Lecture - Codensity Monads

Recall - Given an adjunction

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, bend left, "G"] \end{tikzcd}$$

then $(G \circ F, G\varepsilon F, \eta)$ is a monad on $\mathcal{C}$

and $(F \circ G, -, -)$ is a comonad on $\mathcal{D}$

But sometimes, even if there is no adjunction, there may be a monad anyway.

2 examples

1) $\text{FinSet} \hookrightarrow \text{Set}$

Exercise: Why is there no adjunction here?

The associated codensity monad is the ultrafilter monad and the algebras are the compact hausdorff spaces (Gildenhuys & Kennison)

2) $\text{Vec}_{fd} \hookrightarrow \text{Vec}$

Again, no adjunction.

In this case, the codensity monad is the double dual monad $(\ )^{**} : \mathrm{Vec} \to \mathrm{Vec}$.

The algebras for this monad are the linearly compact spaces, in the sense of Lefschetz. (Leinster)

Lots of others

1) $\mathrm{FinSet} \hookrightarrow \mathrm{Top}$ (Sipos)

discrete topology

2) The Giry monad can be realized as a codensity monad. (Avery)

3) If there is an adjunction, and the codensity construction exists, then we get the usual monad.

Lots of examples in archives.

Density

Intuition: Let $\mathcal{C}$ be a subcat of $\mathcal{D}$. $\mathcal{C}$ is dense if every object of $\mathcal{D}$ is a colimit of objects of $\mathcal{C}$.

But it is best stated in terms of arbitrary functors, not just inclusions.

Defn: A functor $F: S \to C$ is dense if every object $c \in Ob(C)$ is obtained via the following colimit

$$\mathrm{Colim}\left(F/_c \xrightarrow{\pi} S \xrightarrow{F} C \right) = c$$

where $F/_c$ is the comma category

objects are pairs $(d, g: Fd \to c)$

arrows $h: d \to d'$ s.t.

$$\begin{tikzcd} Fd \arrow[rr, "Fh"] \arrow[dr, "g"'] & & Fd' \arrow[dl, "g'"] \\ & c & \end{tikzcd}$$

π is the functor

$$\pi : F/_c \to S$$

defined by $(d, g : Fd \to c) \mapsto d$

Thm: TFAE

1) $F : S \to C$ is dense

2) The restricted yoneda embedding

$$Y : C \to Fun(S^{op}, Set)$$

is full & faithful. Exercise: Figure out what this is.

Examples

1) $\{*\} \hookrightarrow Set$

1 object discrete category

2) $FinSet \hookrightarrow Set$

3) $Grp_{Fg} \hookrightarrow Grp$

4) The Yoneda embedding

$$Y : \mathcal{C} \to \mathrm{Func}(\mathcal{C}^{op}, \mathrm{Set})$$

every functor is a colimit of representables.

Some weird facts

1) Dense functors are not closed under composition

2) Top, the category of topological spaces & continuous maps, has no small dense subcategory.

Defn: $F : A \to B$ is <u>codense</u> if $\forall\, b \in Ob(B)$

$$\mathrm{Lim}\big((b \downarrow F) \xrightarrow{\Pi} A \xrightarrow{F} B\big) = b$$

Codense functors are harder to come by.

Codensity monads are a way of measuring its failure to be codense

Def: Let $G : \mathbb{B} \to \mathbb{A}$ be a functor. The <u>codensity monad</u>, if it exists, is defined via <u>ends</u>

$$T^G(A) = \int_{B \in \mathbb{B}} \underbrace{G(B)^{\mathbb{A}(A, GB)}}$$

the power of $G(B)$ $|\mathbb{A}(A, GB)|$-times

This is equivalent to saying

$$T^G(A) = \mathrm{Lim}\left(A \downarrow G \xrightarrow{\pi} \mathbb{B} \xrightarrow{G} \mathbb{A}\right)$$

so, clearly G is codense if and only if $T^G \simeq id_{\mathbb{A}}$

Thm: - T^G is always a monad
- If G has a left adjoint, then T^G is the usual monad

- T^G can be equivalently defined in terms of Kan extensions.

