Linearly Distributive Fox Theorem

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Linearly distributive categories

Motivation: characterization of cartesian LDCs

Definition (Cockett, Seely [5])

A *linearly distributive category*, or LDC, $(\mathbb{X}, \otimes, \top, \oplus, \bot)$ consists of:

- a category $(\mathbb{X}, ;, 1_A)$,
- a *tensor* monoidal structure $(\mathbb{X}, \otimes, \top, \alpha_\otimes, u^R_\otimes, u^L_\oplus)$,
- a *par* monoidal structure $(\mathbb{X}, \oplus, \bot, \alpha_\oplus, u^R_\oplus, u^L_\oplus)$, and
- left and right *linear distributivity* natural transformations

$$\delta^R_{A,B,C} \colon (A \oplus B) \otimes C \to A \oplus (B \otimes C)$$

$$\delta^L_{A,B,C} \colon A \otimes (B \oplus C) \to (A \otimes B) \oplus C$$

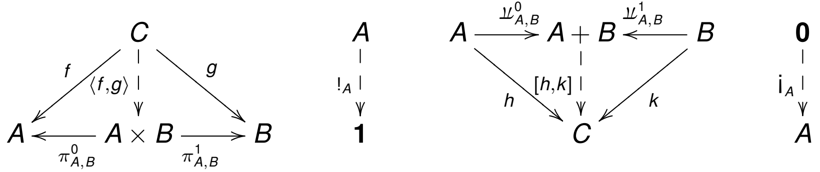
satisfying coherence conditons.

Cartesian linearly distributive categories

Definition (Cockett, Seely [5])

A linearly distributive category $(\mathbb{X}, \otimes, \top, \oplus, \bot)$ is *cartesian* if

- the tensor monoidal structure is cartesian $(\mathbb{X}, \times, \mathbf{1})$, and
- the par monoidal structure is cocartesian $(\mathbb{X}, +, \mathbf{0})$.



Example (Cockett, Seely [5])

A category with finite biproducts is a compact cartesian linearly distributive category.

Cartesian linearly distributive categories

Definition (Cockett [3])

A category is *distributive* if it has finite products and coproducts such that the canonical natural transformations

$$d^R_{A,B,C} = [\amalg^0_{A,B} \times 1_C, \amalg^1_{A,B} \times 1_C] \colon (A \times C) + (B \times C) \to (A + B) \times C$$

$$d^L_{A,B,C} = [1_A \times \amalg^0_{B,C}, 1_A \times \amalg^1_{B,C}] \colon (A \times B) + (A \times C) \to A \times (B + C)$$

$$\rho_A = \langle 1_{\mathbf{0}}, \mathrm{i}_A \rangle \colon \mathbf{0} \to \mathbf{0} \times A \qquad \lambda_A = \langle \mathrm{i}_A, 1_A \rangle \colon \mathbf{0} \to A \times \mathbf{0}$$

are isomorphisms.

Proposition (Cockett, Seely [5])

A cartesian linearly distributive category is a distributive category if and only if it is a preorder.

Cartesian linearly distributive categories

Definition

Given a distributive category $\mathcal{X}$, define the exception monad by

$$(\quad) + \mathbf{1} \colon \mathcal{X} \to \mathcal{X} \qquad \eta_A = \amalg^0_{A,\mathbf{1}} \qquad \mu_A = [1_{A+\mathbf{1}}, \amalg^1_{A,\mathbf{1}}]$$

Proposition (Cockett, Seely [5])

The Kleisli category $\mathcal{X}_{+\mathbf{1}}$ of the exception monad is an isomix cartesian linearly distributive category, with

- *Initial object and binary coproducts in $\mathcal{X}_{+\mathbf{1}}$ are the same as in $\mathcal{X}$,*
- *Terminal object in $\mathcal{X}_{+\mathbf{1}}$ is the initial object $\mathbf{0}$ in $\mathcal{X}$, and unique arrows*

$$!_A ; {\amalg^1}_{\mathbf{0},\mathbf{1}} \colon A \to \mathbf{0} + \mathbf{1}$$

- *Binary products in $\mathcal{X}_{+\mathbf{1}}$ are $A \,\&\, B = (A + B) + (A \times B)$, and projections*

$$\overline{\pi}^0_{A,B} = [[\eta_A, !_B ; \amalg^1_{A,\mathbf{1}}], \pi^0_{A,B} ; \eta_A] \colon (A + B) + (A \times B) \to A + \mathbf{1}$$
$$\overline{\pi}^1_{A,B} = [[!_A ; \amalg^1_{B,\mathbf{1}}, \eta_B], \pi^1_{A,B} ; \eta_A] \colon (A + B) + (A \times B) \to B + \mathbf{1}$$

