Categorical and homological tools in computational problems

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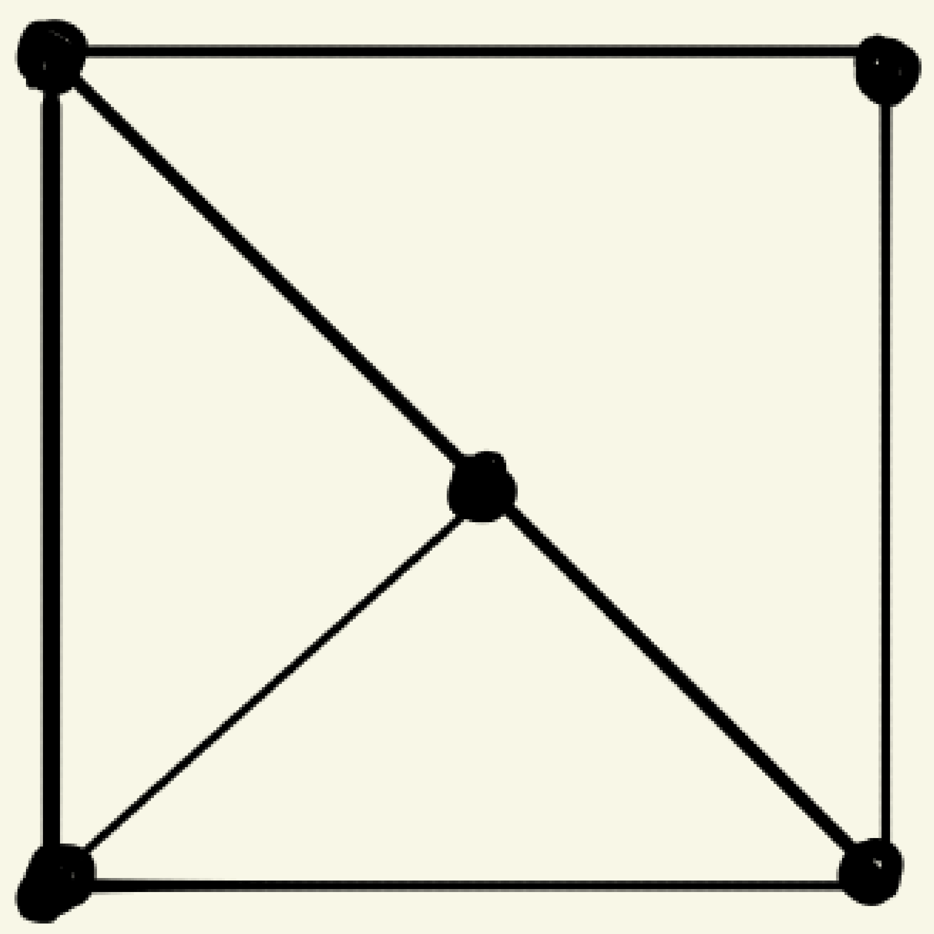
Support: CAPES

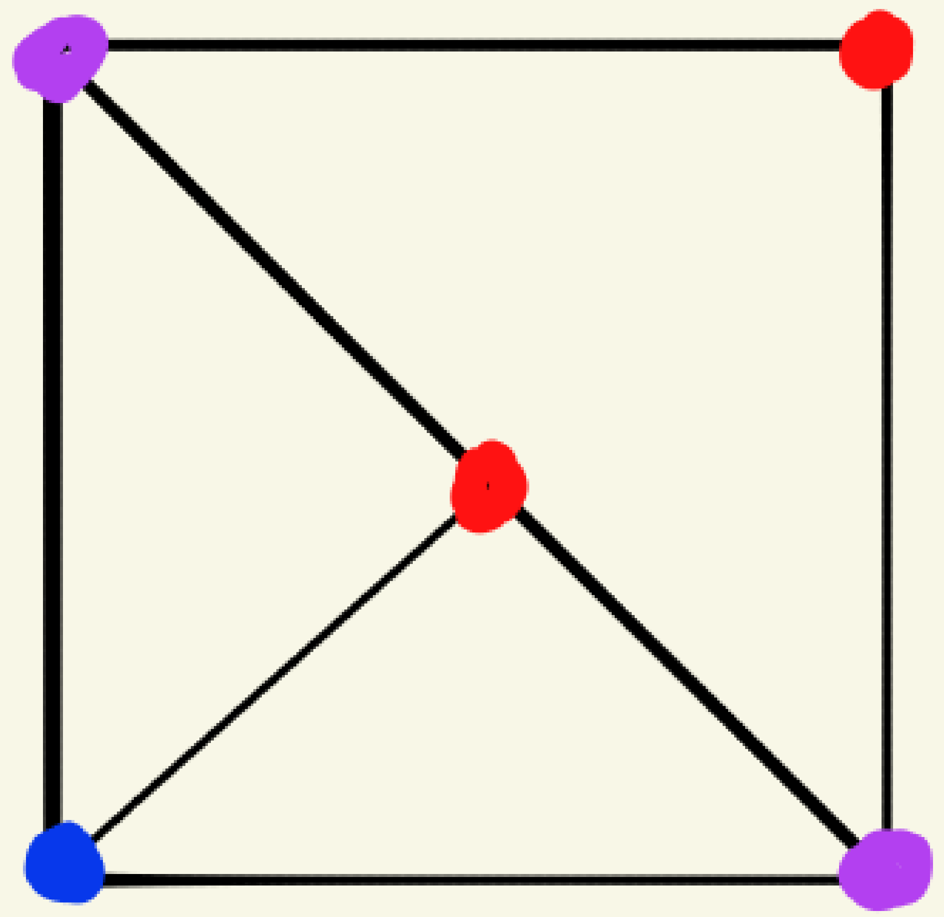
Overview

- Computational problems : presheaves (2 examples)
- Sheaves : divide and conquer
  - ↳ fast FPT (fixed parameter tractable) algorithm [1]
- Measuring the failure of being a sheaf : cohomology
  - ↳ obstruction to algorithm compositionality
- Obstructions to the existence of a solution

