COMONADS ON SET

with Aaron Fairbanks

Algebra

Space

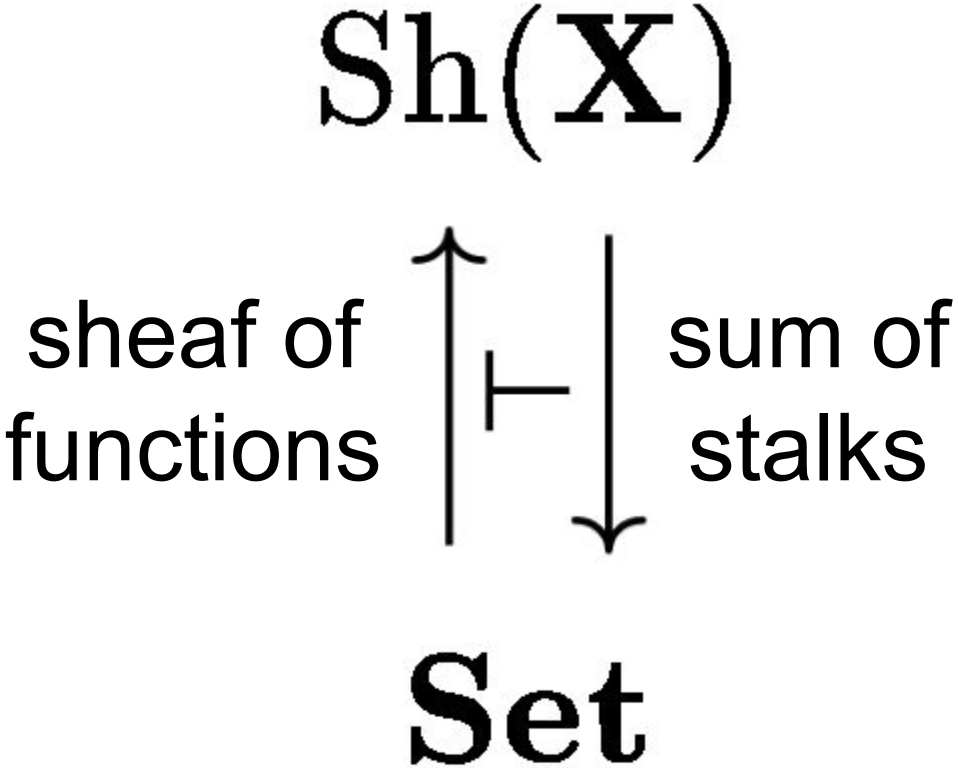
Algebra

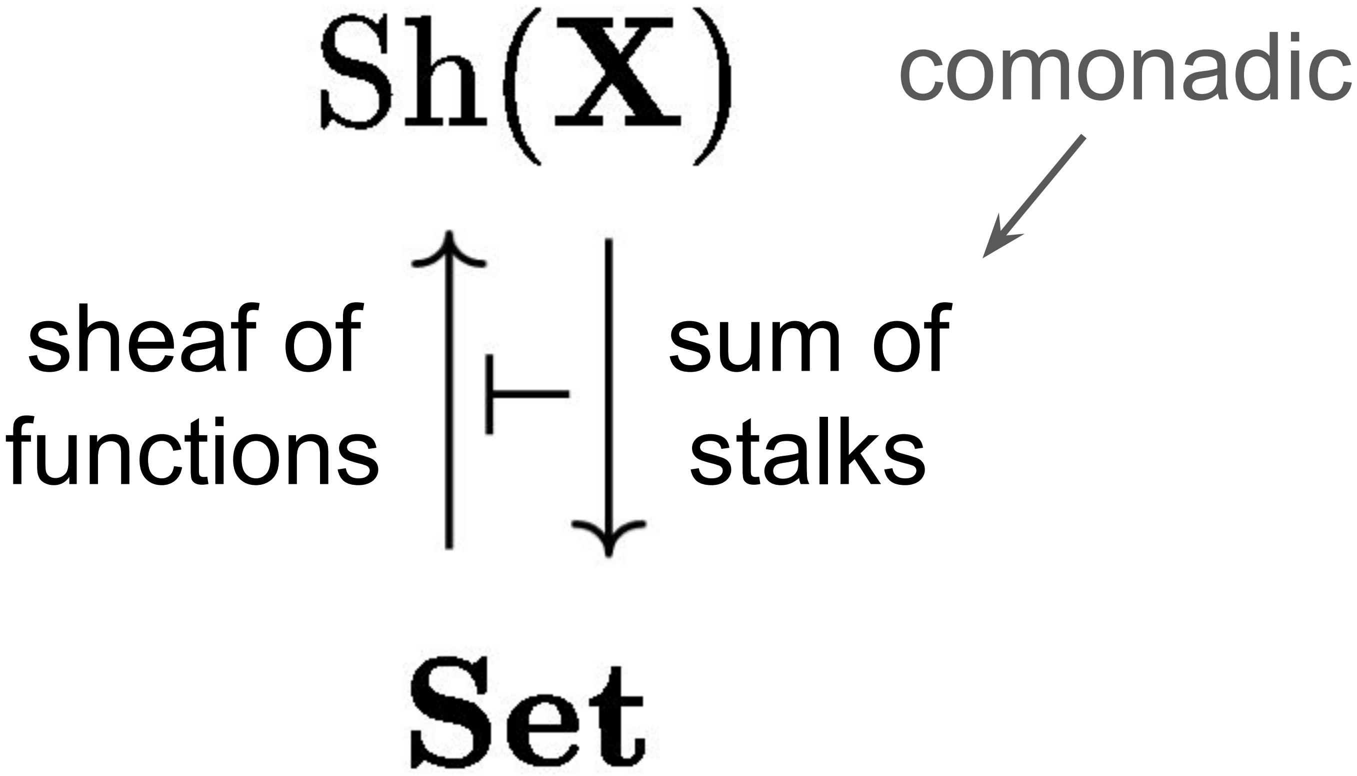
Space

Monads on $\mathbf{Set}$

Algebra	Space
Monads on $\mathbf{Set}$	Comonads on $\mathbf{Set}$

