The Para construction as a Wreath product.

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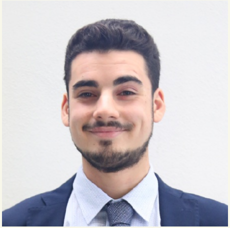

"You are now about to witness the strength of Street knowledge"

Double Categories of Contextualized Maps

<table>
<tr><th>Construction</th><th>Input</th><th>Output $\begin{tikzcd} A \arrow[r, "p", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\alpha \Downarrow"] & C \arrow[d, "g"] \\ B \arrow[r, "q"', "\shortmid"{marking}] & D \end{tikzcd}$</th></tr>
<tr><td>Span</td><td>A cat $\mathcal{A}$ with pullbacks
(a class $\mathcal{E}$ of display maps</td><td>$\begin{tikzcd} A \arrow[d, "f"'] & P \arrow[l] \arrow[r, "p"] \arrow[d, "\alpha"'] & C \arrow[d, "g"] \\ B & Q \arrow[l] \arrow[r, "q"'] & D \end{tikzcd}$</td></tr>
<tr><td>Cokliesli</td><td>A comonad
$\mathbb{C} : \mathcal{A} \longrightarrow \mathcal{A}$</td><td>$\begin{tikzcd} \mathbb{C}A \arrow[r, "p"] \arrow[d, "\mathbb{C}f"'] \arrow[dr, phantom, "\circlearrowright"] & C \arrow[d, "g"] \\ \mathbb{C}B \arrow[r, "q"'] & D \end{tikzcd}$
α is the fact that this commutes</td></tr>
<tr><td>Para</td><td>An actegory
$\odot : \mathcal{E} \times \mathcal{A} \longrightarrow \mathcal{A}$
with $(\mathcal{E}, \otimes, \mathbb{1})$ a monoidal cat</td><td>$\begin{tikzcd} P \odot A \arrow[r, "p"] \arrow[d, "\alpha \odot f"'] & C \arrow[d, "g"] \\ Q \odot B \arrow[r, "q"'] & D \end{tikzcd}$</td></tr>
</table>

Composition is by combining contexts

Span

$$
\begin{tikzcd}
A_1 \arrow[ddd, "a"'] & P_1 \arrow[l] \arrow[r, "p"] \arrow[ddd, "\alpha"'] & B_1 & Q_1 \arrow[l] \arrow[r, "q"] \arrow[ddd, "\beta"] & C_1 \arrow[ddd, "c"] \\
& & P_1 \times_{B_1} Q_1 \arrow[ul] \arrow[ur, "p \times Q_1"] \arrow[d, "\alpha \times_b \beta"] & & \\
& & P_2 \times_{B_2} Q_2 \arrow[dl] \arrow[dr] & & \\
A_2 & P_2 \arrow[l] \arrow[r] & B_2 & Q_2 \arrow[l] \arrow[r] & C_2
\end{tikzcd}
$$

Cokliesli

$$
\begin{tikzcd}
CA_1 \arrow[r, "\delta"] \arrow[d, "Ca"'] \arrow[dr, phantom, "\circlearrowright"] & CCA_1 \arrow[r, "CP_1"] \arrow[d, "CCa"'] \arrow[dr, phantom, "C\alpha"] & CB_1 \arrow[r, "q_1"] \arrow[d, "Cb"'] \arrow[dr, phantom, "\beta"] & C_1 \arrow[d, "c"] \\
CA_2 \arrow[r, "\delta"] & CCA_2 \arrow[r, "CP_2"'] & CB_2 \arrow[r, "q_2"'] & C_2
\end{tikzcd}
$$

Para

$$
\begin{tikzcd}
(Q_1 \otimes P_1) \odot A_1 \arrow[r, "\sim"] \arrow[d, "(\beta \otimes \alpha) \odot a"'] \arrow[dr, phantom, "\circlearrowright"] & Q_1 \odot (P_1 \odot A_1) \arrow[r, "Q_1 \odot p_1"] \arrow[d, "\beta \odot (\alpha \odot a)"'] & Q_1 \odot B_1 \arrow[r, "q_1"] \arrow[d, "\beta \odot b"'] & C_1 \arrow[d, "c"] \\
(Q_2 \otimes P_2) \odot A_2 \arrow[r, "\sim"] & Q_2 \odot (P_2 \odot A_2) \arrow[r, "Q_2 \odot P_2"'] & Q_1 \odot B_2 \arrow[r, "q_2"'] & C_2
\end{tikzcd}
$$