Computational problems as presheaves

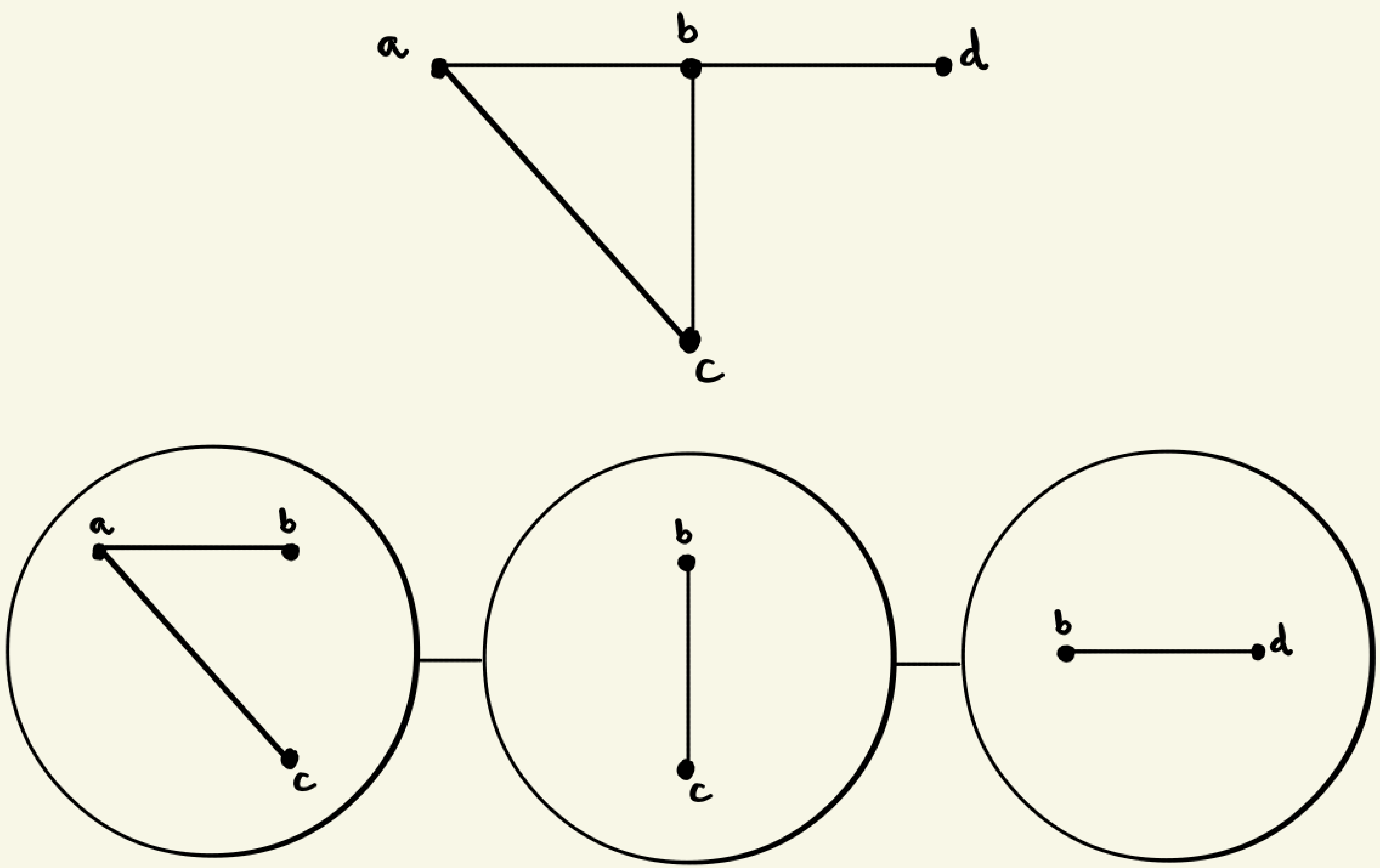
Coloring problem



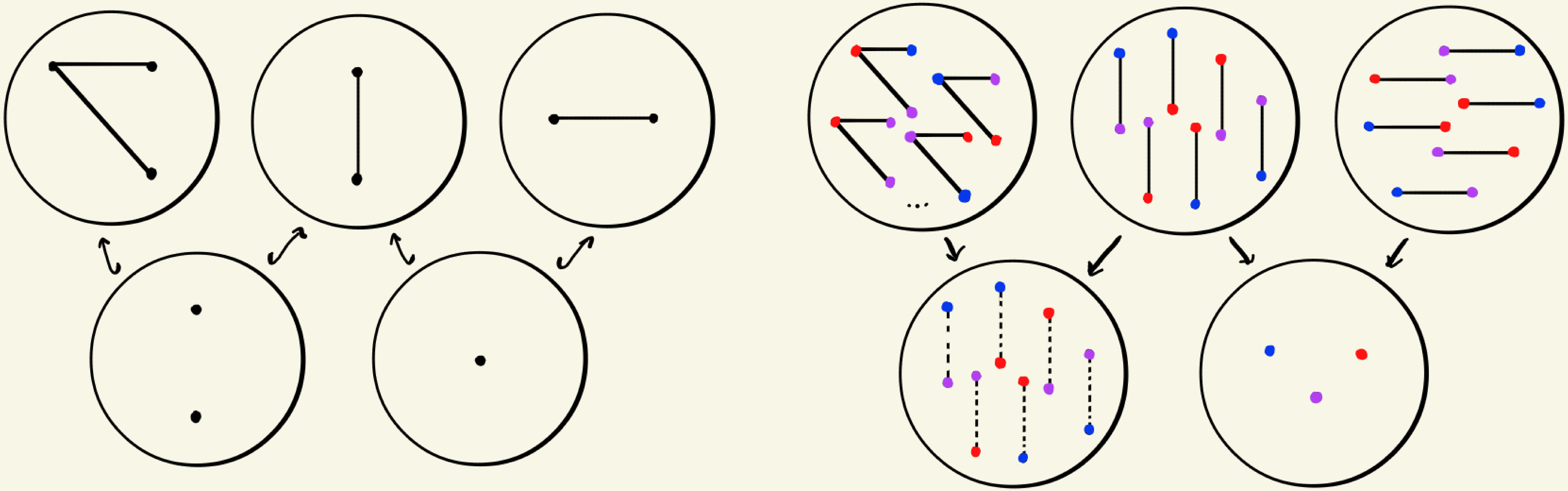


Definition: a k-coloring of a graph G is an assignment of k colors to the vertices of G such that adjacent vertices have different colors.