Comonads on $\mathbf{Set}$ strictly generalize topological spaces.


Sh(X)
sheaf of functions
⊣
sum of stalks
Set



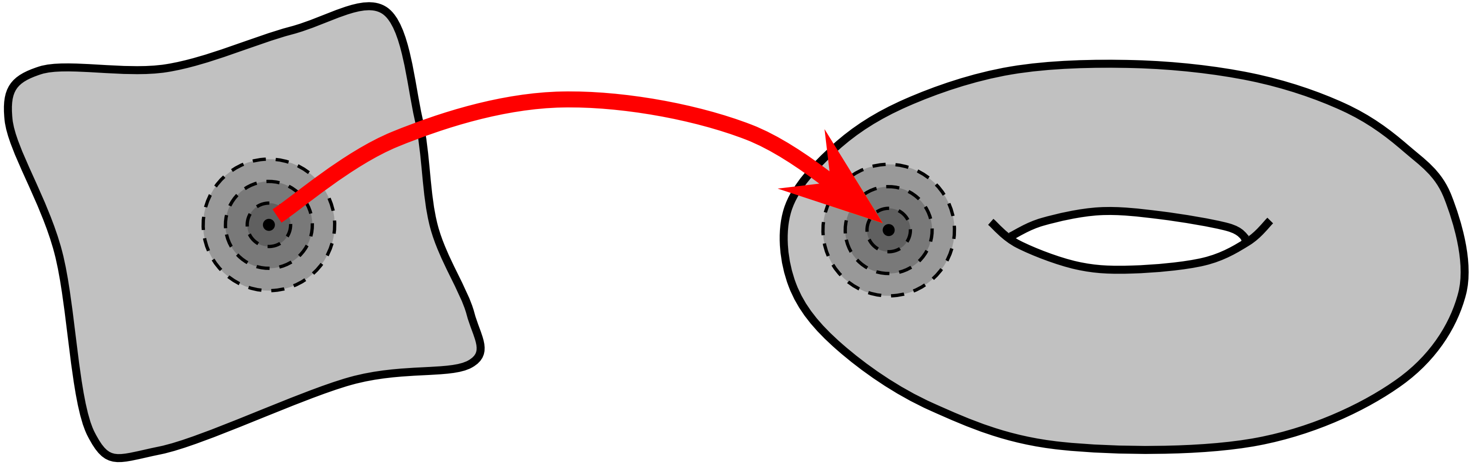
Sh(X)
comonadic
sheaf of functions
⊢
sum of stalks
Set

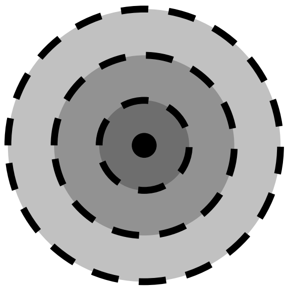
What is a general comonad on $\mathbf{Set}$?