The goal of this talk is to unify these three constructions (and their duals) into

$$\mathrm{Ctx} : \{\text{Contextads}\} \longrightarrow \mathrm{DoubleCat}_h$$

— or —

$$\mathrm{Cokleisli} : \left\{\begin{matrix}\text{Dependently graded} \\ \text{comonads}\end{matrix}\right\} \longrightarrow \mathrm{DoubleCat}_h$$

— or —

$$\mathrm{Para} : \left\{\begin{matrix}\text{fibred colax} \\ \text{actions}\end{matrix}\right\} \longrightarrow \mathrm{DoubleCat}_h$$

The Ctx-construction will be the wreath product of pseudomonads in Span(Cat),

So also in Span($\mathbb{K}$) for other 2-cats, including $\mathrm{Alg}_{colax}(T)$, giving us colaxly T-structure double cats

Dbl cats are pseudomonads in Span(Cat) — Distributive Law?

Constr	Squares	Category of Squares

Span

$$\begin{tikzcd}
A \arrow[d, "f"'] & P \arrow[l] \arrow[r, "p"] \arrow[d, "\alpha"'] & C \arrow[d, "g"] \\
B & Q \arrow[l] \arrow[r, "q"'] & D
\end{tikzcd}$$

$$\begin{tikzcd}
& & \mathrm{Span}(\mathbb{A})_1 \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\vee"] \arrow[ddll, bend right] \arrow[ddrr, bend left] & & \\
& \mathbb{A}^{\downarrow} \arrow[dl, "\mathrm{cod}"'] \arrow[dr, "\mathrm{dom}"] & & \mathbb{A}^{\downarrow} \arrow[dl, "\mathrm{dom}"'] \arrow[dr, "\mathrm{cod}"] & \\
\mathbb{A} & & \mathbb{A} & & \mathbb{A}
\end{tikzcd}$$

Double category

Cokliesli

$$\begin{tikzcd}
\mathcal{C}A \arrow[r, "p"] \arrow[d, "\mathcal{C}f"'] & C \arrow[d, "g"] \\
\mathcal{C}B \arrow[r, "q"'] & D
\end{tikzcd}$$

$$\begin{tikzcd}
& & \mathrm{Cokliesli}(\mathcal{C})_1 \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\vee"] \arrow[ddll, bend right] \arrow[ddrr, bend left] & & \\
& \mathbb{A} \arrow[dl, equal] \arrow[dr, "\mathcal{C}"] & & \mathbb{A}^{\downarrow} \arrow[dl, "\mathrm{dom}"'] \arrow[dr, "\mathrm{cod}"] & \\
\mathbb{A} & & \mathbb{A} & & \mathbb{A}
\end{tikzcd}$$

Double category

Para

$$\begin{tikzcd}
P \odot A \arrow[r, "p"] \arrow[d, "\alpha \odot f"'] & C \arrow[d, "g"] \\
Q \odot B \arrow[r, "q"'] & D
\end{tikzcd}$$

$$\begin{tikzcd}
& & \mathrm{Para}(\odot)_1 \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\vee"] \arrow[ddll, bend right] \arrow[ddrr, bend left] & & \\
& \mathcal{C} \times \mathbb{A} \arrow[dl, "\pi"'] \arrow[dr, "\odot"] & & \mathbb{A}^{\downarrow} \arrow[dl, "\mathrm{dom}"'] \arrow[dr, "\mathrm{cod}"] & \\
\mathbb{A} & & \mathbb{A} & & \mathbb{A}
\end{tikzcd}$$

