# Joyal's representation theorem for Heyting categories

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**Theorem (Joyal)**

*For any small Heyting category $\mathcal{H}$, there is a small category $\mathbb{C}$ and a conservative Heyting functor $\mathcal{H} \hookrightarrow \mathbf{Set}^{\mathbb{C}}$.*

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A model theoretic proof has been presented by Makkai and Reyes in 1977.

**Goal :** to provide a categorical approach

- Posetal case : **Heyting algebras**
  *For any Heyting algebra $\mathcal{H}$, there is a poset $X$ and an injective homomorphism of Heyting algebras $\mathcal{H} \hookrightarrow \mathbb{2}^X$*

- **Stone representation theorem**
  *For any Boolean algebra $\mathcal{B}$, there is a set $X$ and an injective Boolean homomorphism $\mathcal{B} \hookrightarrow 2^X$*

**Representation theorems $\Leftrightarrow$ completeness theorems**

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through the construction of *syntactic categories*, build from theories.

- For any coherent theory $\mathbb{T}$, there is a coherent category $\mathcal{C}_{\mathbb{T}}$ such that

$$\mathrm{Coh}\left(\mathcal{C}_{\mathbb{T}}, \mathbf{Set}\right) \simeq \mathrm{Mod}(\mathbb{T})$$

- A finitely complete category $\mathcal{C}$ is *regular* if and only if :
  - any arrow in $\mathcal{C}$ factorizes as a regular epimorphism followed by a monomorphism;
  - these factorizations are pullback-stable.
- A *coherent category* is a regular category in which posets of subobjects $\mathrm{Sub}(a)$ have finite unions (i.e coproducts) and each pullback functor $f^{-1} : \mathrm{Sub}(b) \to \mathrm{Sub}(a)$ preserves them.

# Introduction - Heyting categories

- A *Heyting category* is a coherent category in which for each map $f : a \to b$, the pullback functor $f^{-1} : \mathrm{Sub}(b) \to \mathrm{Sub}(a)$ has a right adjoint $\forall_f$ :

$$\mathrm{Sub}(b) \underset{\forall_f}{\overset{f^{-1}}{\underset{\bot}{\rightleftarrows}}} \mathrm{Sub}(a)$$

**Example :** Any presheaf category. In particular, $\mathrm{PSh}(\mathbb{C}^{\mathrm{op}}) = \mathbf{Set}^{\mathbb{C}}$

Let $\mathcal{H}$ be a small coherent category.

**Theorem**

*For any morphism $f : a \to b$ in $\mathcal{H}$, if $M(a) \cong M(b)$ for all coherent functor $M : \mathcal{H} \to \mathbf{Set}$, then $f$ is an isomorphism.*

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$\Rightarrow$ Gödel completeness for first-order logic

## Sketch of the proof

**Notations :**

- $\mathrm{Lex}(\mathcal{C}, \mathcal{D})$ : category of left exact functors between finitely complete categories $\mathcal{C}$ and $\mathcal{D}$
- $\mathrm{Coh}(\mathcal{C}, \mathcal{D})$ : category of coherent functors between coherent categories $\mathcal{C}$ and $\mathcal{D}$

### Theorem (Joyal)

*For any small Heyting category $\mathcal{H}$, there is a small category $\mathbb{C}$ and a conservative Heyting functor $\mathcal{H} \hookrightarrow \mathbf{Set}^{\mathbb{C}}$.*

For the proof, we show that $\mathbb{C}$ can be taken to be the category of coherent functors $\mathrm{Coh}(\mathcal{H}, \mathbf{Set})$ and that the functor is given by :

$$ev : \mathcal{H} \hookrightarrow \mathrm{Set}^{\mathbb{C}}$$

$$\begin{array}{ccc} a & \longmapsto & (F \mapsto F(a)) \\ \downarrow f & & \downarrow ev(f)_F = Ff \\ b & \longmapsto & (F \mapsto F(b)) \end{array}$$

- Conservativity of the functor : Deligne's theorem
- Coherence of the functor : $\mathbb{C} = \mathrm{Coh}(\mathcal{H}, \mathbf{Set})$
- Preservation of the Heyting structure : we need to show that for any $f : a \to b$, and for any $u \in \mathsf{Sub}(a)$,

$$\forall_{ev(f)}(ev(u)) = ev(\forall_f(u))$$

On objects : for any coherent functor $M \in \mathbb{C}$, we need to show that

$$\forall_{ev(f)}(ev(u))(M) = M(\forall_f(u))$$

.