What is a generalized space?

A continuous map of spaces

"sends infinitesimal neighborhoods to infinitesimal neighborhoods"





"infinitesimal neighborhood"

=

formal limit of sets

formal limit of sets

$=^*$

object of $(\mathbf{Set}^{\mathbf{Set}})^{\mathrm{op}}$

*Technically, $(\mathbf{Set}^{\mathbf{Set}})^{\mathrm{op}}$ additionally includes some *large* formal limits.

Definition

The **halo** of a point x in a topological space $\mathbf{X}$

$$\mathrm{Halo}_x \in (\mathbf{Set}^{\mathbf{Set}})^{\mathrm{op}}$$

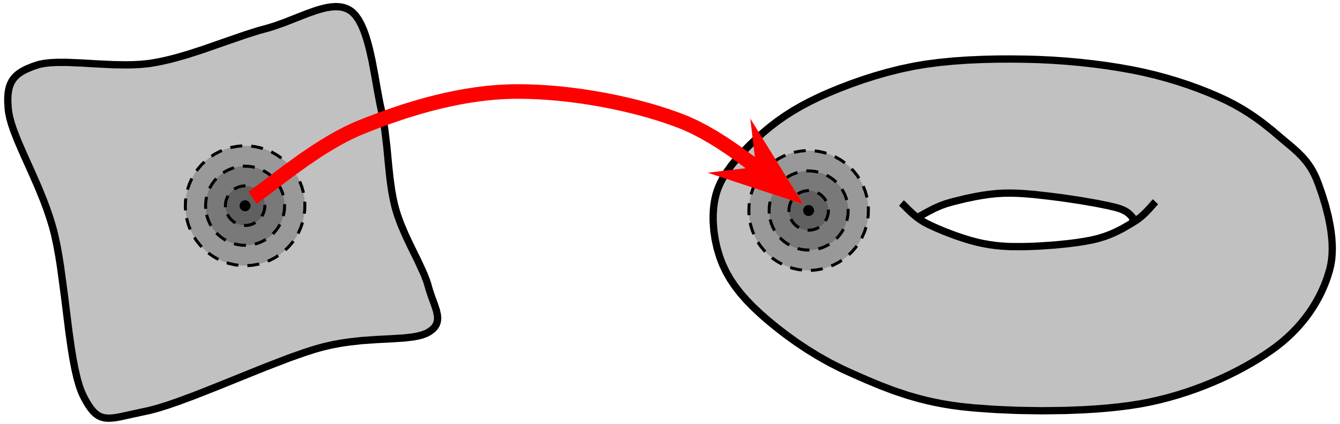
is the limit of the diagram

$$\mathcal{N}_x \xrightarrow{\text{points}} \mathbf{Set} \xrightarrow[\text{よ}]{\text{Yoneda}} (\mathbf{Set}^{\mathbf{Set}})^{\mathrm{op}}$$