Double category

Pseudomonads in $f\mathrm{Span}^{\Rightarrow}$: Combining contexts

Span

$$\begin{tikzcd} A \arrow[r, equal] & A \end{tikzcd}$$

$$\begin{tikzcd} A & P \arrow[l] & Q \arrow[l] \arrow[ll, bend right] \end{tikzcd}$$

$$\begin{tikzcd}
A \arrow[r, equal] \arrow[d, equal] & A \arrow[r, equal] \arrow[d, "id"'] & A \arrow[d, equal] \\
A & A^{\downarrow} \arrow[l, "cod"] \arrow[r, "dom"'] & A
\end{tikzcd}$$

$$\begin{tikzcd}
A \arrow[dd, equal] & A^{\downarrow} \arrow[l, "cod"'] \arrow[r, "dom"] & A & A^{\downarrow} \arrow[l, "cod"'] \arrow[r, "dom"] & A \arrow[dd, equal] \\
 & & \hat{A}^{\downarrow} \arrow[ul] \arrow[ur] \arrow[d, "\circ"'] & & \\
A & & A^{\downarrow} \arrow[ll, "cod"] \arrow[rr, "dom"'] & & A
\end{tikzcd}$$

Cokleisli

$$\mathbb{C}A \xrightarrow{\varepsilon} A$$

$$\mathbb{C}A \xrightarrow{\delta} \mathbb{C}\mathbb{C}A$$

$$\begin{tikzcd}
A \arrow[r, equal] \arrow[d, equal] & A \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\varepsilon \Nearrow"] & A \arrow[d, equal] \\
A \arrow[r, equal] & A \arrow[r, "\mathbb{C}"'] & A
\end{tikzcd}$$

$$\begin{tikzcd}
A \arrow[r, equal] \arrow[dd, equal] & A \arrow[r, "\mathbb{C}"] & A \arrow[r, equal] & A \arrow[r, "\mathbb{C}"] & A \arrow[dd, equal] \\
 & & \hat{A} \arrow[ul, equal] \arrow[ur, "\mathbb{C}"'] \arrow[d, equal] \arrow[drr, phantom, "\delta \Nearrow"] & & \\
A \arrow[rr, equal] & & A \arrow[rr, "\mathbb{C}"'] & & A
\end{tikzcd}$$

Para

$$\mathbb{1} \odot A \underset{\sim}{\xrightarrow{\varepsilon}} A$$

$$\begin{tikzcd}
(C_2 \otimes C_1) \odot A \arrow[d, "\delta"', "\wr"] \\
C_2 \odot (C_1 \odot A)
\end{tikzcd}$$

$$\begin{tikzcd}
A \arrow[r, equal] \arrow[d, equal] & A \arrow[r, equal] \arrow[d, "\mathbb{1}\times A"'] \arrow[dr, phantom, "\varepsilon \Nearrow \wr"] & A \arrow[d, equal] \\
A & e\times A \arrow[l] \arrow[r, "\odot"'] & A
\end{tikzcd}$$

$$\begin{tikzcd}
A \arrow[dd, equal] & e\times A \arrow[l, "\pi"'] \arrow[r, "\odot"] & A & e\times A \arrow[l, "\pi"'] \arrow[r, "\odot"] & A \arrow[dd, equal] \\
 & & \widehat{e\times e\times A} \arrow[ul] \arrow[ur, "e\times\odot"'] \arrow[d, "\otimes\times A"'] \arrow[drr, phantom, "\delta \Nearrow \wr"] & & \\
A & & e\times A \arrow[ll] \arrow[rr, "\odot"'] & & A
\end{tikzcd}$$

Dbl cats are pseudomonads in Span(Cat) — Distributive Law?

Constr — Squares — Category of Squares

Span

Squares

$$\begin{tikzcd}
A \arrow[d, "f"'] & P \arrow[l] \arrow[r, "p"] \arrow[d, "\alpha"] & C \arrow[d, "g"] \\
B & Q \arrow[l] \arrow[r, "q"'] & D
\end{tikzcd}$$

Category of Squares

Fibration!

$$\begin{tikzcd}
 & & \mathrm{Span}(A)_1 \arrow[dl] \arrow[dr] \arrow[dll, bend right] \arrow[drr, bend left] \arrow[d, phantom, "\vee"] & & \\
 & A^{\downarrow} \arrow[dl, "\mathrm{cod}"'] \arrow[dr, "\mathrm{dom}"] & {} & A^{\downarrow} \arrow[dl, "\mathrm{dom}"'] \arrow[dr, "\mathrm{cod}"] & \\
A & & A & & A
\end{tikzcd}$$