Divide and conquer



How to relate local solutions?



Coloring as a presheaf

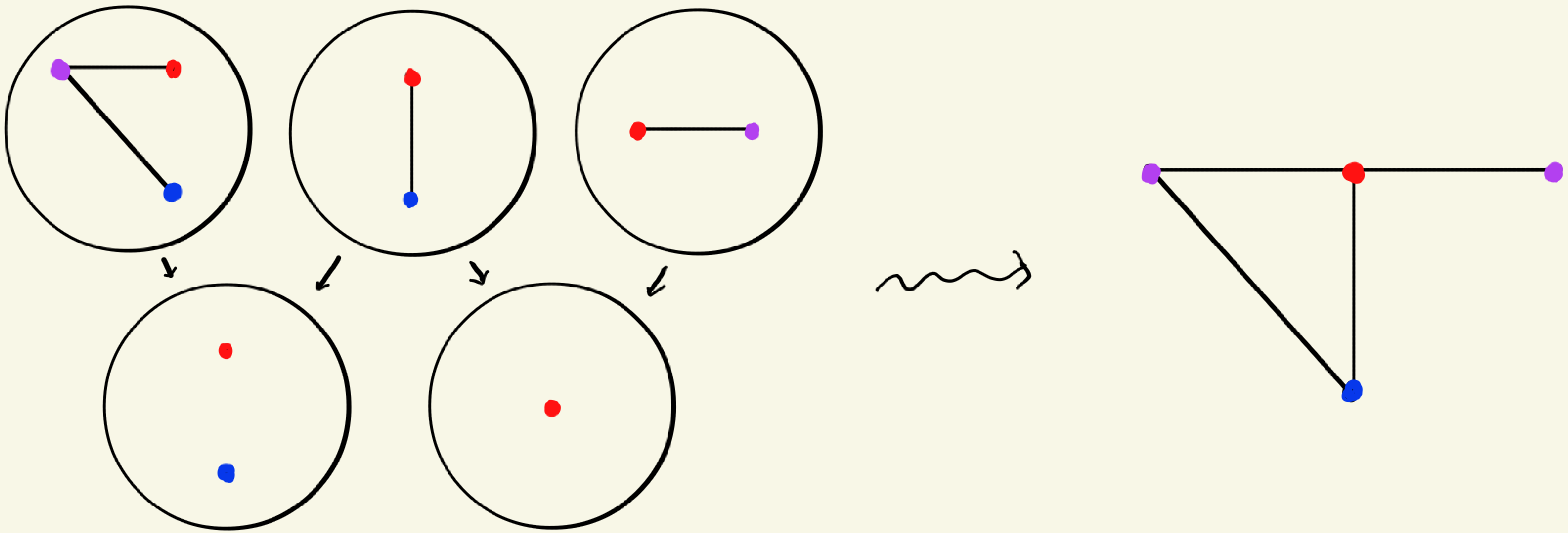
$$\mathcal{C}_K : \mathrm{Sub}(G)^{op} \longrightarrow \mathrm{Set}$$

$$\mathcal{C}_K(H) = \{ f : K \longrightarrow V(H) \; : \; f \text{ is a coloring of } H \}$$