The left linearly distributivity in $\mathcal{X}_{+\mathbf{1}}$ is the map in $\mathcal{X}$

$$\delta^L_{A,B,C} \colon (A + (B + C)) + (A \times (B + C)) \to (((A + B) + (A \times B)) + C) + \mathbf{1}$$

defined as the map induced by coproduct universal properties:

$$A \xrightarrow{\amalg^0_{A,B}} A + B \xrightarrow{\amalg^0_{A+B,A\times B}} A \,\&\, B \xrightarrow{\amalg^0_{A\,\&\,B,C}} (A \,\&\, B) + C \xrightarrow{\eta_{(A\,\&\,B)+C}} ((A \,\&\, B) + C) + \mathbf{1}$$

$$B \xrightarrow{\amalg^1_{A,B}} A + B \xrightarrow{\amalg^0_{A+B,A\times B}} A \,\&\, B \xrightarrow{\amalg^0_{A\,\&\,B,C}} (A \,\&\, B) + C \xrightarrow{\eta_{(A\,\&\,B)+C}} ((A \,\&\, B) + C) + \mathbf{1}$$

$$C \xrightarrow{\amalg^1_{A\,\&\,B,C}} (A \,\&\, B) + C \xrightarrow{\eta_{(A\,\&\,B)+C}} ((A \,\&\, B) + C) + \mathbf{1}$$

$$A \times (B + C) \xrightarrow{{d^L}^{-1}_{A,B,C}} (A \times B) + (A \times C) \xrightarrow{1_{A\times B} + !_{A\times C}} (A \times B) + \mathbf{1}$$

$$\xrightarrow{\pi^1_{A+B,A\times B} + 1_{\mathbf{1}}} (A \,\&\, B) + \mathbf{1} \xrightarrow{\pi^0_{A\,\&\,B,C} + 1_{\mathbf{1}}} ((A \,\&\, B) + C) + \mathbf{1}$$

Cartesian linearly distributive categories

Theorem

Every classical distributive restriction category (in the sense of Cockett and Lemay [4]) is an isomix cartesian linearly distributive category.

Example (Cockett, Lemay [4])

- PAR, the category of sets and partial functions
- k-$\mathsf{CALG}_\bullet^{op}$, the opposite category of commutative k-algebras and the non-unital k-algebra morphisms
- $\mathsf{TOP}_\bullet^{clopen}$, the category of topological spaces and partial continuous maps defined on clopen sets
- $\mathsf{STONE}_\bullet^{clopen}$, the category of Stone spaces and partial continuous functions defined on clopen sets $\Rightarrow$ opposite category of Boolean algebras and maps which preserve meets, joins and the bottom

Fox's theorem

Definition

Consider a symmetric monoidal category, or SMC, $(\mathcal{X}, \oslash, I, \alpha, \rho, \gamma)$.

- A *cocommutative comonoid* is a triple $\langle A, \Delta_A, t_A \rangle$ of an object A in $\mathcal{X}$ equipped with two morphisms, the *diagonal* and the *counit*

$$\Delta_A : A \to A \oslash A \qquad t_A : A \to I$$

  such that

$$\Delta_A; (1_A \oslash \Delta_A) = \Delta_A; (\Delta_A \oslash 1_A); \alpha_{A,A,A}$$

$$\Delta_A; (1_A \oslash t_A) = \rho_A \qquad \Delta_A; \gamma_{A,A} = \Delta_A$$

- A *comonoid morphism* $f : \langle A, \Delta_A, t_A \rangle \to \langle B, \Delta_B, t_B \rangle$ is a morphism $f : A \to B$ in $\mathcal{X}$ such that

$$f; \Delta_B = \Delta_A; (f \oslash f) \qquad f; t_B = t_A$$

Fox's theorem

Let $\mathbb{C}(\mathcal{X})$ denote the category of cocommutative comonoids and comonoid morphisms.

Proposition (Fox [8])

Given cocommutative comonoids $\langle A, \Delta_A, t_A \rangle$ and $\langle B, \Delta_B, t_B \rangle$, then $\langle A \oslash B, \Delta_{A \oslash B}, t_{A \oslash B} \rangle$ defined by

$$\Delta_{A \oslash B} = A \oslash B \xrightarrow{\Delta_A \oslash \Delta_B} (A \oslash A) \oslash (B \oslash B) \xrightarrow{s^{\oslash}_{A,A,B,B}} (A \oslash B) \oslash (A \oslash B)$$

$$t_{A \oslash B} = A \oslash B \xrightarrow{t_A \oslash t_B} I \oslash I \xrightarrow{\sim} I$$

is a cocommutative comonoid and $\mathbb{C}(\mathcal{X})$ is a cartesian category with this monoidal product.

$$s^{\oslash}_{A,B,C,D} \colon (A \oslash B) \oslash (C \oslash D) \xrightarrow{\sim} A \oslash (B \oslash (C \oslash D)) \xrightarrow{\sim} A \oslash ((B \oslash C) \oslash D) \xrightarrow{\sim} A \oslash ((C \oslash B) \oslash D) \xrightarrow{\sim} A \oslash (C \oslash (B \oslash D)) \xrightarrow{\sim} (A \oslash C) \oslash (B \oslash D)$$

Fox's theorem

Lemma (Fox [8])

Given a symmetric strong monoidal functor $F\colon \mathcal{X} \to \mathcal{Y}$ and a commutative comonoid $\langle A, \Delta_A, t_A \rangle$ in $\mathcal{X}$, the triple $\langle F(A), \Delta_{F(A)}, t_{F(A)} \rangle$ defined by

$$\Delta_{F(A)} = F(A) \xrightarrow{F(\Delta_A)} F(A \oslash A) \xrightarrow{m_{\oslash\ A,A}^{-1}} F(A) \oslash F(A)$$

$$t_{F(A)} = F(A) \xrightarrow{F(t_A)} F(I) \xrightarrow{m_I^{-1}} I$$

is a cocommutative comonoid. As such, F canonically extends to a cartesian functor $\mathrm{C}(F)\colon \mathrm{C}(\mathcal{X}) \to \mathrm{C}(\mathcal{X})$.