- Using the definition of the universal quantification in presheaves and the definition of $ev$ :

$$\forall_{ev(f)}(ev(u))(M) = \{x \in M(b) \mid \text{ for all } h : M \to N \text{ in } \mathrm{Coh}(\mathcal{H}, \mathbf{Set}),$$

$$\text{for all } y \in N(a), \text{ if } Nf(y) = h_b(x) \text{ then } y \in N(u)\}$$

- Therefore, to show $\forall_{ev(f)}(ev(u))(M) \subseteq M(\forall_f(u))$, we assume $x \in M(b)$, $x \notin M(\forall_f u)$ and we need to show that :

$$\text{there exists } N \in \mathrm{Coh}(\mathcal{H}, \mathbf{Set}),\ h : M \to N \text{ and } y \in N(a) \text{ such that } Nf(y) = h_b(x) \text{ but } y \notin N(u)$$

We have $f : a \to b$, $u \in \mathrm{Sub}(a)$ and $M \in \mathrm{Coh}(\mathcal{H}, \mathbf{Set})$

## Sketch of the proof

We have $f : a \to b$, $u \in \mathrm{Sub}(a)$ and $M \in \mathrm{Coh}(\mathcal{H}, \mathbf{Set})$

$$\begin{array}{c} \mathsf{y}b \\ {\scriptstyle \mathsf{y}f}\big\downarrow \\ \mathsf{y}a \\ \big\downarrow \\ \mathsf{y}u \end{array}$$

We have $f : a \to b$, $u \in \mathrm{Sub}(a)$ and $M \in \mathrm{Coh}(\mathcal{H}, \mathbf{Set})$

Suppose $x \in M(b)$ but $x \notin M(\forall_f u)$

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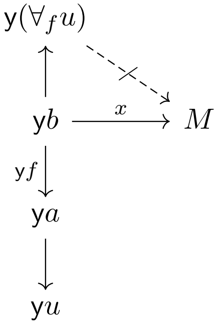

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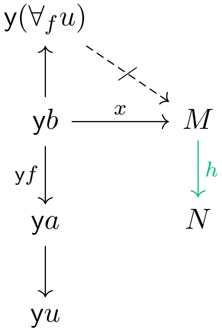


**WTS :** there exists $N \in \mathrm{Coh}(\mathcal{H}, \mathbf{Set})$, $h : M \to N$

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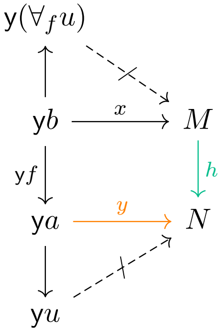


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# First step : Finite limit preserving functors

- Since $\mathcal{H}$ has finite limits, the category $\mathrm{Lex}(\mathcal{H}, \mathbf{Set})$ is cocomplete

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$$\begin{array}{ccc} \mathsf{y}b & \xrightarrow{x} & M \\ {\scriptstyle \mathsf{y}f}\downarrow & & \downarrow{\scriptstyle \tilde{h}} \\ \mathsf{y}a & \xrightarrow{\tilde{y}} & \tilde{N} \end{array}$$

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{\scriptstyle \mathsf{y}f}\big\downarrow & \ulcorner & \big\downarrow{\scriptstyle \tilde{h}} \\
\mathsf{y}a & \xrightarrow{\;\tilde{y}\;} & \tilde{N}
\end{array}
$$