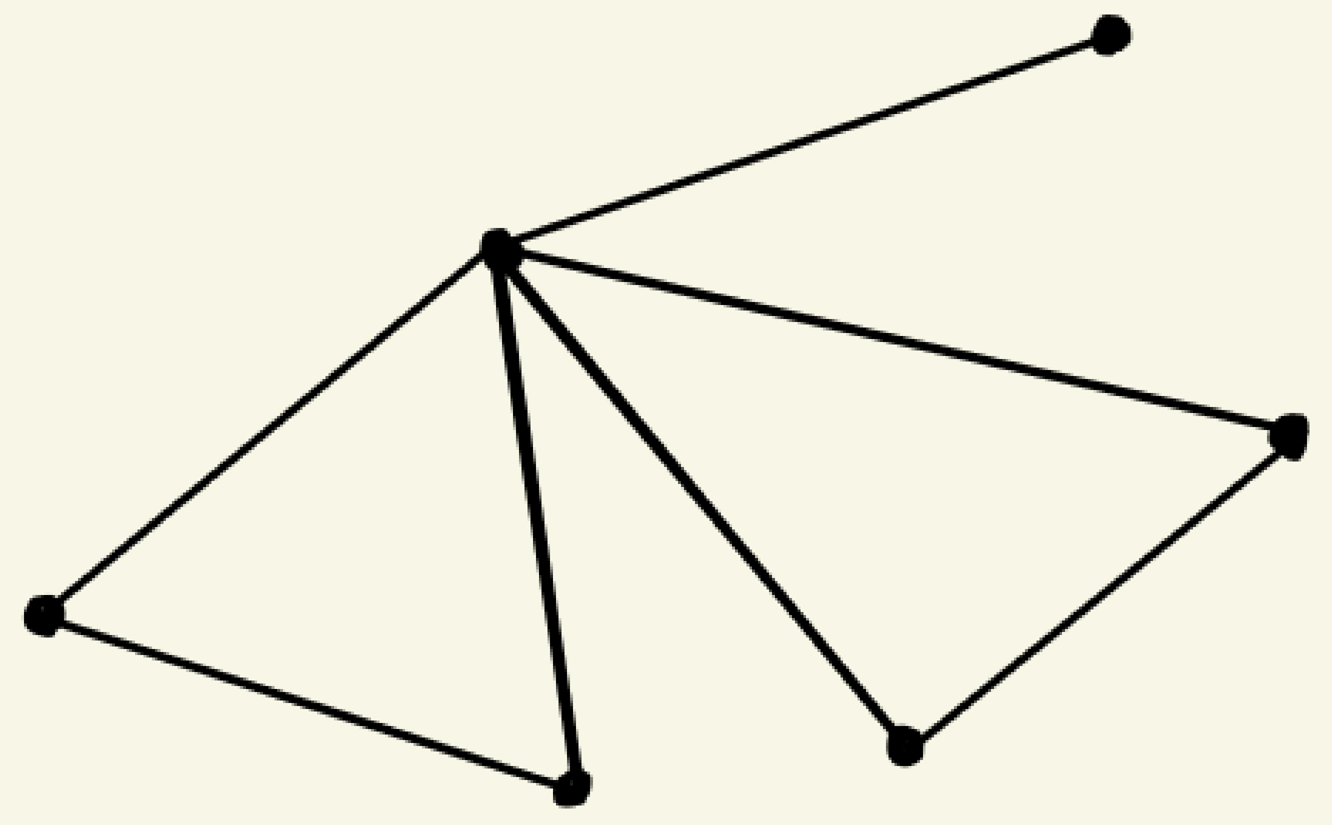
$$\mathcal{C}_K : \mathrm{Sub}(G)^{op} \longrightarrow \mathrm{Set}$$

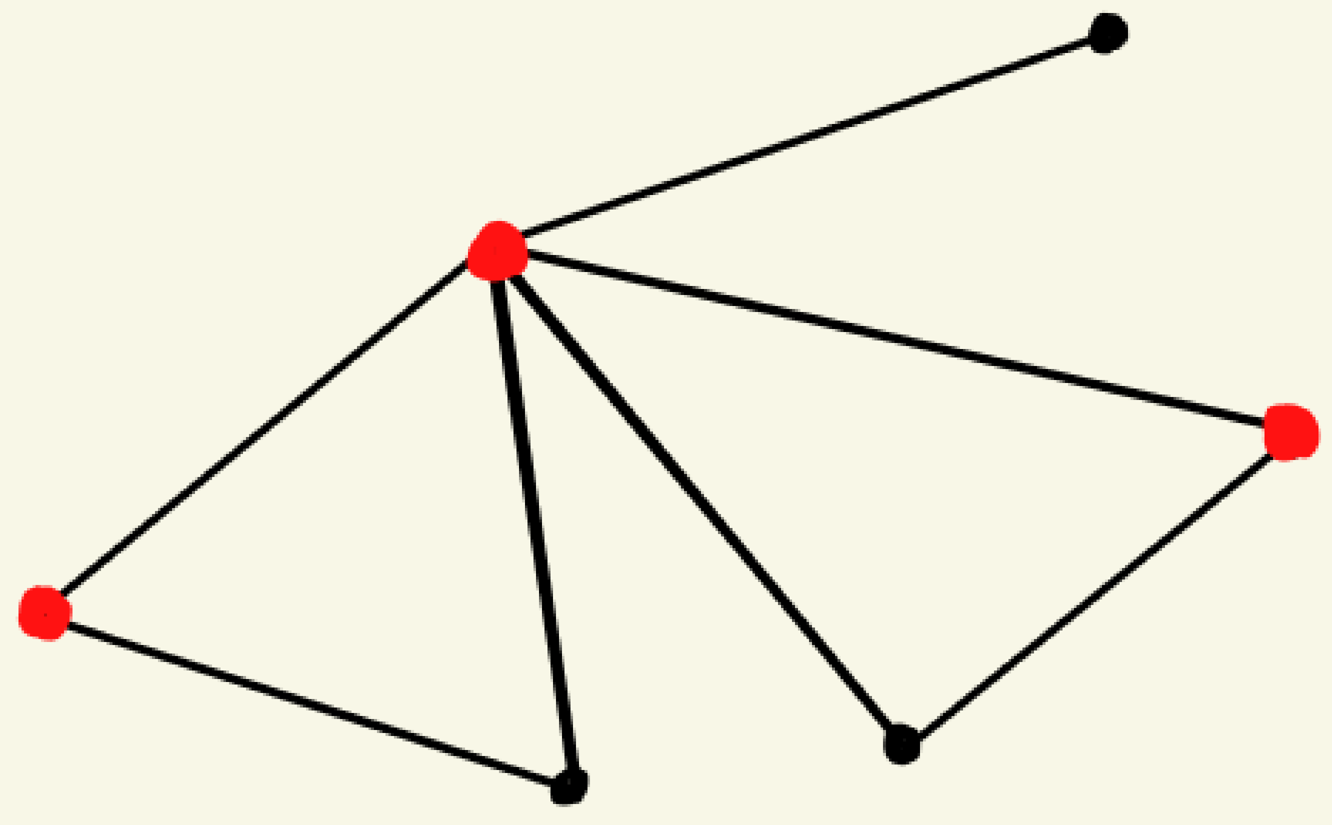
$$\begin{array}{ccc} H' & & \mathcal{C}_K(H') \\ \Big\downarrow & \longmapsto & \Big\uparrow \\ H & & \mathcal{C}_K(H) \end{array}$$

Gluing solutions: coloring is a sheaf!