Theorem (Fox [8])

The functor $\mathrm{C}(-)\colon \underline{\mathsf{SMON}} \to \underline{\mathsf{CART}}$ is right adjoint to the inclusion.

Corollary (Fox [8])

A SMC $\mathcal{X}$ is cartesian if and only if it is isomorphic to its category of cocommutative comonoids $\mathrm{C}(\mathcal{X})$.

Corollary (Heunen, Vicary [10])

A SMC $\mathcal{X}$ is cartesian if and only if there are natural transformations

$$t_A \colon A \to I \qquad \Delta_A \colon A \to A \oslash A$$

such that (A, Δ_A, t_A) is cocommutative comonoid and

$$t_{A \oslash B} = (t_A \oslash t_B); \rho_I^{-1} \qquad t_I = 1_I$$
$$\Delta_{A \oslash B} = (\Delta_A \oslash \Delta_B); s^{\oslash}_{A,A,B,B} \qquad \Delta_I = \rho_I.$$

Is there a Fox-like theorem for cartesian LDCs?

By Fox's theorem and its dual statement:

Proposition

A SLDC $\mathbb{X}$ is cartesian if and only if there are natural transformations

$$\Delta_A : A \to A \otimes A \qquad t_A : A \to \top \qquad \nabla_A : A \oplus A \to A \qquad s_A : \bot \to A$$

such that, $\forall A, B \in \mathbb{X}$,

- $\langle A, \Delta_A, t_A \rangle$ *determines a $\otimes$-cocommutative comonoid,*
- $\langle A, \nabla_A, s_A \rangle$ *determines a $\oplus$-commutative monoid, and*

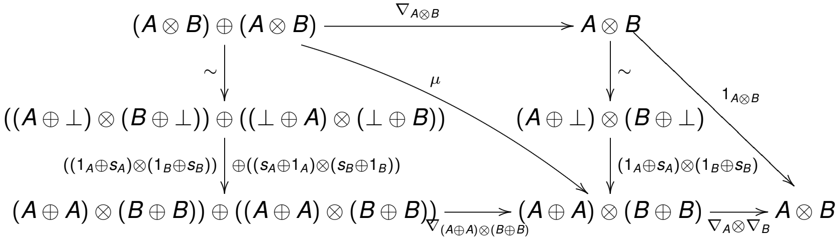
$$\Delta_{A \otimes B} = (\Delta_A \otimes \Delta_B); s^{\otimes}_{A,A,B,B} \qquad t_{A \otimes B} = (t_A \otimes t_B); {u^R_{\otimes\top}}^{-1}$$

$$\nabla_{A \oplus B} = s^{\oplus}_{A,B,A,B}; (\nabla_A \oplus \nabla_B) \qquad s_{A \oplus B} = {u^R_{\oplus\bot}}^{-1}; (s_A \oplus s_B)$$

$$\Delta_\top = u^R_{\otimes\top} \qquad t_\top = 1_\top \qquad \nabla_\bot = u^R_{\oplus\bot} \qquad s_\bot = 1_\bot$$

Given a cartesian linearly distributive category $\mathbb{X}$, there is a map

$$\nabla_{A\otimes B} : (A \otimes B) \oplus (A \otimes B) \to A \otimes B$$



$$\Rightarrow \nabla_{A\otimes B} : (A \otimes B) \oplus (A \otimes B) \xrightarrow{\mu} (A \oplus A) \otimes (B \oplus B) \xrightarrow{\nabla_A \otimes \nabla_B} A \otimes B$$

Similarly for $s_{A\otimes B}$, $t_{A\oplus B}$ and $\Delta_{A\oplus B}$

Duoidal categories

We need a symmetric linearly distributive category $\mathbb{X}$ with arrows

$$(A \otimes B) \oplus (C \otimes D) \to (A \oplus C) \otimes (B \oplus D)$$

$$\bot \to \bot \otimes \bot \qquad \top \oplus \top \to \top \qquad \bot \to \top$$

Answer: **duoidal categories**

Definition (Aguiar, Mahajan [1])

A *duoidal category* $(\mathcal{X}, \diamond, I, \star, J,)$ is category $\mathcal{X}$ with two monoidal structures $(\mathcal{X}, \star, J)$ and $(\mathcal{X}, \diamond, I)$ equipped with morphisms

$$\Delta_I : I \to I \star I \qquad \mu_J : J \diamond J \to J \qquad \iota : I \to J$$

and an *interchange* natural transformation

$$\zeta_{A,B,C,D} : (A \star B) \diamond (C \star D) \to (A \diamond C) \star (B \diamond D)$$

satisfying some coherence conditions.