CoKliesli

Squares

$$\begin{tikzcd}
CA \arrow[r, "p"] \arrow[d, "Cf"'] \arrow[dr, phantom, "\circlearrowright"] & C \arrow[d, "g"] \\
CB \arrow[r, "q"'] & D
\end{tikzcd}$$

Category of Squares

Fibration!

$$\begin{tikzcd}
 & & \mathrm{CoKliesli}(C)_1 \arrow[dl] \arrow[dr] \arrow[dll, bend right] \arrow[drr, bend left] \arrow[d, phantom, "\vee"] & & \\
 & A \arrow[dl, equal] \arrow[dr, "C"] & {} & A^{\downarrow} \arrow[dl, "\mathrm{dom}"'] \arrow[dr, "\mathrm{cod}"] & \\
A & & A & & A
\end{tikzcd}$$

Para

Squares

$$\begin{tikzcd}
P \odot A \arrow[r, "p"] \arrow[d, "\alpha \odot f"'] & C \arrow[d, "g"] \\
Q \odot B \arrow[r, "q"'] & D
\end{tikzcd}$$

Category of Squares

Fibration!

$$\begin{tikzcd}
 & & \mathrm{Para}(\odot)_1 \arrow[dl] \arrow[dr] \arrow[dll, bend right] \arrow[drr, bend left] \arrow[d, phantom, "\vee"] & & \\
 & \mathcal{C} \times A \arrow[dl, "\pi"'] \arrow[dr, "\odot"] & {} & A^{\downarrow} \arrow[dl, "\mathrm{dom}"'] \arrow[dr, "\mathrm{cod}"] & \\
A & & A & & A
\end{tikzcd}$$

Composing Pseudo-monads — Distributive Law?

Constr	λ	Category of Squares

Span

λ:

$$\begin{tikzcd}
A & P \arrow[l] \arrow[r, "p"] & B & Q \arrow[l, "b"'] \arrow[r] & C \\
& P\,{}_p\!\times_b Q \arrow[u, "\hat{}"] \arrow[urr, "p\times_b Q"'] & & &
\end{tikzcd}$$

Category of Squares:

$$\begin{tikzcd}
& (A^\downarrow)^\downarrow \arrow[rr, "dom^\downarrow"] & & A^\downarrow \arrow[dr, "dom"] & \\
A^\downarrow{}_{cod}\!\times_{cod} A^\downarrow \arrow[ur, "lift"] \arrow[r, dashed, "\lambda"] \arrow[drrr, bend right, "pull"'] & A^\downarrow{}_{dom}\!\times_{dom} A^\downarrow \arrow[urr] \arrow[drr] & & & A \\
& & & A^\downarrow \arrow[ur, "dom"'] &
\end{tikzcd}$$

Cokleisli

λ:

$\mathbb{C}A \xrightarrow{p} B, \quad \mathbb{C}B \to C$

$$\begin{tikzcd}
& \mathbb{C}\mathbb{C}A \arrow[r, "\mathbb{C}p"] & \mathbb{C}B \arrow[r] & C \\
\mathbb{C}A \arrow[ur] & & &
\end{tikzcd}$$

Category of Squares:

$$\begin{tikzcd}
& A^\downarrow \arrow[rr, "\mathbb{C}^\downarrow"] & & A^\downarrow \arrow[dr, "dom"] & \\
A^\downarrow{}_{cod}\!\times_{id} A \arrow[ur, "lift"] \arrow[r, dashed, "\lambda"] \arrow[drrr, bend right, "pull"'] & A_{\mathbb{C}}\!\times_{dom} A^\downarrow \arrow[urr] \arrow[drr] & & & A \\
& & & A \arrow[ur, "\mathbb{C}"'] &
\end{tikzcd}$$

Para

λ:

$P\odot A \xrightarrow{p} B, \quad Q\odot B \to C$

$$\begin{tikzcd}
Q\odot(P\odot A) \arrow[r, "Q\odot P"] & Q\odot B \arrow[d] \\
(Q\otimes P)\odot A \arrow[u] & C
\end{tikzcd}$$