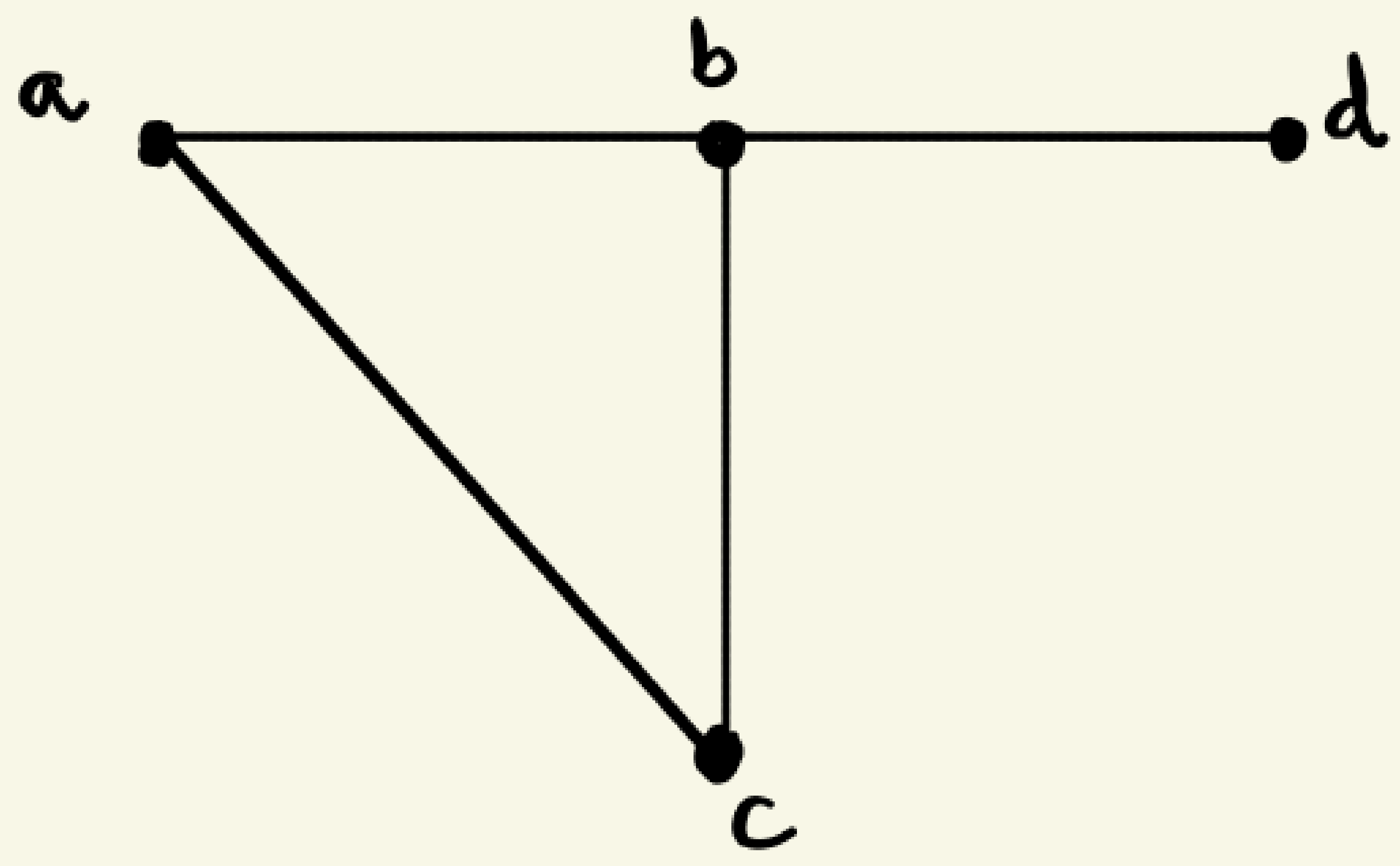
Vertex cover problem

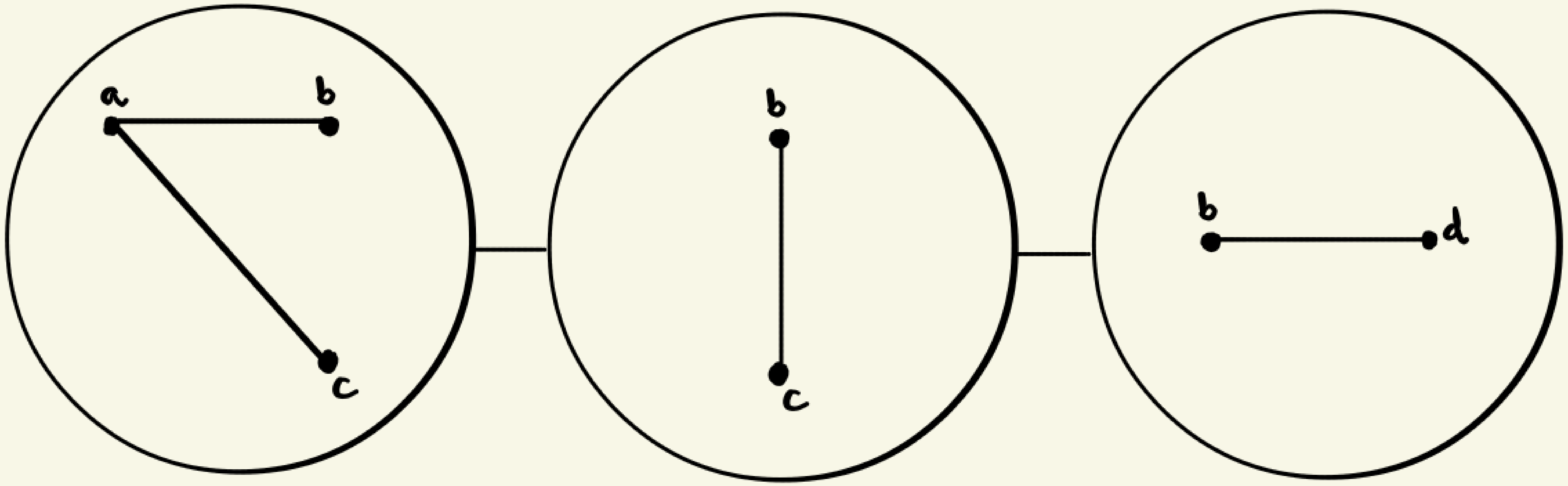




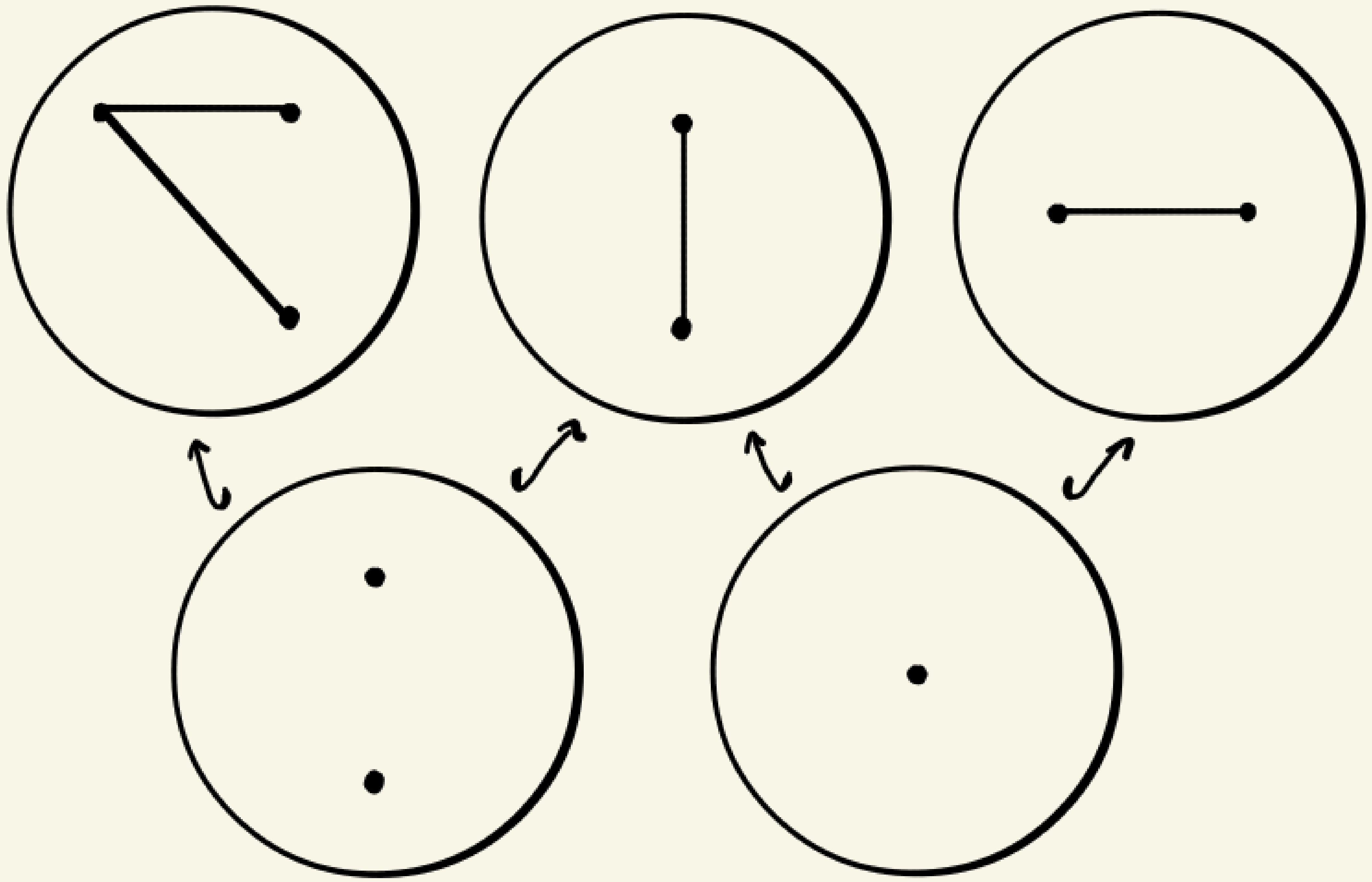
Definition: $A \subseteq V(G)$ is a vertex cover of a graph G if its vertices touch every each edge of G. Eq. if $G - A$ is edgeless. minimum vertex cover