Duoidal categories

Example (Aguiar, Mahajan [1])

Consider a category $\mathcal{X}$ with finite products and coproducts, then $(\mathcal{X}, +, \mathbf{0}, \times, \mathbf{1})$ is a duoidal category with structure maps

$$\Delta_{\mathbf{0}} = \mathrm{i}_{\mathbf{0}\times\mathbf{0}} \colon \mathbf{0} \to \mathbf{0} \times \mathbf{0} \qquad \mu_I = !_{\mathbf{1}+\mathbf{1}} \colon \mathbf{1} + \mathbf{1} \to \mathbf{1} \qquad \iota = !_{\mathbf{0}} = \mathrm{i}_{\mathbf{1}} \colon \mathbf{0} \to \mathbf{1}$$

$$\zeta_{A,B,C,D} = [\amalg^0_{A,C} \times \amalg^0_{B,D}, \amalg^1_{A,C} \times \amalg^1_{B,D}] \colon (A \times B) + (C \times D) \to (A + C) \times (B + D)$$

Definition (Aguiar, Mahajan [1])

A *bicommutative duoidal bimonoid* in $\mathcal{X}$ is a quintuple $\langle A, \mu, \eta, \Delta, \epsilon \rangle$ where

- $\langle A, \mu, \eta \rangle$ is a commutative $\diamond$-monoid, and
- $\langle A, \Delta, \epsilon \rangle$ is a cocommutative $\star$-comonoid, such that

$$(\Delta \diamond \Delta); \zeta_{A,A,A,A}; (\mu \star \mu) = \mu; \Delta \qquad \eta; \epsilon = \iota$$

$$\mu; \epsilon = (\epsilon \diamond \epsilon); \mu_J \qquad \Delta_I; (\eta \star \eta) = \eta; \Delta$$

A *morphism of duoidal bimonoids* is a morphism of the underlying monoids and comonoids.

Medial linearly distributive categories

Definition

A *medial linearly distributive category*, or MLDC, $(\mathbb{X}, \otimes, \top, \oplus, \bot)$ consists of:

- a category $(\mathbb{X}, ;, 1_A)$,
- a *tensor* monoidal structure $(\mathbb{X}, \otimes, \top)$.
- a *par* monoidal structure $(\mathbb{X}, \oplus, \bot)$,
- *nullary medial*, *nullary comedial* and *mix* morphisms,

$$\nabla_\top \colon \top \oplus \top \to \top \qquad \Delta_\bot \colon \bot \to \bot \otimes \bot \qquad m \colon \bot \to \top$$

- a *medial* natural transformation,

$$\mu_{A,B,C,D} \colon (A \otimes B) \oplus (C \otimes D) \to (A \oplus C) \otimes (B \oplus D)$$

- left and right *linear distributivity* natural transformations

$$\delta^R_{A,B,C} \colon (A \oplus B) \otimes C \to A \oplus (B \otimes C) \quad \delta^L_{A,B,C} \colon A \otimes (B \oplus C) \to (A \otimes B) \oplus C$$

satisfying the coherence conditions on the next slide.

Medial linearly distributive categories

Definition

- $(\mathbb{X}, \otimes, \top, \oplus, \bot)$ is a linearly distributive category,
- $(\mathbb{X}, \oplus, \bot, \otimes, \top)$ is a duoidal category, and
- the *medial maps* interact coherently with the linear distributivities:

$$\begin{array}{ccccc}
((A \otimes B) \oplus (C \otimes D)) \otimes X & \xrightarrow{\mu \otimes 1} & ((A \oplus C) \otimes (B \oplus D)) \otimes X & \xrightarrow{\alpha_\otimes} & (A \oplus C) \otimes ((B \oplus D) \otimes X) \\
\Big\downarrow {\scriptstyle \delta^R} & & & & \Big\downarrow {\scriptstyle 1 \otimes \delta^R} \\
(A \otimes B) \oplus ((C \otimes D) \otimes X) & \xrightarrow[1 \oplus \alpha_\otimes]{} & (A \otimes B) \oplus (C \otimes (D \otimes X)) & \xrightarrow[\mu]{} & (A \oplus C) \otimes (B \oplus (D \otimes X))
\end{array}$$

and

$$(1 \otimes \mu); {\alpha_\otimes}^{-1}; (\delta^L \otimes 1) = \delta^L; ({\alpha_\otimes}^{-1} \oplus 1); \mu$$

$$(\delta^R \oplus 1); {\alpha_\oplus}^{-1}; (1 \oplus \mu) = \mu; ({\alpha_\oplus}^{-1} \otimes 1); \delta^R$$

$$(1 \oplus \delta^L); \alpha_\oplus; (\mu \oplus 1) = \mu; (1 \otimes \alpha_\oplus); \delta^L$$

Examples of medial linearly distributive categories

Example

1. Symmetric monoidal categories $(\mathcal{X}, \oslash, I, \oslash, I)$

$$\nabla_I = \rho_I^{-1} = \lambda_I^{-1} \colon I \oslash I \to I \qquad \Delta_I = \rho_I = \lambda_I \colon I \to I \oslash I \qquad m = 1_I \colon I \to I$$

$$\mu_{A,B,C,D} = s^{\oslash}_{A,B,C,D} \colon (A \oslash B) \oslash (C \oslash D) \to (A \oslash C) \oslash (B \oslash D)$$
$$\delta^R_{A,B,C} = \alpha_{A,B,C} \colon (A \oslash B) \oslash C \to A \oslash (B \oslash C)$$

