Enriched homotopy colimits

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The approach we pursue is based on the emergence of a ”canonical” homotopy theory on $[\Delta^{\circ}, \mathcal{V}]$ behaving as a $\mathcal{V}$-enriched version of Kan’s homotopy theory on sSet.

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(ii) as colimits which are coherent relatively to a chosen subcategory $\mathcal{W}$.

Point (ii) is more fundamental; in fact, every type of ”category theory” should admit a ”relative” version (where you consider categories with a specified class of weak equivalences) with a corresponding a notion of ”homotopy colimits” which give the correct way of glueing objects. For example, one can consider ∞-categlories with weak equivalences, and thus have homotopy colimits in the setting of ∞-categories.

For a relative 1-category $(\mathcal{C}, \mathcal{W})$, take the ordinary cocompletion $[\mathcal{C}^\circ, \mathrm{Set}]$, pass to simplicial objects $[\Delta^\circ, [\mathcal{C}^\circ, \mathrm{Set}]]$, with a choice of weak equivalences that includes pointwise weak homotopy equivalences of simplicial sets as well as all the maps between representables $h_c \to h_{c'}$ coming from a weak equivalence $c \to c'$ in $\mathcal{C}$. This gives you the homotopy cocompletion of $(\mathcal{C}, \mathcal{W})$.

Denote $\widehat{C}$ the homotopy cocompletion of $(\mathcal{C}, \mathcal{W})$.

We can define homotopy bimodules: a homotopy bimodule $\mathcal{A} \nrightarrow \mathcal{B}$ is a homotopical functor $\mathcal{A} \to \widehat{\mathcal{B}}$. Every homotopical functor $f : \mathcal{A} \to \mathcal{B}$ has an associated homotopy bimodule $h_f : \mathcal{A} \to \widehat{\mathcal{B}}$ represented by f, with a dual "up to homotopy" $h^f : \mathcal{B} \to \widehat{\mathcal{A}}$.

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We define what it means for a category with weak equivalences $\mathcal{M}$ to be homotopy $\mathcal{D}$-cocomplete in terms of extenstion properties with respect to $\widehat{\mathcal{D}}$.

And for every $\mathcal{C}$, we have $\widehat{\mathcal{C}}$ is homotopy $\mathcal{D}$-cocomplete for every small $\mathcal{D}$ and universal with this property.

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We need an homotopy theory on $[\Delta^\circ, \mathcal{V}]$ playing the role of the homotopy theory of spaces for enriched relative 1-categories. We might call the fibrant objects $\mathcal{V}$-spaces or Kan $\mathcal{V}$-complexes.

We focus on choosing the (trivial) fibrations on $[\Delta^{\circ}, \mathcal{V}]$. Using the ”underlying” functor $\mathcal{V}(1, -) : \mathcal{V} \to \mathrm{Set}$ *does not work*, because we loose too much information.

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We call a morphism $f : X \to Y$ a $\mathcal{V}$-fibration (resp. trivial) when the morphism $\mathcal{V}(v, X) \to \mathcal{V}(v, Y)$ is a Kan fibration (resp. trivial fibration) of simplicial sets *for every* $v \in \mathcal{V}$.

Where $\mathcal{V}(v, X)$ is the composition $\Delta^\circ \xrightarrow{X} \mathcal{V} \xrightarrow{\mathcal{V}(v,-)} \text{Set}$.

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- in the context of relative homological algebra, they are called *absolute* fibrations (Christensen-Hovey);
- with a view towards homotopy type theory and constructive mathematics, they are called *effective* fibrations (Gambino-Henry-Sattler-Szumilo).

In particular, for $[\Delta^\circ, \mathrm{GSet}]$ we have the ”genuine” or ”fine” homotopy theory of G-spaces, where fibrations and weak equivalences are detected mapping out of every orbit.

The GSet-enriched homotopy cocompletion of a point is the genuine homotopy theory of G-spaces.

Elmendorf theorem becomes the following statement:

the GSet-enriched homotopy cocompletion of a point is equivalent to the ordinary homotopy cocompletion of the category of orbits of G.

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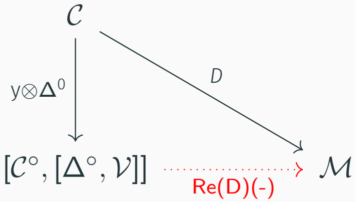
As a sanity check for our point of view, we want to show that the enriched version of Dugger theorem holds, with the universal property in its full strenght.

We need a definition of enriched model structure retrieving the ordinary definition over the base $\mathcal{V} = \mathrm{Set}$: we replace weak factorization systems with $\mathcal{V}$-*enriched* w.f.s. (Riehl)

We can prove an enriched version of Dugger's theorem:

> every enriched functor from a small $\mathcal{V}$-category into an enriched model category factorizes up to weak equivalence along the Yoneda embedding via a $\mathcal{V}$-enriched left Quillen functor, and the space of such factorizations is contractible.