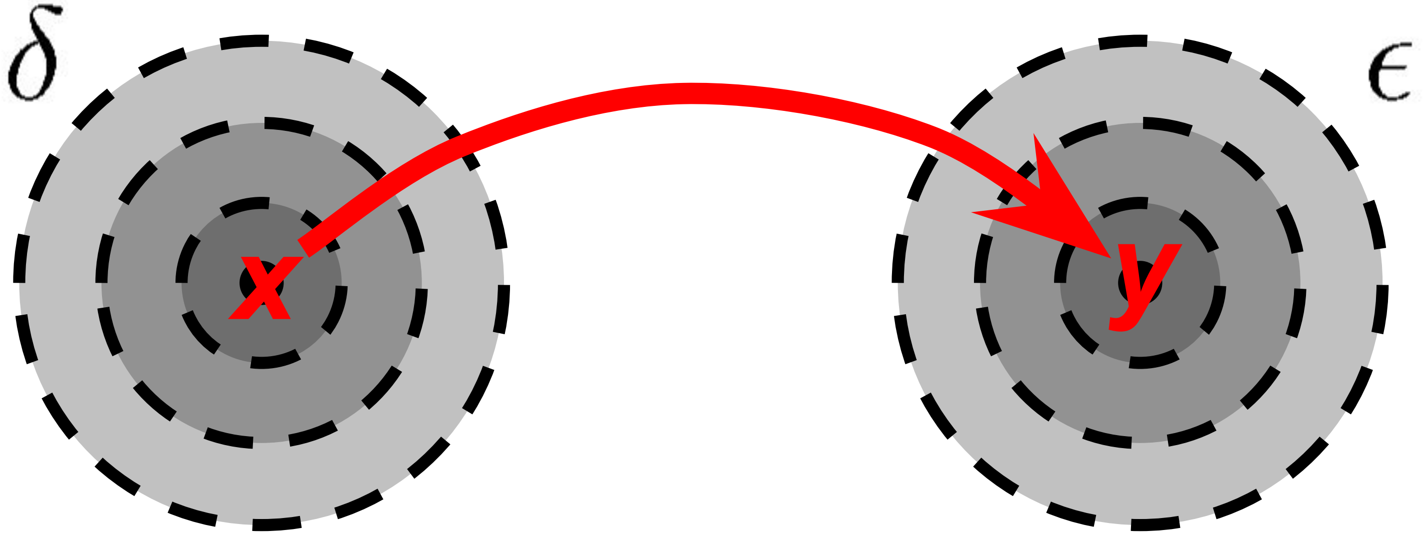
where $\mathcal{N}_x$ is the poset of neighborhoods of x.

It is the *formal limit* of neighborhoods of x.

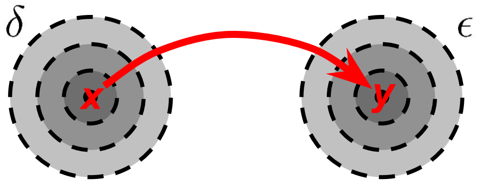
A function $f : \mathbf{X} \to \mathbf{Y}$ is **continuous** when it induces a map of formal limits of sets $\mathrm{Halo}_x \to \mathrm{Halo}_{f(x)}$ for each x in $\mathbf{X}$.



Continuity



$$\forall \epsilon \quad \exists \delta \quad \delta \to \epsilon$$



$$(\mathbf{Set}^{\mathbf{Set}})^{\mathrm{op}}(\mathrm{Halo}_x, \mathrm{Halo}_y)$$

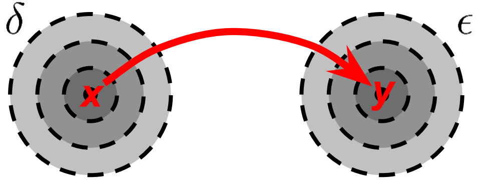
$$=$$

$$(\mathbf{Set}^{\mathbf{Set}})^{\mathrm{op}}(\lim_{\delta \in \mathcal{N}_x} \text{よ}\,\delta, \lim_{\epsilon \in \mathcal{N}_y} \text{よ}\,\epsilon)$$

$$\cong$$

$$\lim_{\epsilon \in \mathcal{N}_y} \operatorname{colim}_{\delta \in \mathcal{N}_x} \mathbf{Set}(\delta, \epsilon)$$

"$\forall \epsilon \quad \exists \delta \quad \delta \to \epsilon$"



$$(\mathbf{Set}^{\mathbf{Set}})^{\mathrm{op}}(\mathrm{Halo}_x, \mathrm{Halo}_y)$$

$$=$$

$$(\mathbf{Set}^{\mathbf{Set}})^{\mathrm{op}}(\lim_{\delta\in\mathcal{N}_x} よ\delta, \lim_{\epsilon\in\mathcal{N}_y} よ\epsilon)$$

$$\cong$$

$$\lim_{\epsilon\in\mathcal{N}_y} \operatorname{colim}_{\delta\in\mathcal{N}_x} \mathbf{Set}(\delta, \epsilon)$$

"$\forall\epsilon \quad \exists\delta \quad \delta\to\epsilon$" ✓

Spaces may be defined in terms of their halos.