2. Cartesian linearly distributive category $(\mathbb{X}, \times, \mathbf{1}, +, \mathbf{0})$

$$\nabla_{\mathbf{1}} = !_{\mathbf{1}+\mathbf{1}} \colon \mathbf{1} + \mathbf{1} \to \mathbf{1} \qquad \Delta_{\mathbf{0}} = \text{i}_{\mathbf{0}\times\mathbf{0}} \colon \mathbf{0} \to \mathbf{0} \times \mathbf{0} \qquad m = !_{\mathbf{0}} = \text{i}_{\mathbf{1}} \colon \mathbf{0} \to \mathbf{1}$$

$$\mu_{A,B,C,D} = [\iota^0_{A,C} \times \iota^0_{B,D}, \iota^1_{A,C} \times \iota^1_{B,D}] = \langle \pi^0_{A,B} + \pi^0_{C,D}, \pi^1_{A,B} + \pi^1_{C,D} \rangle \colon$$
$$(A \times B) + (C \times D) \to (A + C) \oslash (B + D)$$

Definition (Yau [11])

A symmetric monoidal category $(\mathcal{X}, \oslash, I)$ is *distributive* if

- $\mathcal{X}$ has finite coproducts, and
- the following canonical natural transformations

$$d^L_{A,B,C} = [1_A \oslash \iota^0_{B,C}, 1_A \oslash \iota^1{}_{B,C}] \colon (A \oslash B) + (C \oslash D) \to A \oslash (B + C)$$

$$\lambda^\bullet_A = \mathsf{i}_{\mathbf{0} \oslash A} \colon \mathbf{0} \to \mathbf{0} \oslash A$$

  are isomorphisms.

Theorem

Given a distributive symmetric monoidal category $(\mathcal{X}, \oslash, I)$ with a zero object $\emptyset$, then $(\mathcal{X}, \curlyvee, \emptyset, +, \emptyset)$ is an isomix symmetric medial LDC where

$$- \curlyvee - = (- + -) + (- \oslash -) \colon \mathcal{X} \times \mathcal{X} \to \mathcal{X}$$

known as the "either-or-both" product.

3. Distributive symmetric monoidal category with a zero object
   i. Distributive restriction category with zero object
      - Classical distributive restriction categories, e.g. PAR, k-$\mathrm{CALG}_\bullet^{op}$, $\mathrm{TOP}_\bullet^{clopen}$ and $\mathrm{STONE}_\bullet^{clopen}$
        Warning! these are not cartesian LDCs as $\curlyvee$ is not the categorical product
      - $\mathrm{TOP}_\bullet$, the category of topological spaces and partial continuous maps
        $\rightarrow$ non-classical distributive restriction category (Cockett, Lemay [4])
   ii. Symmetric monoidal closed categories $\mathcal{X}$ with finite coproducts and a zero object (Elgueta [7])
      - Mod(R), the category of R-modules and module homomorphisms
      - Q-Rel, the category of sets and Q-relations
   iii. Symmetric 2-Rig (in the sense of Baez, Moeller, Trimble [2])
      - $\mathrm{FinRep}_k(G)$, the category of representations of a group G on finite-dimensional vector spaces
      - $\mathrm{FinVect}_k(X)$, the category of finite-dimensional vector bundles over a topological space X

Warning! Normal duoidal categories do not provide examples of medial linearly distributive categories.

Medial bimonoids

Let $\mathbb{X}$ be a symmetric medial linearly distributive category.

Definition

- A *bicommutative medial bimonoid* in $\mathbb{X}$ is a bicommutative duoidal bimonoid in $(\mathbb{X}, \oplus, \bot, \otimes, \top)$, i.e. a quintuple $\langle A, \Delta_A, t_A, \nabla_A, s_A \rangle$ consisting of an object A and four morphisms

$$\Delta_A : A \to A \otimes A \qquad t_A : A \to \top \qquad \nabla_A : A \oplus A \to A \qquad s_A : \bot \to A$$

  satisfying coherence conditions.
- A *medial bimonoid morphism* in $\mathbb{X}$ is a morphism of duoidal bimonoids in $(\mathbb{X}, \oplus, \bot, \otimes, \top)$.

Proposition

$\langle \top, u^R_{\otimes_\top}, 1_\top, \nabla_\top, m \rangle$ and $\langle \bot, \Delta_\bot, m, u^R_{\oplus_\bot}, 1_\bot \rangle$ are bicommutative medial bimonoids.

Medial bimonoids

Proposition

Given two bicommutative medial bimonoids $\langle A, \Delta_A, t_A, \nabla_A, s_A \rangle$ and $\langle B, \Delta_B, t_B, \nabla_B, s_B \rangle$ in $\mathbb{X}$, then $\langle A \otimes B, \Delta_{A\otimes B}, t_{A\otimes B}, \nabla_{A\otimes B}, s_{A\otimes B} \rangle$ defined by

$$\Delta_{A\otimes B} = (\Delta_A \otimes \Delta_B); s^{\otimes}_{A,A,B,B} \qquad t_{A\otimes B} = (t_A \otimes t_B); {u^R_{\otimes\top}}^{-1}$$

$$\nabla_{A\otimes B} = \mu_{A,B,A,B}; (\nabla_A \otimes \nabla_B) \qquad s_{A\otimes B} = \Delta_\perp; (s_A \otimes s_B) \,,$$

and $\langle A \oplus B, \Delta_{A\oplus B}, t_{A\oplus B}, \nabla_{A\oplus B}, s_{A\oplus B} \rangle$ defined by

$$\Delta_{A\oplus B} = (\Delta_A \oplus \Delta_B); \mu_{A,A,B,B} \qquad t_{A\oplus B} = (t_A \oplus t_B); \nabla_\top$$

$$\nabla_{A\oplus B} = s^{\oplus}_{A,B,A,B}; (\nabla_A \oplus \nabla_B) \qquad s_{A\oplus B} = {u^R_{\oplus\perp}}^{-1}; (s_A \oplus s_B) \,,$$

are bicommutative medial bimonoids.