As a corollary, we also obtain a $\mathcal{V}$-enriched version of Dwyer-Kan mapping spaces:

for every object $M \in \mathcal{M}$, the map $M : 1 \to \mathcal{M}$ induces a homotopically unique right Quillen $\mathcal{V}$-functor

$$\mathsf{Map}\,(M, -) : \mathcal{M} \rightleftarrows [\Delta^\circ, \mathcal{V}]$$

taking values in the subcategory of Kan $\mathcal{V}$-complexes

where $\mathcal{M}$ is an enriched model category.

We can prove that Kan $\mathcal{V}$-complexes have the following properties:

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and a version over $\mathcal{V}$ of a fundamental result of Joyal:

> for every internal category $\mathcal{C}$ in $\mathcal{V}$, the internal nerve of $\mathcal{C}$ is a Kan $\mathcal{V}$-complex if and only if $\mathcal{C}$ is an internal groupoid.

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Combining our results with Gepner-Haugseng's and Lurie's we can prove that:

the underlying ∞-category of an enriched model category is enriched over the ∞-category of Kan $\mathcal{V}$-complexes.

We obtain a concrete model for weighted colimits in these enriched ∞-categories.

Shulman was the first to study homotopy colimits in the enriched context (see also Riehl for an elaboration); there are also definitions by Vokřínek and Lack & Rosický.

> As we have seen, the difference of these other references with our approach can be summarized by saying that we do not assume a homotopical structure on $\mathcal{V}$, but instead enrich over a canonical structure on $[\Delta^\circ, \mathcal{V}]$.

Fixing a structure on $\mathcal{V}$, the (derived) homotopy colimit functor is not necessarily $\mathcal{V}$-enriched.

> This issue disappears in our approach.

Another way of dealing with it (Lack & Rosický) is to restrict the base of enrichment to a combinatorial monoidal model category $\mathcal{V}$ that *has all objects cofibrant* (see also Shulman).

In this slide, we suppose that $\mathcal{V}$ is endowed with a good enough model structure μ in which in particular all objects are cofibrant (example: Cisinski model structure on $\mathcal{V}$ a topos), and such that $\mathcal{V}$ is enriched in the model structure μ on itself.

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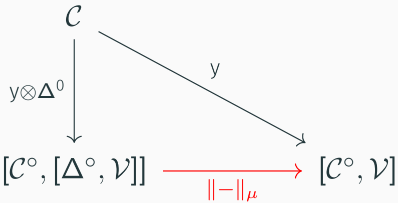
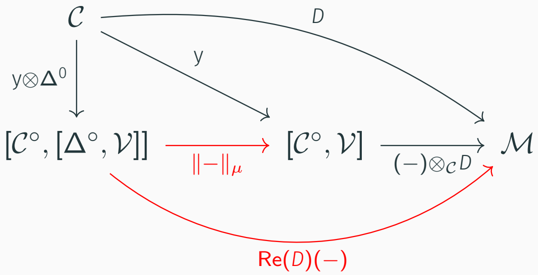
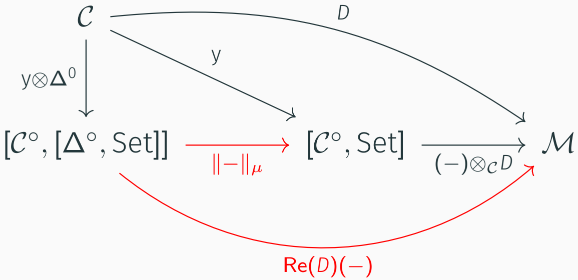
Then for every such choice of model structure μ on $\mathcal{V}$ we have a left Quillen $\mathcal{V}$-functor $\|-\|_\mu : [\mathcal{C}^\circ, [\Delta^\circ, \mathcal{V}]] \to [\mathcal{C}^\circ, \mathcal{V}]$ induced by the Yoneda embedding $\mathrm{y} : \mathcal{C} \to [\mathcal{C}^\circ, \mathcal{V}]$.



Then we have the following comparison between our notion of homotopy $\mathcal{V}$-colimit and the approach based on deriving the weighted colimit functor:

> for every model structure μ on $\mathcal{V}$ as above, for every $\mathcal{V}$-diagram D valued in cofibrant objects in a μ-enriched model category $\mathcal{M}$, our realization functor $\mathrm{Re}(D)(-)$ is Quillen homotopic to the weighted colimit $\| - \|_\mu \otimes_{\mathcal{C}} D$





To retrieve ordinary and enriched colimits as particular cases, consider $\mathcal{V} = \mathrm{Set}$, $\mathcal{M}$ cocomplete and take μ the trival model structure where $\mathcal{W} =$ isomorphisms. In this setting, a Quillen homotopy is just an isomorphism. Moreover, for any cofibrant replacement of the terminal weight one has $\|\mathcal{Q}(*)\| \cong *$, hence

$$\mathrm{Re}(D)(\mathcal{Q}(*)) \cong \|\mathcal{Q}(*)\| \otimes_{\mathcal{C}} D \cong * \otimes_{\mathcal{C}} D \cong \operatorname{colim} D$$

Similarly for enriched colimits.

Thank you!