Category of Squares:

$$\begin{tikzcd}
& (\mathcal{C}\times A)^\downarrow \arrow[rr, "\odot^\downarrow"] & & A^\downarrow \arrow[dr, "dom"] & \\
A^\downarrow{}_{cod}\!\times_{\pi} \mathcal{C}\times A \arrow[ur, "lift"] \arrow[r, dashed, "\lambda"] \arrow[drrr, bend right, "pull"'] & \mathcal{C}\times A_{\odot}\!\times_{dom} A^\downarrow \arrow[urr] \arrow[drr] & & & A \\
& & & \mathcal{C}\times A \arrow[ur, "\odot"'] &
\end{tikzcd}$$

Contextuals

Def: A *paradise* $(\mathbb{K}, \mathbb{D})$ consists of a 2-cat $\mathbb{K}$ (think Cat) equipped with a class $\mathbb{D}$ of "display maps", st

1. $\mathbb{D}$ contains all isomorphisms and is stable under strict pullback,
2. $\mathbb{K}$ has arrow objects $A^{\downarrow}$, and $\mathrm{dom} : A^{\downarrow} \to A$ is in $\mathbb{D}$

Def: Given a paradise $\mathbb{K}$, define the 2-categories

$$\mathrm{dSpan}^{\Rightarrow}(\mathbb{K})(A_1, A_2) := \left\{ \begin{tikzcd} A_1 \arrow[d, equal] & C_1 \arrow[l, "\pi_1"'] \arrow[r, "f_1"] \arrow[d, bend right, "c_1"'] \arrow[d, bend left, "c_2"] \arrow[d, phantom, "\overset{\alpha}{\Rightarrow}"] \arrow[dr, phantom, "\overset{\gamma_1}{\Rightarrow}\ \overset{\gamma_2}{\Rightarrow}"] & A_2 \arrow[d, equal] \\ A_1 & C_2 \arrow[l, "\pi_1"] \arrow[r, "f_2"'] & A_2 \end{tikzcd} \right\}$$

And tricategories

$$\mathrm{Span}(\mathbb{K}) := \{ \cdot\cdot \mid \gamma = \mathrm{id} \},$$

$$\mathrm{fSpan}^{\Rightarrow}(\mathbb{K}) := \left\{ \cdots \;\middle|\; \begin{array}{l} \pi \text{ is a fibration} \\ C \text{ is cartesian} \end{array} \right\}$$

Contextads

Def: A category internal to $\mathbb{K}$ ("Double category") is a pseudo-monad in $\mathbb{S}\mathrm{pan}(\mathbb{K})$

Def: A Contextad is a pseudo-monad in $\mathrm{fSpan}^{\to}(\mathbb{K})$

$$\begin{tikzcd}
A \arrow[r, equal] \arrow[d, equal] & A \arrow[r, equal] \arrow[d, phantom, "\mathbb{I}\Downarrow"] \arrow[dr, phantom, "\Nearrow\varepsilon"] & A \arrow[d, equal] \\
A & e \arrow[l, "\pi"] \arrow[r, "\odot"'] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[dd, equal] & e \arrow[l] \arrow[r] & A & e \arrow[l] \arrow[r, "\odot"] & A \arrow[dd, equal] \\
 & & e \times_A e \arrow[ul] \arrow[ur] \arrow[d, phantom, "\otimes\Downarrow"] & \arrow[d, phantom, "\Nearrow\delta"] & \\
A & & e \arrow[ll, "\pi"] \arrow[rr, "\odot"'] & & A
\end{tikzcd}$$

So, Ctx should take pseudo-monads in $\mathrm{fSpan}^{\Rightarrow}(\mathbb{K})$ to pseudo-monads in $\mathbb{S}\mathrm{pan}(\mathbb{K})$...

... by "distributing over $A \xleftarrow{\mathrm{dom}} A^{\downarrow} \xrightarrow{\mathrm{cod}} A$", the double category of squares in A.