Suppose $G : \mathcal{B} \to Set$

One can prove:

$$T^G(A) = \{\rho : \rho \text{ is a natural transformation } G^A \to G\}$$

a set

Double Dualization

$( )^{**} : Vec \longrightarrow Vec$ is a monad

It is induced by the self-adjoint

$$( )^* : Vec^{op} \to Vec$$

Calculation:

$$\mathrm{Hom}_{Vec}(A, B^*) \cong \mathrm{Hom}_{Vec}(A, B \to k)$$

$$\cong \mathrm{Hom}_{Vec}(A \otimes B, k) \cong \mathrm{Hom}_{Vec}(B \otimes A, k) \cong \mathrm{Hom}_{Vec}(B, A^*)$$

$$\cong \mathrm{Hom}_{Vec^{op}}(A^*, B)$$

This adjunction gives a monad on Vec

It's a much more general result:

Thm: In any symmetric monoidal closed category, for any object c, the functor

$(-) \multimap c : \mathcal{C}^{op} \to \mathcal{C}$ is self-adjoint and induces a monad on $\mathcal{C}$.

<u>Another approach</u>

Kleisli Triples (Ernie Manes)

Equivalent to a monad

$$T : Ob(\mathcal{C}) \to Ob(\mathcal{C})$$

arrows for all A $\quad \eta_A : A \to TA$

lifting $\quad \dfrac{A \to TB}{f^{\#} : TA \to TB}$

satisfying some equations.

Exercise for students. Construct lifting operator.

Thm (Leinster) Doubledualization is in fact the codensity monad for the functor

$$\mathrm{Vec}_{fd} \hookrightarrow \mathrm{Vec}$$

The proof is a calculation directly from the definition.

What are the T-algebras?

Def'n: Let $G: \mathcal{C} \to \mathcal{D}$ be a functor with a left adjoint F. Then F factors through the forgetful functor

$$U: \mathcal{C}^T \to \mathcal{C}$$

So we get

$$\begin{tikzcd} F: \mathcal{D} \arrow[r, "F"] \arrow[dr, "\hat{F}"'] & \mathcal{C} \\ & \mathcal{C}^T \arrow[u, "U"'] \end{tikzcd}$$

G is <u>monadic</u> if $\hat{F}$ is an equivalence of categories.

There's a thm to check for monadicity, called Beck's Monadicity Thm for determining monadicity.

Thm: $U: \mathcal{C} \to \mathcal{D}$ is monadic iff

1) U has a left adjoint
2) U reflects isomorphisms, i.e. $U(f)$ an iso implies f an iso
3) $\mathcal{C}$ has coequalizers of U-split parallel pairs, and U preserves those coequalizers

Thm (Leinster)

$$(\ )^*: \mathrm{Vec}^{op} \to \mathrm{Vec} \quad \text{is monadic}$$

Corollary

The category of T-algebras for the double dualization monad is equivalent to Vec^{op}

Solomon Lefschetz defined the notion of linear topology to have an ∞-dimensional strong duality

His category LTopMod is symmetric monoidal closed AND the usual embedding

$$\rho : V \longrightarrow V^{**}$$

is always a bijection

When one restricts to all objects such that ρ is an isomorphism, one gets a $*$-autonomous category

This is the category which led to the definition (Barr)

A topology on a k-vector space V is linear if it has a nbhd basis of 0 consisting of open linear subspaces. We'll use Lefschetz for linear

Cat is called LTopMod. NB Mod is the discrete subcategory

Lemma: LTopMod is an Ab-category

Thm TFAE for a Lefschetz space A

1) A is complete, and all its open subspaces (linear) are of finite codimension
2) A is isomorphic to the dual V^* of a discrete vector space V, with pointwise topology
3) A is iso to k^X with X a discrete set and with pointwise topology
4) A is the closure of a linear span of a compact subset of A.
5) Every filter of closed subspaces of A has nonempty intersection.

Def: A space satisfying the above is called linearly compact

T3

Thm (Lefschetz)

$$\mathrm{Vec}^{op} \simeq \underbrace{\mathrm{LCPT}}$$

linearly compact spaces
& linearly ~~compact~~ continuous functions

See Prop 24.8 of Bergman & Hausknecht

$$\mathrm{Vec}^{op} \longrightarrow \mathrm{LCPT}$$

$V \mapsto V^*$ topologized with topology of pointwise convergence

$$\longleftarrow$$

$$\mathrm{Hom}_{\mathrm{LCPT}}(A, k) \longleftarrow\!\shortmid A$$

We have the following analogy

~~Finite~~

sets $\longrightarrow$ compact Hausdorff spaces

vector spaces $\longrightarrow$ linearly compact vector spaces

Can this analogy be generalized?

References

1) Tom Leinster: Codensity & the ultrafilter monad

2) Tom Avery: Codensity & The Giry Monad

3) Bergman & Hausknecht: Cogroups & Co-rings in categories of associative rings