- $\tilde{N}f(\tilde{y}) = \tilde{h}_b(x)$ by commutation of the diagram
- $\tilde{y} \notin \tilde{N}(u)$ by our assumption $x \notin M(\forall_f u)$ and the adjunction $f^{-1} \dashv \forall_f$

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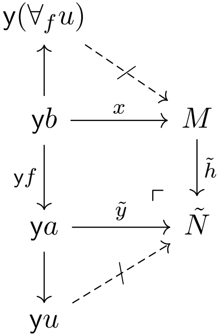

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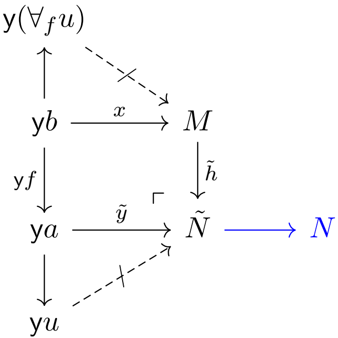

## Second step : A new category $\mathcal{H}(\tilde{N})$

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- $\mathcal{H}(\tilde{N})^{op} \hookrightarrow \tilde{N}/\operatorname{Lex}(\mathcal{H}, \mathbf{Set})$ is the full subcategory
- Its object are pushouts of maps in $\mathcal{H}$, e.g. :

$$
\begin{array}{ccc}
\mathsf{y}d & \longrightarrow & \tilde{N} \\
\big\downarrow & & \big\downarrow \tilde{h} \\
\mathsf{y}c & \longrightarrow & Y
\end{array}
\quad (\ulcorner)
$$

for a map $c \to d$ in $\mathcal{H}$.

- $\tilde{N} \in \mathrm{Lex}(\mathcal{H}, \mathbf{Set}) \Rightarrow \tilde{N} \cong \mathrm{colim}_{i \in I}\, \mathsf{y}\tilde{N}_i$ for $I$ a filtered category.
- $\mathcal{H}(\tilde{N}) \simeq \mathrm{colim}_{i \in I}\, \mathcal{H}/\tilde{N}_i$

**Lemma**

*A filtered colimit of coherent categories and coherent functors between them is coherent.*

$\Rightarrow \mathcal{H}(\tilde{N})$ is coherent.

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**Lemma**

*A filtered colimit of coherent categories and coherent functors between them is coherent.*

$$\Rightarrow \mathcal{H}(\tilde{N}) \text{ is coherent.}$$

Moreover, $\mathrm{Coh}(\mathcal{H}(\tilde{N}), \mathbf{Set}) \simeq \tilde{N}/\,\mathrm{Coh}(\mathcal{H}, \mathbf{Set})$

## Third step : Deligne's theorem, again

- There is an object $Y$ in $\mathcal{H}(\tilde{N})$ : $\begin{array}{ccc} \mathrm{y}a & \xrightarrow{\tilde{y}} & \tilde{N} \\ \downarrow & \ulcorner & \downarrow \\ \mathrm{y}u & \longrightarrow & Y \end{array}$ such that $Y \ncong \tilde{N}$.

**Theorem (Deligne)**

*For any morphism $f : a \to b$ in $\mathcal{H}$, if $M(a) \cong M(b)$ for all $M \in \mathrm{Coh}(\mathcal{H}, \mathbf{Set})$, then $f$ is an isomorphism.*

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*For any morphism $f : a \to b$ in $\mathcal{H}$, if $M(a) \cong M(b)$ for all $M \in \mathrm{Coh}(\mathcal{H}, \mathbf{Set})$, then $f$ is an isomorphism.*

- There exists $N \in \mathrm{Coh}(\mathcal{H}(\tilde{N}), \mathbf{Set})$ such that $N(Y) \not\cong N(\tilde{N})$
- Since $\mathrm{Coh}(\mathcal{H}(\tilde{N}), \mathbf{Set}) \simeq \tilde{N} / \mathrm{Coh}(\mathcal{H}, \mathbf{Set})$, we have

$$\begin{array}{ccc} \tilde{N} & \longrightarrow & {\color{blue} N} \\ \downarrow & \nearrow\!\!\!\!\!/ & \\ Y & & \end{array}$$