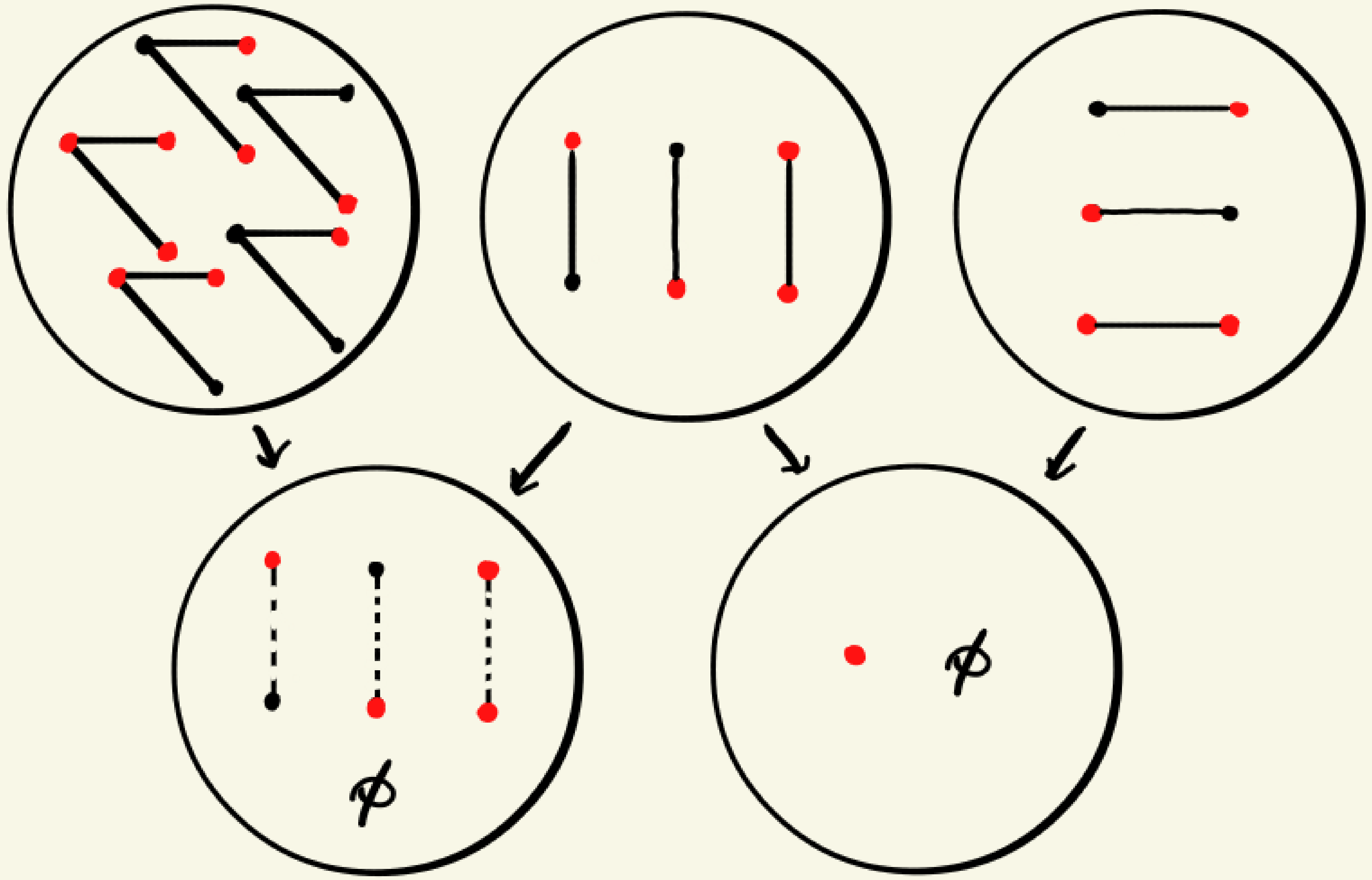
Divide and conquer





Local solutions





Vertex cover as a presheaf

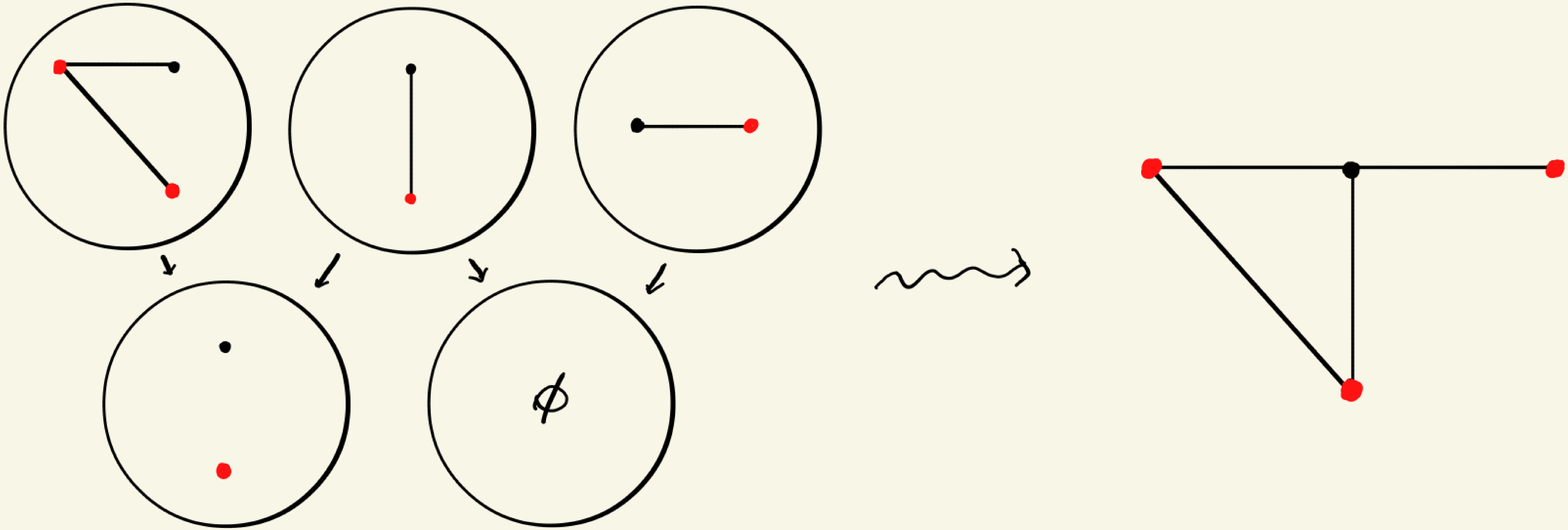
$$\mathcal{V}_{\leq k} : \mathrm{Sub}(G)^{op} \longrightarrow \mathrm{Set}$$

$$\mathcal{V}_{\leq k}(H) = \{ A \subseteq V(H) : A \text{ is a vertex cover of } H, |A| \leq k \}$$

$$\mathcal{V}_{\leq k} : \mathrm{Sub}(G)^{op} \longrightarrow \mathrm{Set}$$

$$\begin{array}{ccc} H' & & \mathcal{V}_{\leq k}(H') \\ \downarrow & \longmapsto & \uparrow \\ H & & \mathcal{V}_{\leq k}(H) \end{array}$$

Is $\mathcal{V}_{\leq 2}$ a sheaf? No!