The Ctx-Construction as a Wreath Product

Lemma (M.-Capucci):

$$\mathrm{dSpan}^{\Rightarrow}(A,B) \cong \mathrm{Kl}\left(- \times_{\mathrm{dom}} B^{\downarrow}, \mathrm{dSpan}(A,B)\right)$$

$$\left\{\begin{tikzcd}
A \arrow[d, equal] & e_1 \arrow[l, two heads, "\pi_1"'] \arrow[r, "f_1"] \arrow[d, bend right, "c_1"'] \arrow[d, bend left, "c_2"] \arrow[d, phantom, "\Rightarrow \alpha"] \arrow[dr, phantom, "\Rightarrow \gamma_1,\ \Rightarrow \gamma_2"] & B \arrow[d, equal] \\
A & e_2 \arrow[l, two heads, "\pi_1"] \arrow[r, "f_2"'] & B
\end{tikzcd}\right\} \cong \left\{\begin{tikzcd}
A \arrow[d, equal] & e_1 \arrow[l, two heads, "\pi_1"'] \arrow[r, "f_1"] \arrow[d, bend right, "(c_1,\gamma_1)"'] \arrow[d, bend left, "(c_2,\gamma_2)"] \arrow[d, phantom, "\Rightarrow (\alpha, id)"] & B \arrow[d, equal] \\
A & e_2 \times_{f_2, \mathrm{dom}} B^{\downarrow} \arrow[l, two heads, "\pi_1"] \arrow[r, "\mathrm{cod}"'] & B
\end{tikzcd}\right\},$$

Theorem (Street; M.-Capucci):

$$\mathrm{fSpan}^{\Rightarrow}(A,B) \cong \mathrm{Alg}\left(A^{\downarrow} \times_{\mathrm{cod}} -, \mathrm{dSpan}^{\Rightarrow}(A,B)\right)$$

$$\begin{tikzcd}
A \arrow[d, equal] & A^{\downarrow} \times_{\mathrm{cod}, \pi} e \arrow[l, two heads, "\mathrm{dom}"'] \arrow[r, "f"] \arrow[d, "\mathrm{pull}"'] \arrow[dr, phantom, "\Uparrow \mathrm{flift}"] & B \arrow[d, equal] \\
A & e \arrow[l, two heads, "\pi"] \arrow[r, "f"'] & B
\end{tikzcd}$$

Lemma (M.-Capucci):

$$\mathbb{d}\mathrm{Span}^{\Rightarrow}(A,B) \cong \mathrm{Kl}(- \times_{dom} B^{\downarrow}, \mathbb{d}\mathrm{Span}(A,B))$$

Theorem (Street; M.-Capucci):

$$\mathrm{fSpan}^{\Rightarrow}(A,B) \cong \mathrm{Alg}(A^{\downarrow} \times_{cod} -, \mathbb{d}\mathrm{Span}^{\Rightarrow}(A,B))$$

Putting these together:

$$\mathrm{fSpan}^{\Rightarrow}(A,B) \cong \mathrm{Alg}(A^{\downarrow} \times_{cod} -, \mathrm{Kl}(- \times_{dom} B^{\downarrow}, \mathbb{d}\mathrm{Span}(\mathbb{K})))$$

That is: $\mathrm{fSpan}^{\Rightarrow} \xhookrightarrow{ff} \mathrm{KL}(\mathbb{d}\mathrm{Span})$

where KL is the free-cocompletion under Kleisli objects of pseudomonads, following Lack-Street and using Adrian Miranda's thesis.

The Ctx-Construction as a Wreath Product

Thm: $; : \mathrm{fSpan}^{\Rightarrow} \xhookrightarrow{\text{ff}} \mathrm{KL}(\mathrm{dSpan})$

Def: The Ctx construction is

$$\mathrm{KL}(\mathrm{fSpan}^{\Rightarrow}(\mathbb{K})) \xrightarrow{\mathrm{KL}(;)} \mathrm{KL}(\mathrm{KL}(\mathrm{dSpan}(\mathbb{K}))) \xrightarrow{\text{wreath product}} \mathrm{KL}(;\mathrm{Span}(\mathbb{K}))$$