Linearly distributive category of medial bimonoids

Definition

Define $\mathrm{B}(\mathbb{X})$ to be the category of bicommutative medial bimonoids and bimonoid morphisms in $\mathbb{X}$.

Lemma

$\mathrm{B}(\mathbb{X})$ is a linearly distributive category.

By our earlier characterization of cartesian LDCs:

Theorem

$\mathrm{B}(\mathbb{X})$ is a cartesian linearly distributive category.

Linear functors and transformations

Let $\mathbb{X}$ and $\mathbb{Y}$ be LDCs.

Definition (Cockett, Seely [6])

A *linear functor* $F = (F_\otimes, F_\oplus) \colon \mathbb{X} \to \mathbb{Y}$ consists of:

- a monoidal functor $(F_\otimes, m_\top, m_\otimes) \colon (\mathbb{X}, \otimes, \top) \to (\mathbb{Y}, \otimes, \top)$, equipped

$$m_\top \colon \top \to F_\otimes(\top) \qquad {m_\otimes}_{A,B} \colon F_\otimes(A) \otimes F_\otimes(B) \to F_\otimes(A \otimes B)$$

- a comonoidal functor $(F_\oplus, n_\perp, n_\oplus) \colon (\mathbb{X}, \oplus, \perp) \to (\mathbb{Y}, \oplus, \perp)$, equipped

$$n_\perp \colon F_\oplus(\perp) \to \perp \qquad {n_\oplus}_{A,B} \colon F_\oplus(A \oplus B) \to F_\oplus(A) \oplus F_\oplus(B)$$

- four natural transformations, known as *linear strengths*,

$$v^R_{\otimes\,A,B} \colon F_\otimes(A \oplus B) \to F_\oplus(A) \oplus F_\otimes(B) \quad v^L_{\otimes\,A,B} \colon F_\otimes(A \oplus B) \to F_\otimes(A) \oplus F_\oplus(B)$$

$$v^R_{\oplus\,A,B} \colon F_\otimes(A) \otimes F_\oplus(B) \to F_\oplus(A \otimes B) \quad v^L_{\oplus\,A,B} \colon F_\oplus(A) \otimes F_\otimes(B) \to F_\oplus(A \otimes B)$$

subject to various coherence conditions.

Linear functors and transformations

Definition (Cockett, Seely [6])

Let $F, G\colon \mathbb{X} \to \mathbb{Y}$ be linear functors between LDCs. A *linear transformation* $\alpha = (\alpha_\otimes, \alpha_\oplus)\colon F \Rightarrow G$ consists of:

- a monoidal transformation $\alpha_\otimes\colon (F_\otimes, m^F_\top, m^F_\otimes) \Rightarrow (G_\otimes, m^G_\top, m^G_\otimes)$ and
- a comonoidal transformation $\alpha_\oplus\colon (G_\oplus, n^G_\perp, n^G_\oplus) \Rightarrow (F_\oplus, n^F_\perp, n^F_\oplus)$,

which commute coherently with the linear strengths of F and G.

Proposition (Cockett, Seely [6])

Linearly distributive categories, linear functors, and linear transformations form a 2-category, which is denoted by **LDC***.*

Cartesian linear functors and transformations

Definition (Cockett, Seely [6])

If $\mathbb{X}$ and $\mathbb{Y}$ are symmetric LDCs, then a linear functor $F = (F_\otimes, F_\oplus)$ is *symmetric* if $F_\otimes$ is a symmetric monoidal functor, $F_\oplus$ is a symmetric comonoidal functors, and

$$v^L_{\otimes A,B} = F_\otimes(\gamma_{\oplus A,B}); v^R_{\otimes B,A}; \gamma_{\oplus F_\oplus(B),F_\otimes(A)}$$

$$v^L_{\oplus A,B} = \gamma_{\otimes F_\oplus(A),F_\otimes(B)}; v^R_{\oplus B,A}; F_\oplus(\gamma_{\otimes B,A})$$

Definition

A linear functor $F = (F_\otimes, F_\oplus)\colon \mathbb{X} \to \mathbb{Y}$ is *strong* if $F_\otimes$ is a strong monoidal functor and $F_\oplus$ is a strong comonoidal functor.

Corollary

Cartesian linearly distributive categories, strong symmetric linear functors and linear transformations form a 2-category, denoted **CLDC***.*

Duoidal functors and transformations

Let $\mathcal{X}$ and $\mathcal{Y}$ be duoidal categories.