A “*space*” consists of…

A “*space*” consists of…

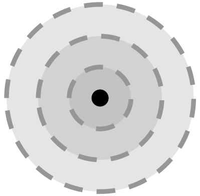
1) a formal limit of sets

$$\mathcal{X} = \prod_{x \in X} \mathrm{Halo}_x \in (\mathbf{Set}^{\mathbf{Set}})^{\mathrm{op}}$$

or equivalently a formal product of formal connected limits (“halos”)

A “*space*” consists of…

2) a distinguished point (per “halo”)



$$e : * \to \mathcal{X} \quad \left(= \prod_{x \in X} \mathrm{Halo}_x\right)$$

defined as a map in $(\mathbf{Set}^{\mathbf{Set}})^{\mathrm{op}}$ where

$*$ is represented by the set with one point

A “*space*” consists of…

3) *“for every neighborhood* U

(set in the diagram that defines the formal limit $\mathcal{X}$ *)*

there is some neighborhood V *and*

for every v *in* V

there is some neighborhood W_v

mapping back to U *”*

A "*space*" consists of…

3) a map

$$m\colon \mathcal{X} \circ \mathcal{X} \to \mathcal{X}$$

in $(\mathbf{Set}^{\mathbf{Set}})^{\mathrm{op}}$

where $\circ$ is composition of functors $\mathbf{Set} \to \mathbf{Set}$

A “*space*” consists of…

1) $\mathcal{X} \in (\mathbf{Set}^{\mathbf{Set}})^{\mathrm{op}}$

2) $e\colon * \to \mathcal{X}$

3) $m\colon \mathcal{X} \circ \mathcal{X} \to \mathcal{X}$

+ associativity and unit laws

A “*space*” consists of…

a monoid in formal limits of sets

“space”

=

monoid in formal limits of sets

“space”

=

monoid in formal limits of sets

=

monoid in $(\mathbf{Set}^{\mathbf{Set}})^{\mathrm{op}}$

“space”

=

monoid in formal limits of sets

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monoid in $(\mathbf{Set}^{\mathbf{Set}})^{\mathrm{op}}$

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comonoid in $\mathbf{Set}^{\mathbf{Set}}$

“space”

=

monoid in formal limits of sets

=

monoid in $(\mathbf{Set}^{\mathbf{Set}})^{\mathrm{op}}$

=

comonoid in $\mathbf{Set}^{\mathbf{Set}}$

=

comonad on $\mathbf{Set}$

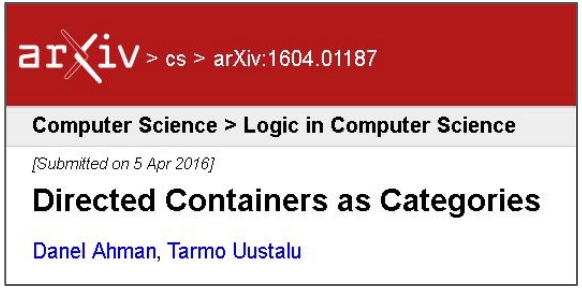
More examples

besides topological spaces

monoids

=

representable
comonads on $\mathbf{Set}$


arXiv > cs > arXiv:1604.01187

Computer Science > Logic in Computer Science

[Submitted on 5 Apr 2016]

Directed Containers as Categories

Danel Ahman, Tarmo Uustalu

categories

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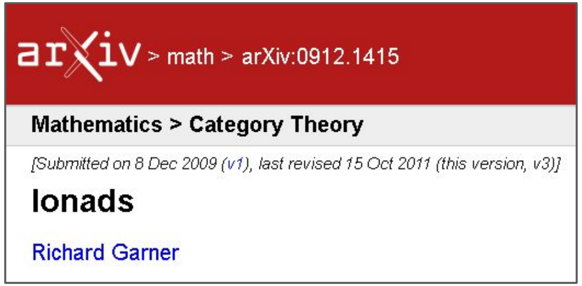
polynomial comonads on $\mathbf{Set}$


arXiv > math > arXiv:0912.1415

Mathematics > Category Theory

[Submitted on 8 Dec 2009 (v1), last revised 15 Oct 2011 (this version, v3)]

Ionads

Richard Garner

ionads

(a.k.a. toposes equipped with enough points)

=

pullback-preserving

comonads on $\mathbf{Set}$

Topological spaces	Comonads on $\mathbf{Set}$	Categories
Basis	$B\colon \mathbf{C}^{\mathrm{op}} \to \mathbf{Set}$	$\sum_{x\in\mathbf{C}} \mathrm{Hom}(-, x)$
Space $\mathbf{X}$	Density comonad $\mathcal{C}$ of B	Category $\mathbf{C}$
Point x of $\mathbf{X}$	$x \in \mathcal{C}(1)\ \ (= \operatorname{colim} B)$	Object x of $\mathbf{C}$
Basic open neighborhoods of point x	Connected component $\mathrm{El}_x(B)$ of $\mathrm{El}(B)$	Arrows into object x
Halo of x	Formal limit of $\mathrm{El}_x \to \mathbf{C}^{\mathrm{op}} \xrightarrow{B} \mathbf{Set}$	Arrows out of object x
Sheaf	Coalgebra	$\mathbf{C}$-set

Let $\mathcal{B}$ be a basis of a topological space.