Measuring the failure of being a sheaf

Sheaves categorically

$(\mathcal{C}, J)$ site, $F : \mathcal{C}^{op} \longrightarrow \mathrm{Set}$ presheaf, $\mathcal{U} = (u_i \xrightarrow{f_i} x) \in \mathrm{Cov}(x)$

$\mathcal{C} = \mathrm{Sub}(G)$ $\quad \mathcal{U} = (H_i \overset{f_i}{\hookrightarrow} H)$ s.t. $\bigcup H_i = H$

$$\begin{tikzcd}
\mathrm{Match}(x, \mathcal{U}) = \mathrm{Eq}(p, q) \arrow[r, hook] & \prod_{i \in I} F(u_i) \arrow[r, shift left, "p"] \arrow[r, shift right, "q"'] & \prod_{i,j} F(u_i \times_x u_j) \\
F(x) \arrow[u, dashed, "\exists! \xi"] \arrow[ur, "\delta^{-1}"'] & &
\end{tikzcd}$$

Sheaves categorically

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F(x) \arrow[u, dashed, "\exists! \, \xi"] \arrow[ur, "\delta^{-1}"'] & &
\end{tikzcd}$$

If for all objects $x \in \mathscr{C}$ and all covers $\mathcal{U}$ the arrow ξ is

- mono, then F is separated
- epi, then F is lavish
- iso, then F is sheaf

Sheaf $\Leftrightarrow$ $\mathrm{Eq}(p, q) \cong F(x)$

Failure of being a sheaf

- We use equalizer to represent when

$$p = q \iff p - q = 0 \qquad \mathrm{Match}(x, \mathcal{U}) = \mathrm{Ker}(p - q)$$

- The map $F(x) \xrightarrow{\delta^{-1}} \prod_{i \in I} F(u_i)$ is the restriction

"to be a glue" = "to be an element of $\mathrm{Im}\,\delta^{-1}$"

$$H^0(x, F) := \frac{\mathrm{Ker}(p - q)}{\mathrm{Im}(\delta^{-1})}$$

(in Abelian categories we can talk about kernel and cokernel)

Thm 1. $F : \mathcal{C}^{op} \longrightarrow A$ separated, then

$$H^0(-, F) = \operatorname{coker}\left(F \xrightarrow{\eta_F} F^+\right)$$

where $\eta : F \Rightarrow F^+$ is the unit of the adjunction given by sheafification.

Prop 2. $\vartheta : \mathrm{Sub}(G)^{op} \longrightarrow \mathrm{Set}$ is the sheafification of $\vartheta_{\leq k}$, for $k \geq 2$

$$\vartheta(H) = \{ A \subseteq V(G) : A \text{ is a vertex cover of } H \}$$

Cohomology of Vertex Cover

Free Abelinization: $\mathrm{Sub}(G)^{op} \xrightarrow{\mathcal{V}_{\leq k}} \mathrm{Set} \xrightarrow{\mathbb{Z}[-]} \mathrm{Ab}$

Prop. 3. If $k \geq 2$, then $H^0(X, \mathcal{V}_{\leq k}) = \mathbb{Z}[\{A \in \mathcal{V}(X) : |A| > k\}]$.

Intuition: size is the only obstruction we need to account for

Other kinds of obstruction

Other kinds of obstruction

- We saw that H^0 can capture obstructions to algorithmic compositionality

- Now we will see that we can use H^0 to talk about obstructions to having <u>any</u> solutions at all

Other kinds of obstruction

Given a problem (presheaf) $F : \mathcal{C}^{op} \longrightarrow Set$

we define

$$\begin{aligned} \mathcal{M}F : \mathcal{C}^{op} &\longrightarrow Set \\ x &\longmapsto \{ x' \hookrightarrow x : F(x') \neq \emptyset \} \end{aligned}$$

Given $f : x \to y$ in $\mathcal{C}$, $\mathcal{M}F(f)$ is defined as:

given $y' \in \mathcal{M}F(y)$, ie, $y' \hookrightarrow y$ s.t. $F(y') \neq \emptyset$, we consider

$$\begin{tikzcd} x' \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & x \arrow[d, "f"] \\ y' \arrow[r, hook] & y \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} F(x') & F(x) \arrow[l] \\ F(y') \arrow[u] & F(y) \arrow[l] \arrow[u, "F(f)"'] \end{tikzcd}$$

$F(y') \neq \emptyset \Rightarrow F(x') \neq \emptyset$, so we put $\mathcal{M}F(f)(y') = x'$

Thm. $\mathcal{M}$ is a covariant functor that maps any presheaf $F : \mathcal{C}^{op} \longrightarrow \mathrm{Set}$, where $\mathcal{C}$ is a category with pullbacks, to a flasque presheaf $\mathcal{M}F$. Moreover, if $\mathcal{C} = \mathrm{Sub}(G)$ and if $F(\bullet\!\!-\!\!\bullet) \neq \emptyset$, then

- $H^0(X, \mathcal{M}F) = \mathbb{Z}[\{X' \subseteq X : F(X') = \emptyset\}]$
- $FX \neq \emptyset$ iff $H^0(X, \mathcal{M}F) = 0$.

Why is this cool?

- Different problems, same language

- We have two different kinds of combinatorial obstructions but we can express them with the same language

Why is this not (so) cool?

- Free Abelinization: when talking about higher cohomology groups, the map $H^0 \longrightarrow H^1$ is trivial.

Future work: study examples of functors defined directly in Abelian categories

References

1. Althus E., Bumpus B., Fairbanks J., Rosiak D. (2024)
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2. Azevedo A., Bumpus B., Capucci M., Fairbanks J., Rosiak D. (2025)
Algorithmic and Extremal Obstructions Through the Language of Cohomology.
10.48550/arXiv.2407.03488