Definition (Aguiar, Mahajan [1])

A *bilax duoidal functor* $(F, p_I, p_\diamond, q_J, q_\star)\colon \mathcal{X} \to \mathcal{Y}$ is a functor $F\colon \mathcal{X} \to \mathcal{Y}$ equipped with

$$p_I\colon I \to F(I) \qquad p_{\diamond_{A,B}}\colon F(A) \diamond F(B) \to F(A \diamond B)$$

$$q_J\colon F(J) \to J \qquad q_{\star_{A,B}}\colon F(A \star B) \to F(A) \star F(B)$$

such that

- $(F, p_I, p_\diamond)\colon (\mathcal{X}, \diamond, I) \to (\mathcal{Y}, \diamond, I)$ is a monoidal functor,
- $(F, q_J, q_\star)\colon (\mathcal{X}, \star, J) \to (\mathcal{Y}, \star, J)$ is a comonoidal functor,

satisfying coherence conditions.

Proposition (Aguiar, Mahajan [1])

A bilax duoidal functor preserves bimonoids and morphisms between bimonoids.

Duoidal functors and transformations

Definition (Aguiar, Mahajan [1])

A *bilax duoidal transformation* $\alpha : (F, p^F_I, p^F_\diamond, q^F_J, q^F_\star) \Rightarrow (G, p^G_I, p^G_\diamond, q^G_J, q^G_\star)$ is a natural transformation $\alpha\colon F \to G$ such that

- $\alpha\colon (F, p^F_I, p^F_\diamond) \Rightarrow (G, p^G_I, p^G_\diamond)$ is a monoidal transformation and
- $\alpha\colon (F, q^F_J, q^F_\star) \Rightarrow (G, q^G_J, q_\star)$ is a comonoidal transformation.

Proposition (Aguiar, Mahajan [1])

There is a 2-category of duoidal categories, bilax duoidal functors and bilax duoidal transformations, denoted by **Duoid***.*

Medial linear functors and transformations

Definition

Let $\mathbb{X}$ and $\mathbb{Y}$ be symmetric medial LDCs. A *strong symmetric medial linear functor* $F = (F_\otimes, F_\oplus)\colon \mathbb{X} \to \mathbb{Y}$ consists of:

- a functor $F_\otimes \colon \mathbb{X} \to \mathbb{Y}$, equipped with

$$m_\top : \top \xrightarrow{\sim} F_\otimes(\top) \qquad m_{\otimes_{A,B}} \colon F_\otimes(A) \otimes F_\otimes(B) \xrightarrow{\sim} F_\otimes(A \otimes B)$$

$$m_\bot : \bot \to F_\otimes(\bot) \qquad m_{\oplus_{A,B}} \colon F_\otimes(A) \oplus F_\otimes(B) \to F_\otimes(A \oplus B)$$

- a functor $F_\oplus \colon \mathbb{X} \to \mathbb{Y}$, equipped with

$$n_\bot : F_\oplus(\bot) \xrightarrow{\sim} \bot \qquad n_{\oplus_{A,B}} \colon F_\oplus(A \oplus B) \xrightarrow{\sim} F_\oplus(A) \oplus F_\oplus(B)$$

$$n_\top : F_\oplus(\top) \to \top \qquad n_{\otimes_{A,B}} \colon F_\oplus(A \otimes B) \to F_\oplus(A) \otimes F_\oplus(B)$$

- linear strength natural transformations

$$v^R_{\otimes_{A,B}} \colon F_\otimes(A \oplus B) \to F_\oplus(A) \oplus F_\otimes(B) \qquad v^R_{\oplus_{A,B}} \colon F_\otimes(A) \otimes F_\oplus(B) \to F_\oplus(A \otimes B)$$

satisfying the coherence conditions on the next slide.

Medial linear functors and transformations

Definition

- $F = (F_\otimes, F_\oplus)$ is a symmetric strong linear functor,
- $(F_\otimes, m_\perp, m_\oplus, m_\top^{-1}, m_\otimes^{-1})$ is a bilax duoidal functor,
- $(F_\oplus, n_\perp^{-1}, n_\oplus^{-1}, n_\top, n_\oplus)$ is a bilax duoidal functor,
- the linear strengths interact coherently with $m_\oplus$ and $n_\otimes$, with the nullary/conullary maps and with the medial transformation, e.g.

$$
\begin{array}{ccc}
F_\otimes((A \otimes B) \oplus (C \otimes D)) & \xrightarrow{F_\otimes(\mu)} & F_\otimes((A \oplus C) \otimes (B \oplus D)) \\
\downarrow{\scriptstyle \nu_\otimes^R} & & \downarrow{\scriptstyle m_\otimes^{-1}} \\
F_\oplus(A \otimes B) \oplus F_\otimes(C \otimes D) & & F_\otimes(A \oplus C) \otimes F_\otimes(B \oplus D) \\
\downarrow{\scriptstyle n_\otimes \oplus m_\otimes^{-1}} & & \downarrow{\scriptstyle \nu_\otimes^R \otimes \nu_\otimes^R} \\
(F_\oplus(A) \otimes F_\oplus(B)) \oplus (F_\otimes(C) \otimes F_\otimes(D))) & \xrightarrow[\mu]{} & (F_\oplus(A) \oplus F_\otimes(C)) \otimes (F_\oplus(B) \oplus F_\otimes(D))
\end{array}
$$

Medial linear functors and transformations

Definition

Let $F, G\colon \mathbb{X} \to \mathbb{Y}$ be strong symmetric medial linear functors.
A *medial linear transformation* $\alpha = (\alpha_\otimes, \alpha_\oplus)\colon F \Rightarrow G$ is a linear transformation such that

- $\alpha_\otimes : (F_\otimes, m^F_\perp, m^F_\oplus) \Rightarrow (G_\otimes, m^G_\perp, m^G_\oplus)$ is a monoidal transformation, and
- $\alpha_\oplus : (G_\oplus, {n_\top}^G, {n_\oplus}^G) \Rightarrow (F_\oplus, {n_\top}^F, {n_\oplus}^F)$ is a comonoidal transformations.