The corresponding comonad on $\mathbf{Set}$ is the *density comonad* of

$$\mathcal{B} \xrightarrow{\text{points}} \mathbf{Set}$$

More examples

besides topological spaces

Let $\mathbf{Surj}$ be the category of sets and surjections.

The density comonad of

$$\mathbf{Surj} \hookrightarrow \mathbf{Set}$$

has *partitioned sets* as coalgebras.

Let $\mathbf{Int}$ be the category of open intervals of $\mathbb{R}$ and distance + orientation preserving maps.

The density comonad (ionad) of

$$\mathbf{Int} \hookrightarrow \mathbf{Set}$$

is a "continuous analogue of the monoid $\mathbb{Z}$".

What are comonad morphisms?

Continuous maps?

Continuous maps?

Functors?

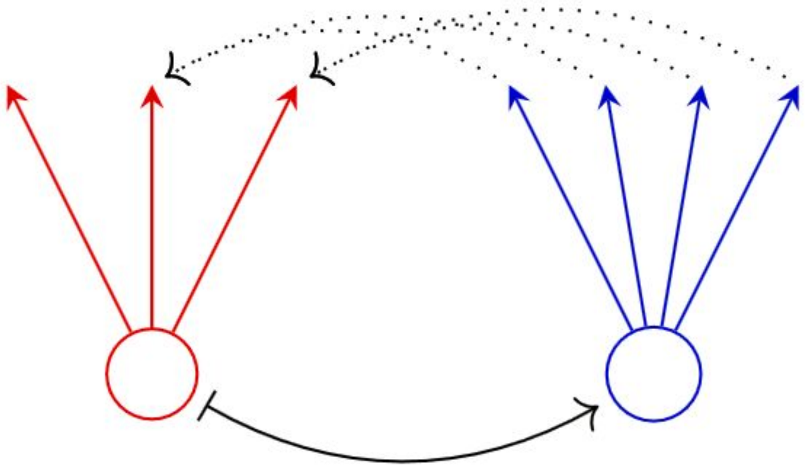
Continuous maps?

Functors?

NO

Comonad morphisms
map points forward,
but map halos *backward*.

Retrofunctors



A retrofunctor from a category $\mathbf{C}$
to the monoid $\mathbb{Z}$ is a “vector field” on $\mathbf{C}$:

for each c in $\mathbf{C}$ and n in $\mathbb{Z}$,
an arrow $c \to d$
with suitable compatibilities.

A comonad morphism from a space $\mathbf{X}$ to the *continuous analogue of* $\mathbb{Z}$ (mentioned earlier) is a “local flow” on $\mathbf{X}$:

for each x in $\mathbf{X}$, a map from an open interval about 0 in $\mathbb{R}$ to $\mathbf{X}$ with suitable compatibilities.

There is a second kind of map between comonads on $\mathbf{Set}$.

There is a second kind of map between comonads on $\mathbf{Set}$.

Functors?

Continuous maps?

There is a second kind of map between comonads on $\mathbf{Set}$.

Functors?

Continuous maps?

YES

“Continuous map”

forward on points,
forward on halos

“Continuous map”

arXiv > math > arXiv:0912.1415

Mathematics > Category Theory

[Submitted on 8 Dec 2009 (v1), last revised 15 Oct 2011 (this version, v3)]

Ionads

Richard Garner

Comonad morphisms
and continuous maps form a
double category
of comonads on $\mathbf{Set}$.

Comonad morphisms
and continuous maps form a
double category
of comonads on $\mathbf{Set}$.

Includes a (non-flat) double category of categories, functors, and retrofunctors.

Stay tuned for upcoming paper

COMONADS ON SET

by Kevin Carlson, Aaron Fairbanks, and David Spivak

Thank you