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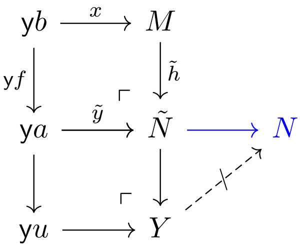

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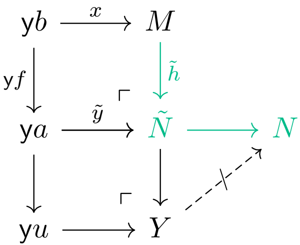

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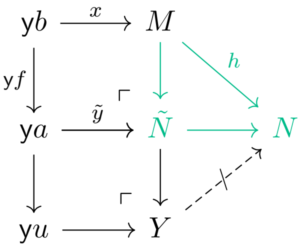

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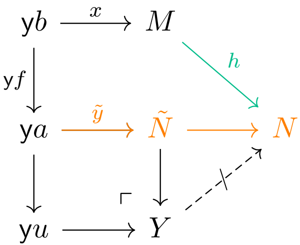

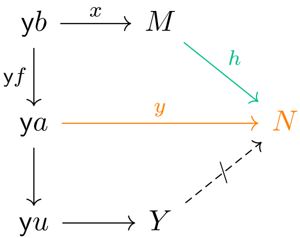
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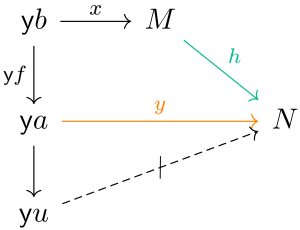

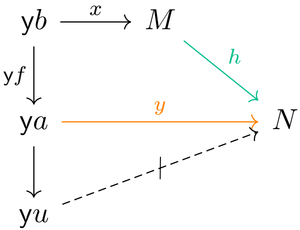


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# Completeness results

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For any intuitionistic first-order theory $\mathbb{T}$, one can construct its *syntactic category* $\mathcal{C}_{\mathbb{T}}$.

- $\mathcal{C}_{\mathbb{T}}$ is a Heyting category.
- $\mathcal{C}_{\mathbb{T}}$ contains a *universal model* $U$ such that a formula (of IFOL) is provable from the axioms of $\mathbb{T}$ if and only if this formula holds in $U$.

$$\mathbb{T} \text{ proves } (\Gamma \mid \varphi) \quad \text{iff} \quad U \models (\Gamma \mid \varphi)$$

# Completeness results

**Theorem (Kripke Completeness of IFOL, 1965)**

*If a formula of IFOL holds in every Kripke model, then it is provable in Heyting predicate calculus.*

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*If a formula of IFOL holds in every Kripke model, then it is provable in Heyting predicate calculus.*

- Kripke semantics agree with semantics in presheaves of the form $\mathbf{Set}^K$, with $K$ a poset
- Diaconescu cover : There exists a poset $K$ and a conservative, Heyting functor $\mathbf{Set}^{\mathbb{C}} \hookrightarrow \mathbf{Set}^K$

# Completeness results

Let $\mathbb{T}$ be an IFO theory, consider Joyal's theorem for its syntactic category

$$ev : \mathcal{C}_{\mathbb{T}} \hookrightarrow \mathbf{Set}^{\mathbb{C}}$$

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- Since $ev$ is a Heyting functor, the image of the universal model $U$ in $\mathcal{C}_{\mathbb{T}}$ under $ev$ is again a model $ev(U)$ of $\mathbb{T}$.

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- Since $ev$ is a Heyting functor, the image of the universal model $U$ in $\mathcal{C}_{\mathbb{T}}$ under $ev$ is again a model $ev(U)$ of $\mathbb{T}$.
- By conservativity of $ev$ : if a formula holds in the model $ev(U)$ it also holds in the model $U$, and therefore is provable.

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