This sends $A \xleftarrow{\pi} e \xrightarrow{\odot} A$, $e \,{}_{\odot}{\times}_{\pi}\, e \xrightarrow{(\otimes,\delta)} e \,{}_{\odot}{\times}_{\mathrm{dom}}\, A^{\downarrow}$ to

$$e \,{}_{\odot}{\times}_{\mathrm{dom}}\, A^{\downarrow} \,{}_{\mathrm{cod}}{\times}_{\pi}\, e \,{}_{\odot}{\times}_{\mathrm{dom}}\, A^{\downarrow} \xrightarrow{e \times \lambda \times A^{\downarrow}} e \,{}_{\odot}{\times}_{\pi}\, e \,{}_{\odot}{\times}_{\mathrm{dom}}\, A^{\downarrow} \,{}_{\mathrm{cod}}{\times}_{\mathrm{dom}}\, A^{\downarrow} \xrightarrow{e \times e \times \circ} e \,{}_{\odot}{\times}_{\pi}\, e \,{}_{\odot}{\times}_{\mathrm{dom}}\, A^{\downarrow}$$

$$\xrightarrow{(\otimes,\delta) \times A^{\downarrow}} e \,{}_{\odot}{\times}_{\mathrm{dom}}\, A^{\downarrow} \,{}_{\mathrm{cod}}{\times}_{\mathrm{dom}}\, A^{\downarrow} \longrightarrow e \,{}_{\odot}{\times}_{\mathrm{dom}}\, A^{\downarrow}$$

Structure on the Ctx Construction

- 2-algebraic structure (e.g. symmetric monoidal) is governed by 2-monads

  Let $\mathrm{Alg}_c(T)$ be the 2-cat of (strict) algebras and colax maps

$$\begin{tikzcd} TA_1 \arrow[r, "Tf"] \arrow[d, "a_1"'] \arrow[dr, phantom, "\overset{\bar{f}}{\nearrow}"] & TA_2 \arrow[d, "a_2"] \\ A_1 \arrow[r, "f"'] & A_2 \end{tikzcd}$$

Def: Let $T : \mathbb{K} \to \mathbb{K}$ be a 2-monad. A colaxly T-structured double category is a pseudomonoid in $\mathrm{dSpan}(\mathbb{K})$

$$\begin{tikzcd} & D_1 \arrow[dl, "s"'] \arrow[dr, "t"] & \\ D_0 & & D_0 \end{tikzcd}$$

where
1. D_0 and D_1 are T-algebras
2. s and t are strict T-morphisms
3. All other structure is colax for T.

E.g. For $T = SMC$, this gives lax-monoidal double cats.

Structure on the Ctx Construction

Thm (M.-Capucci): If $A \xleftarrow{\pi} \mathcal{E} \xrightarrow{\odot} A$ is a contexted in $\mathbb{K}$

st ① A and $\mathcal{E}$ are T-algebras

② π is a ~~strict~~ T-algebra fibration (and normal as a fibration)

③ $\odot, \otimes, \mathbb{I}$ are all colax T-morphisms, and all structure isos are T-2-cells.

Then $\mathrm{Ctx}(\odot)$ is a colaxly T-structured double category.

Lemma: If $(\mathbb{K}, \mathbb{D})$ is a paradise and $T : \mathbb{K} \to \mathbb{K}$ a 2-monad, then $(\mathrm{Alg}_c(T), \{\text{strict display normal fibrations}\})$ is also a paradise.

Proof of thm: Run the Ctx-construction in $\mathrm{Alg}_c(T)$, then check the target is strict!

Structure on the Ctx Construction

Thm (M.-Capucci): If $A \xleftarrow{\pi} \mathcal{C} \xrightarrow{\odot} A$ is a contexted in $\mathbb{K}$

st ① A and $\mathcal{C}$ are T-algebras

② π is a strict T-algebra fibration (and normal as a fibration)

③ $\odot$, $\otimes$, $\amalg$ are all colax T-morphisms, and all structure isos are T-2-cells.

Then $\mathrm{Ctx}(\odot)$ is a colaxly T-structured double category.

Corollaries:

① If A and $\mathcal{C}$ have some limits and π preserves them, then $\mathrm{Ctx}(\odot)$ has them laxly.

If $\odot$, $\otimes$, $\amalg$ preserve them it has them dbl theoretically.

② A colax monoidal comonad has a monoidal Kliesli double cat.

③ A duoidal action of a duoidal cat on a monoidal cat has a lax monoidal Para-Double cat.

- in particular, an SMC acting on itself has SMC Para,

④ $\mathrm{Span}(A)$ is a cartesian double cat.

Thanks

Paper on arXiv soon