Proposition

Symmetric medial linearly distributive categories, strong symmetric medial linear functors, and medial linear transformations form a 2-category, which is denoted by $\mathbf{MLDC_s}$.

Medial linear functors and transformations

Proposition

Inclusion **CLDC** $\to$ $\mathbf{MLDC_s}$ *is a 2-functor.*

Proof.

Let $F = (F_\times, F_+)\colon \mathbb{X} \to \mathbb{Y}$ be a strong symmetric linear functor between cartesian LDCs.

- Functor $F_\times$ canonically becomes a monoidal functor $(F_\times, p_\mathbf{0}, p_+)\colon (\mathbb{X}, +, \mathbf{0}) \to (\mathbb{Y}, +, \mathbf{0})$ with

$$p_\mathbf{0} = \mathrm{i}_{F_\times(\mathbf{0})}\colon \mathbf{0} \to F_\times(\mathbf{0})$$

$$p_{+A,B} = [F_\times(\amalg^0_{A,B}), F_\times(\amalg^1_{A,B})]\colon F_\times(A) + F_\times(B) \to F_\times(A+B)$$

  Similarly, $(F_+, q_\mathbf{1}, q_\times)\colon (\mathbb{X}, \times, \mathbf{1}) \to (\mathbb{Y}, \times, \mathbf{1})$ is a comonoidal functor.

- Functors $F_\times$ and F_+ are then canonically bilax duoidal functors

$$(F_\times, p_\mathbf{0}, p_+, m_\mathbf{1}^{-1}, m_\times^{-1}),\ (F_+, n_\mathbf{0}^{-1}, n_+^{-1}, q_\mathbf{1}, q_\times)\colon (\mathcal{X}, +, \mathbf{0}, \times, \mathbf{1}) \to (\mathcal{Y}, +, \mathbf{0}, \times, \mathbf{1})$$

- $F = (F_\times, F_+)\colon \mathbb{X} \to \mathbb{Y}$ is a strong symmetric medial linear functor.

□

Lemma

Consider a strong symmetric medial linear functor $F = (F_\otimes, F_\oplus)\colon \mathbb{X} \to \mathbb{Y}$ *and a bicommutative medial bimonoid* $\langle A, \Delta_A, t_A, \nabla_A, s_A \rangle$*, then* $\langle F_\otimes(A), \Delta_{F_\otimes(A)}, t_{F_\otimes(A)}, \nabla_{F_\otimes(A)}, s_{F_\otimes(A)} \rangle$ *defined by*

$$\Delta_{F_\otimes(A)} = F_\otimes(A) \xrightarrow{F_\otimes(\Delta_A)} F_\otimes(A \otimes A) \xrightarrow{m_{\otimes}{}_{A,A}^{-1}} F_\otimes(A) \otimes F_\otimes(A)$$

$$t_{F_\otimes(A)} = F_\otimes(A) \xrightarrow{F_\otimes(t_A)} F_\otimes(\top) \xrightarrow{m_\top^{-1}} \top$$

$$\nabla_{F_\otimes(A)} = F_\otimes(A) \oplus F_\otimes(A) \xrightarrow{m_{\oplus A,A}} F_\otimes(A \oplus A) \xrightarrow{F_\otimes(\nabla_A)} F_\otimes(A)$$

$$s_{F_\otimes(A)} = \bot \xrightarrow{m_\bot} F_\otimes(\bot) \xrightarrow{F_\otimes(s_A)} F_\otimes(A)$$

and $\langle F_\oplus(A), \Delta_{F_\oplus(A)}, t_{F_\oplus(A)}, \nabla_{F_\oplus(A)}, s_{F_\oplus(A)} \rangle$ *defined similarly are bicommutative medial bimonoids.*

As such, $F = (F_\otimes, F_\oplus)$ *canonically extends to a strong symmetric linear functor* $\mathrm{B}(F) = (\mathrm{B}(F_\otimes), \mathrm{B}(F_\oplus))\colon \mathrm{B}(\mathbb{X}) \to \mathrm{B}(\mathbb{Y})$.

Linearly distributive Fox theorem

Theorem

$\mathrm{B}(-)\colon \mathbf{MLDC_s} \to \mathbf{CLDC}$ *is right adjoint to the inclusion 2-functor.*

Corollary

A symmetric medial linearly distributive category is cartesian if and only if it is isomorphic to its category of bicommutative medial bimonoids.

Linearly distributive category of medial bimonoids

Proposition

Consider a distributive symmetric monoidal categories $(\mathcal{X}, \oslash, I)$ with biproducts, then $\mathrm{B}(\mathcal{X}) \cong \mathrm{CSGp}_{\oslash}(\mathcal{X})$.

Example

1. B(Vect) is isomorphic to the category of cocommutative $\oslash$-cosemigroups of Vect, where $\oslash$ is the standard tensor product of vector spaces
2. B(Rel) is isomorphic to the category of cocommutative $\oslash$-cosemigroups of Rel, where $\oslash$ is the cartesian product of sets

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