[IS4] If $A \overset{m}{\underset{m'}{\rightrightarrows}} B \in \mathcal{I} \Rightarrow m = m'$

Inclusion systems

Theorem

Every partial category admits an inclusion system:

$\mathcal{F} := \{ \mathrm{id}_A \triangleright U,\ U \leq A \}$

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Theorem

Every partial category admits an inclusion system:

$\mathcal{F} := \{ id_A \circ U, \ U \leq A \}$

Every inclusion system defines a partial structure where:

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Theorem

Every partial category admits an inclusion system:

$\mathcal{F} := \{ id_A \circ U, U \leq A \}$

Every inclusion system defines a partial structure where:

$U \leq A \iff \exists m : U \to A \in \mathcal{F}$

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Theorem

Every partial category admits an inclusion system:

$$\mathcal{F} := \{ \mathrm{id}_A \restriction U,\ U \leq A \}$$

Every inclusion system defines a partial structure where:

$$U \leq A \iff \exists\, m : U \to A \in \mathcal{F}$$

$$\restriction : \frac{f : A \to B,\ U \leq A}{U \xrightarrow{m} A \xrightarrow{f} B}$$

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Theorem

Every partial category admits an inclusion system:

$$\mathcal{F} := \{ id_A \restriction U,\ U \leq A \}$$

Every inclusion system defines a partial structure where:

$$U \leq A \iff \exists\, m : U \to A \in \mathcal{F}$$

$$\restriction :\ \frac{f : A \to B,\ U \leq A}{U \xrightarrow{m} A \xrightarrow{f} B}$$

$$\bullet :\ \frac{f : A \to B,\ V \leq B}{\begin{tikzcd} A \bullet_f V \arrow[r, "f \bullet V"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & V \arrow[d, "m"] \\ A \arrow[r, "f"'] & B \end{tikzcd}}$$

Chapter 5

The Fundamental Theorem of Partiality

The 2-equivalences

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Theorem

Restriction Categories *Partiality on morphisms*	$\simeq$	*Local Categories* *Partiality on objects*
Partial Categories *Operational partiality*	$\simeq$	*Inclusion Systems* *Partiality via inclusions*

Local to inclusion

Lemma

Every local category admits an inclusion system:

$$\mathcal{I} := \{ m : U \to A,\ LU = LA,\ m\eta_A = \eta_U \}$$

Boundedness

Definition

A partial category is bounded when:

$\forall A \ \exists! LA$ s.t.

① $A \leq LA$

② LA is maximal:

If $LA \leq B \Rightarrow B = LA$

Boundedness

Definition

A partial category is bounded when:

$\forall A\ \exists!\, LA$ s.t.

① $A \leq LA$

② LA is maximal:

If $LA \leq B \Rightarrow B = LA$

An inclusion system is bounded when:

$\forall A\ \exists\, m : A \to LA \in \mathcal{F}$ s.t.

if $n : A \to B \in \mathcal{F}$ factors through m:

$$\begin{tikzcd} A \arrow[r, "n"] \arrow[d, "m"'] & B \\ LA \arrow[ur, "x"'] & \end{tikzcd}$$

and $x \in \mathcal{F}$

$\Rightarrow B = LA.$

The 2-equivalences

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Theorem

Restriction Categories *Partiality on morphisms*	$\simeq$	**Local Categories** *Partiality on objects*
$\wr\wr$		$\wr\wr$
Bounded Partial Categories *Operational partiality*	$\simeq$	**Bounded Inclusion Systems** *Partiality via inclusions*

The End.

Thanks