TOWARDS A FORMAL THEORY OF TANGENT OBJECTS

MARCELLO LANFRANCHI

FORMAL MONADS

STREET 1972

PREMISE

A MONAD CONSISTS OF:

A CATEGORY $\mathbb{X}$

AN ENDOFUNCTOR $\mathbb{X} \xrightarrow{S} \mathbb{X}$

A UNIT $id_{\mathbb{X}} \xrightarrow{\eta} S$

A MULTIPLICATION $S^2 \xrightarrow{\mu} S$

FORMAL MONADS

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STREET 1972

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A MULTIPLICATION $S^2 \xrightarrow{\mu} S$

FORMAL MONADS

STREET 1972

PREMISE

A MONAD CONSISTS OF:		A **FORMAL** MONAD IN A 2-CATEGORY CONSISTS OF:
A CATEGORY	$\mathbb{X}$	AN OBJECT
AN ENDOFUNCTOR	$\mathbb{X} \xrightarrow{S} \mathbb{X}$	A 1-MORPHISM
A UNIT	$id_{\mathbb{X}} \xrightarrow{\eta} S$	A UNIT
A MULTIPLICATION	$S^2 \xrightarrow{\mu} S$	A MULTIPLICATION

CAN WE FORMALIZE TANGENT CATEGORY THEORY?

THE PLAN

- TANGENT CATEGORY THEORY 101
- WHY WE NEED TANGENT OBJECTS
- VECTOR FIELDS BUT FORMALLY
- STRUCTURES OF FORMAL VECTOR FIELDS
- NEW IDEAS AND FUTURE WORK

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- NEW IDEAS AND FUTURE WORK

CHAPTER 0

TANGENT CATEGORY THEORY 101

THE OBJECTS OF A TANGENT CATEGORY ARE **LOCALLY LINEAR** GEOMETRIC SPACES.

TANGENT CATEGORY

ROSICKY 1984
COCKETT, CRUTTWELL 2014

DEFINITION

A TANGENT CATEGORY CONSISTS OF:

OBJECTS ARE GEOMETRIC SPACES
MORPHISMS ARE SMOOTH FUNCTIONS

CATEGORY

TANGENT BUNDLE FUNCTOR

PROJECTION

ZERO MORPHISM

SUM MORPHISM

VERTICAL LIFT

CANONICAL FLIP

TANGENT CATEGORY

ROSICKY 1984
COCKETT, CRUTTWELL 2014

DEFINITION

A TANGENT CATEGORY CONSISTS OF:

CATEGORY

TANGENT BUNDLE FUNCTOR

PROJECTION

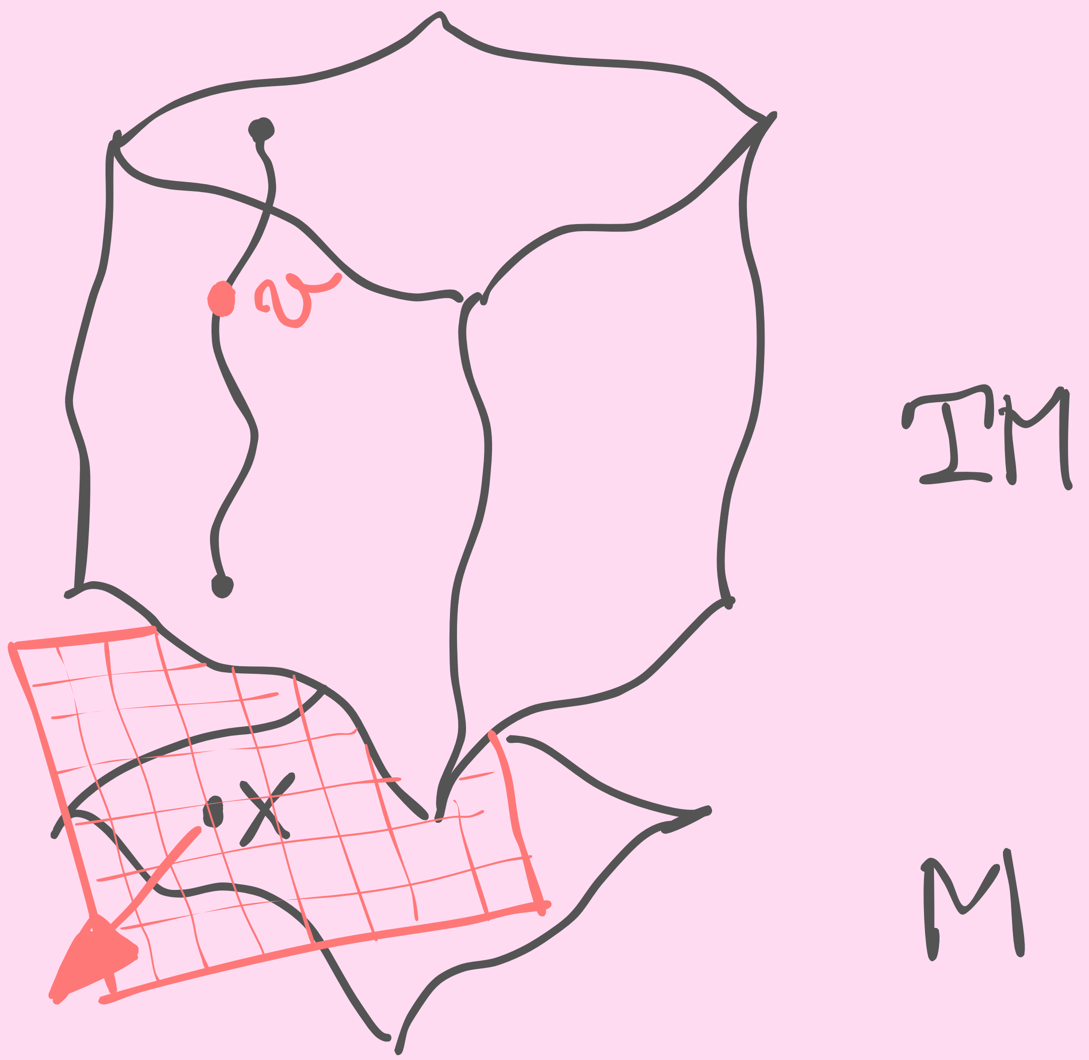
ZERO MORPHISM

SUM MORPHISM

VERTICAL LIFT

CANONICAL FLIP

T SENDS AN OBJECT TO THE COLLECTION OF ITS TANGENT VECTORS



TANGENT CATEGORY

ROSICKY 1984
COCKETT, CRUTTWELL 2014

DEFINITION

A TANGENT CATEGORY CONSISTS OF:

CATEGORY

TANGENT BUNDLE FUNCTOR

PROJECTION

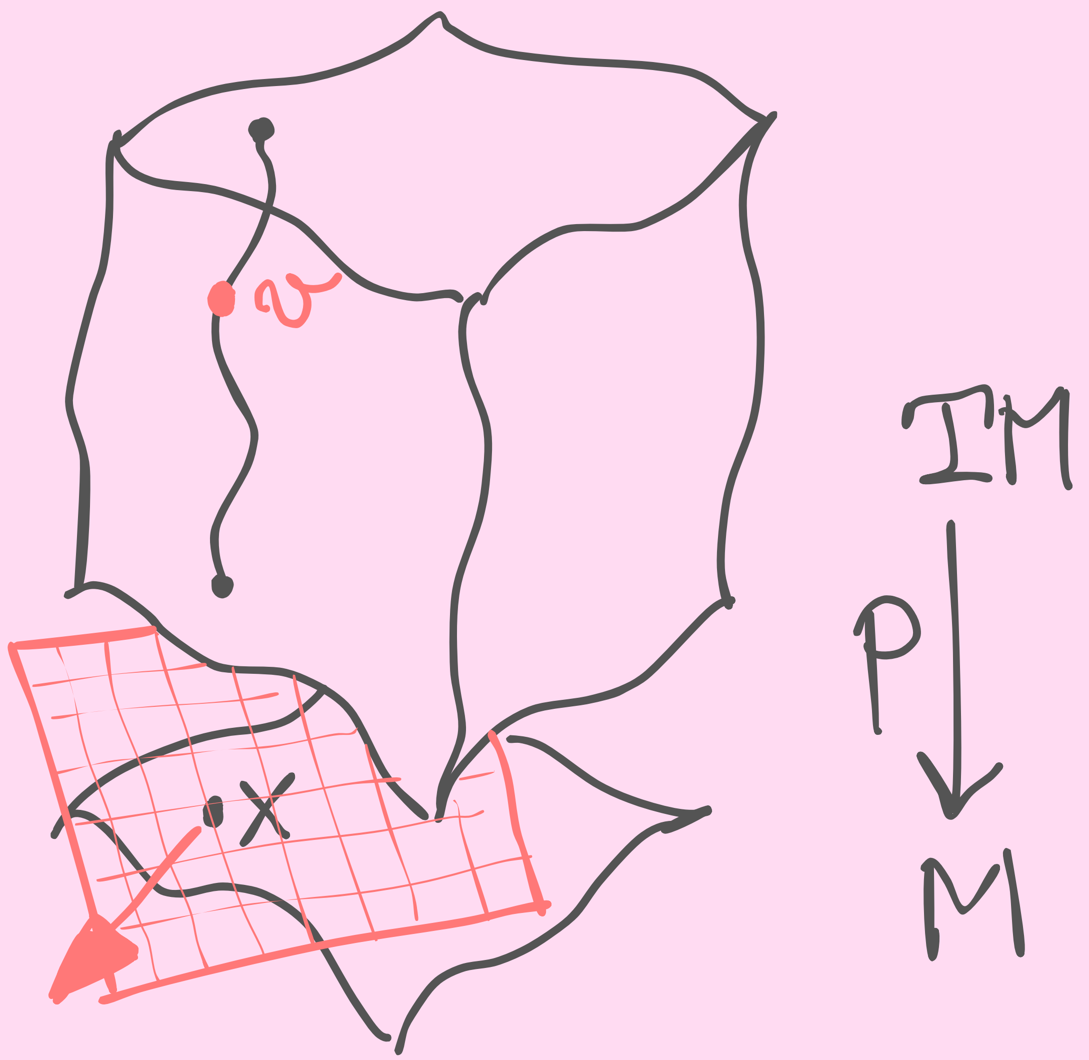
ZERO MORPHISM

SUM MORPHISM

VERTICAL LIFT

CANONICAL FLIP

THE PROJECTION SENDS A VECTOR TO ITS BASE POINT



TANGENT CATEGORY

ROSICKY 1984
COCKETT, CRUTTWELL 2014

DEFINITION

A TANGENT CATEGORY CONSISTS OF:

CATEGORY

TANGENT BUNDLE FUNCTOR

PROJECTION

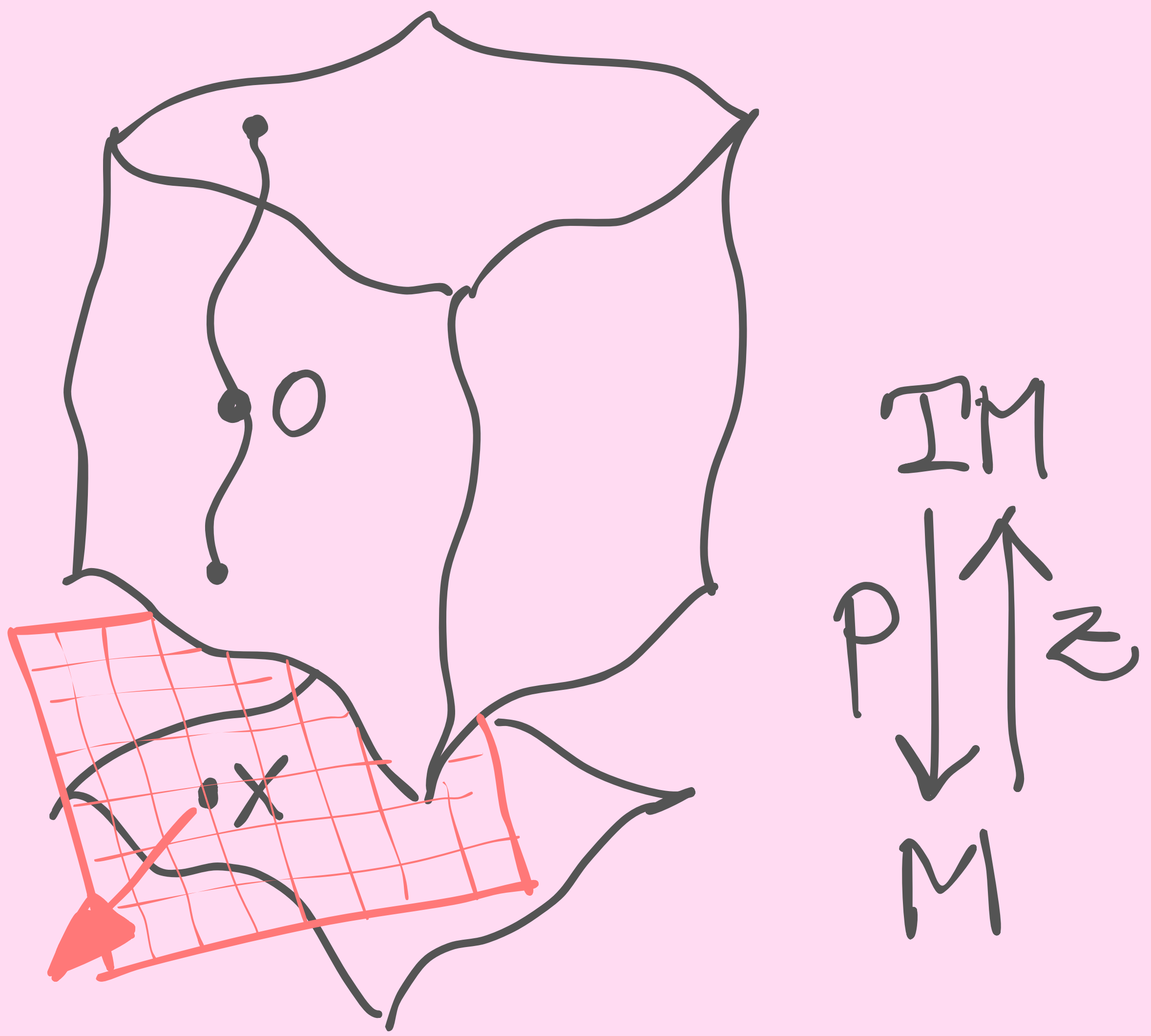
ZERO MORPHISM

SUM MORPHISM

VERTICAL LIFT

CANONICAL FLIP

THE ZERO SENDS A POINT
TO ITS ZERO VECTOR



TANGENT CATEGORY

ROSICKY 1984
COCKETT, CRUTTWELL 2014

DEFINITION

A TANGENT CATEGORY CONSISTS OF:

CATEGORY

TANGENT BUNDLE FUNCTOR

PROJECTION

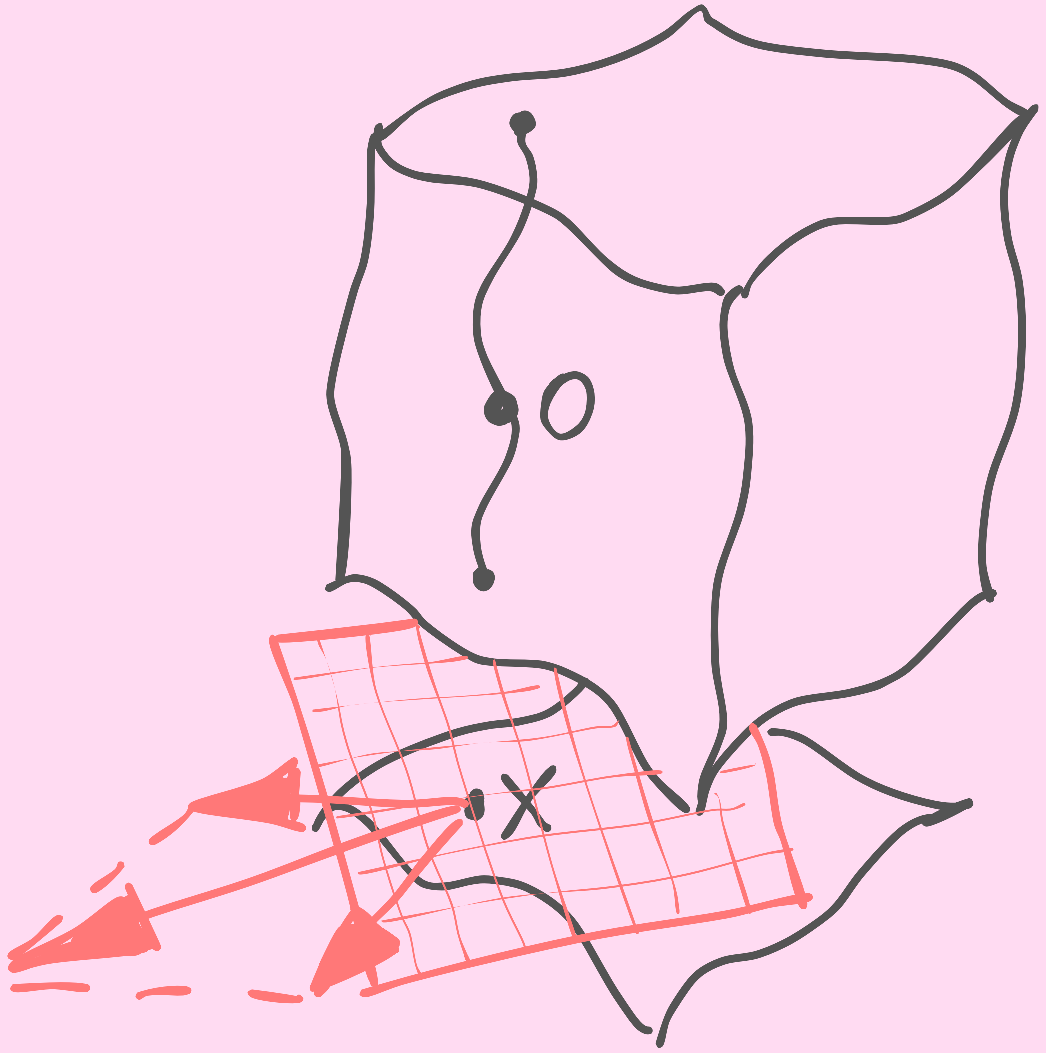
ZERO MORPHISM

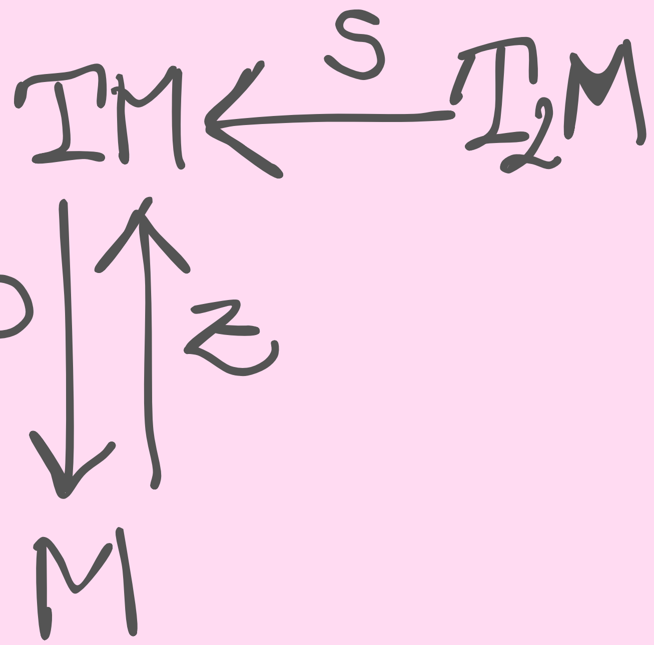
SUM MORPHISM

VERTICAL LIFT

CANONICAL FLIP

THE SUM SUMS VECTOR WITH THE SAME BASE POINT





DEFINITION

TANGENT CATEGORY

ROSICKY 1984
COCKETT, CRUTTWELL 2014

A TANGENT CATEGORY CONSISTS OF:

CATEGORY

TANGENT BUNDLE FUNCTOR

PROJECTION

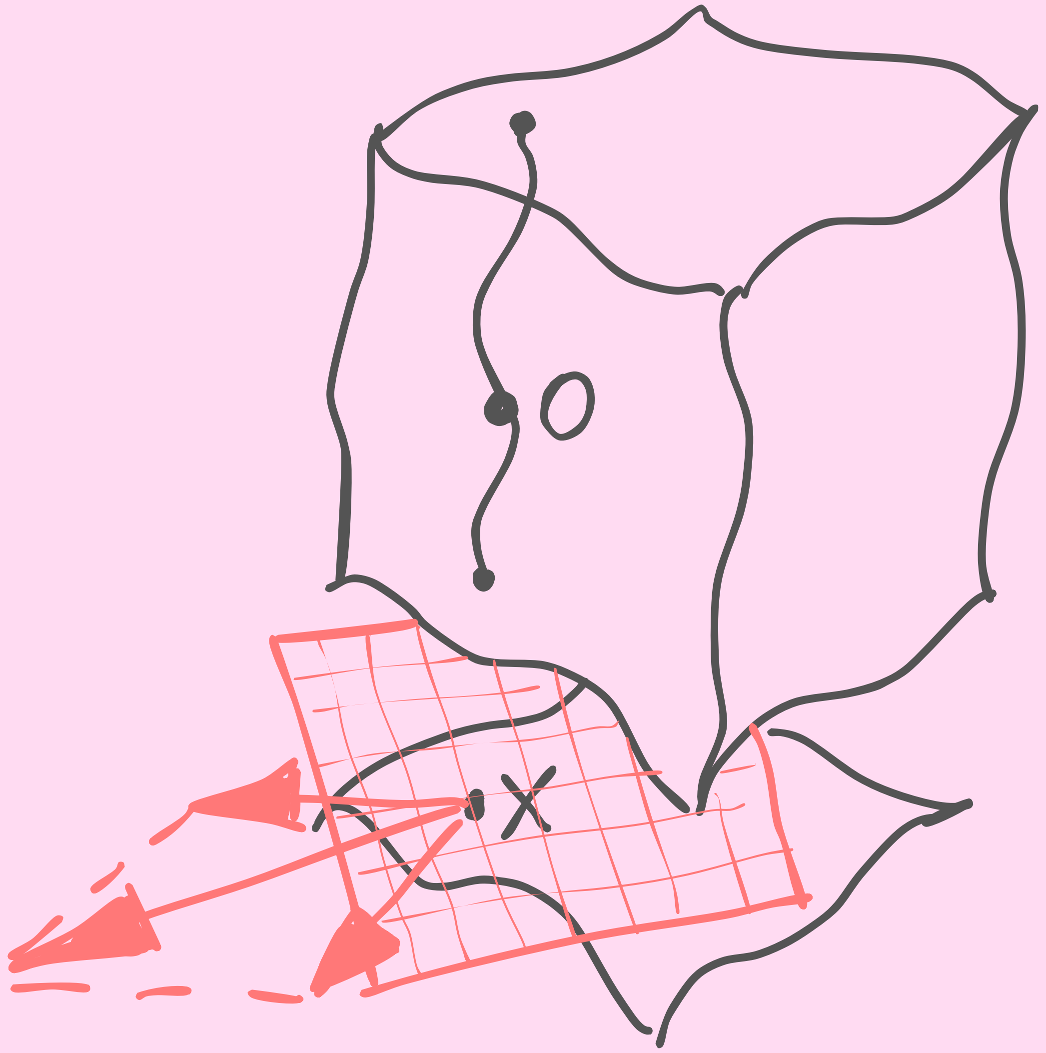
ZERO MORPHISM

SUM MORPHISM

VERTICAL LIFT

CANONICAL FLIP

THE LIFT MAKES TM LOCALLY LINEAR



$$
\begin{tikzcd}
T^2M & \\
TM \arrow[u, "e"] \arrow[d, shift right, "p"'] & T_2M \arrow[l, "s"'] \\
M \arrow[u, shift right, "z"'] &
\end{tikzcd}
$$

DEFINITION

TANGENT CATEGORY

ROSICKY 1984
COCKETT, CRUTTWELL 2014

A TANGENT CATEGORY CONSISTS OF:

CATEGORY

TANGENT BUNDLE FUNCTOR

PROJECTION

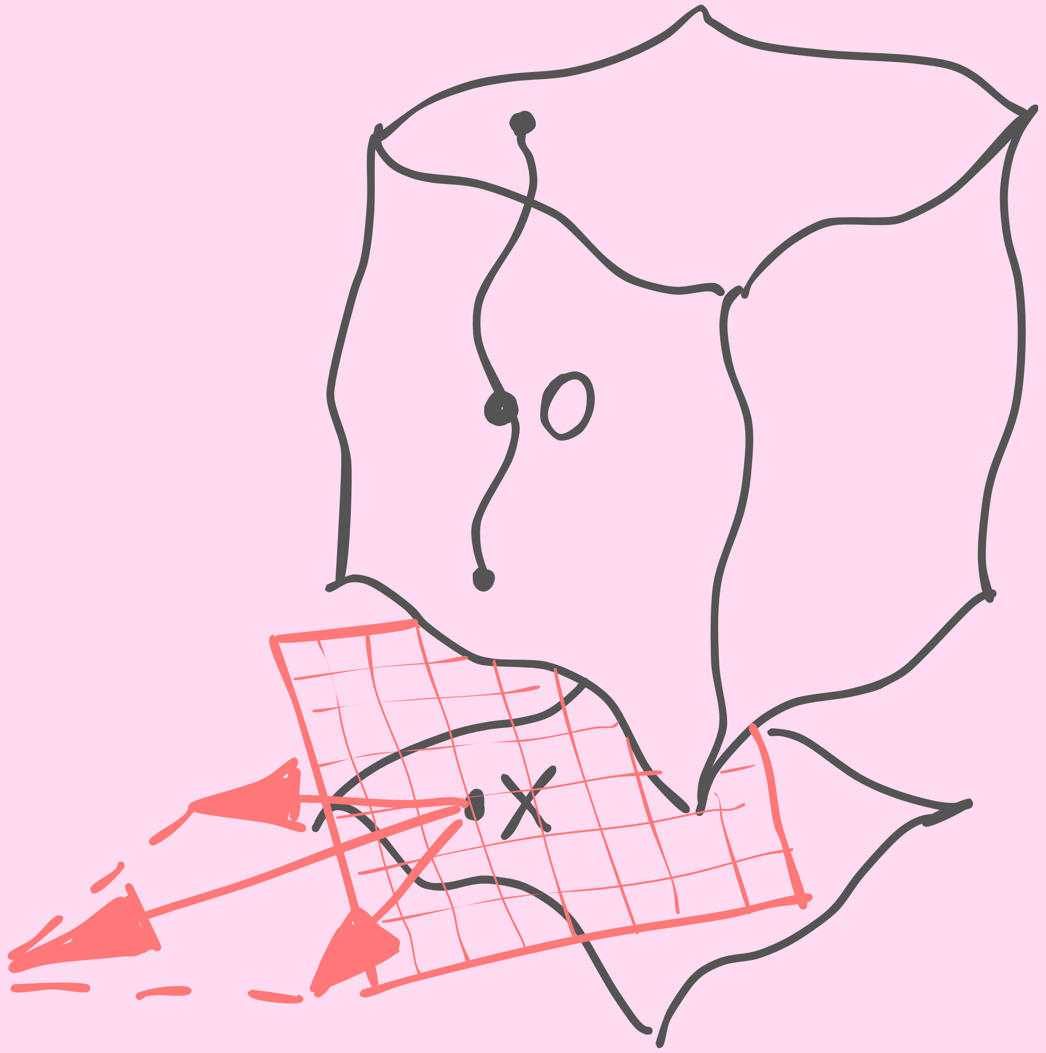
ZERO MORPHISM

SUM MORPHISM

VERTICAL LIFT

CANONICAL FLIP

THE FLIP ENCODES THE SYMMETRY
OF THE HESSIAN MATRIX



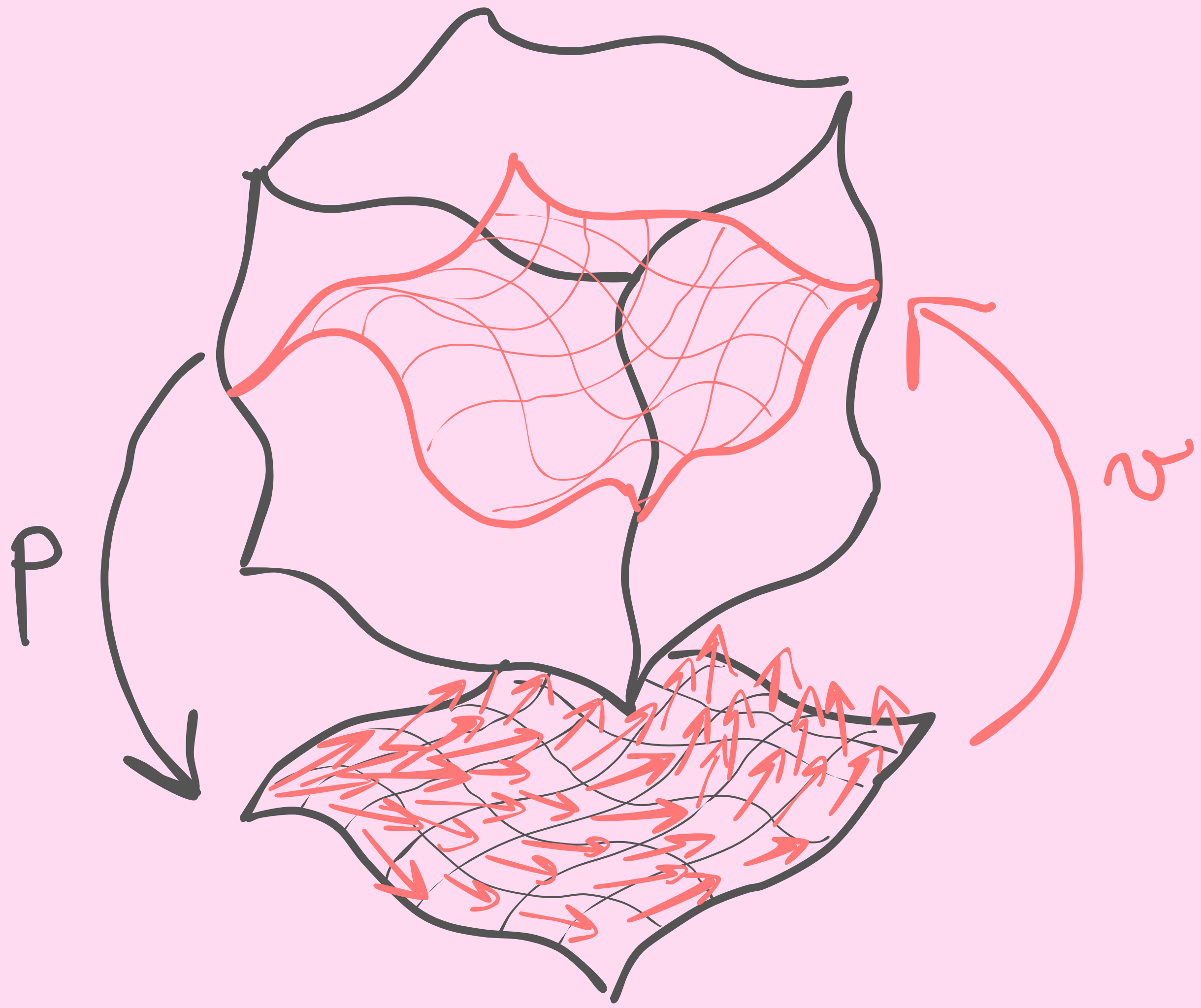
$$\begin{tikzcd} T^2M & T^2M \arrow[l, "c"'] \\ TM \arrow[u, "e"] \arrow[d, shift right, "p"'] & T_2M \arrow[l, "s"'] \\ M \arrow[u, shift right, "z"'] \end{tikzcd}$$

VECTOR FIELDS

ROSICKY 1984

DEFINITION

A VECTOR FIELD IS A
SECTION OF THE PROJECTION

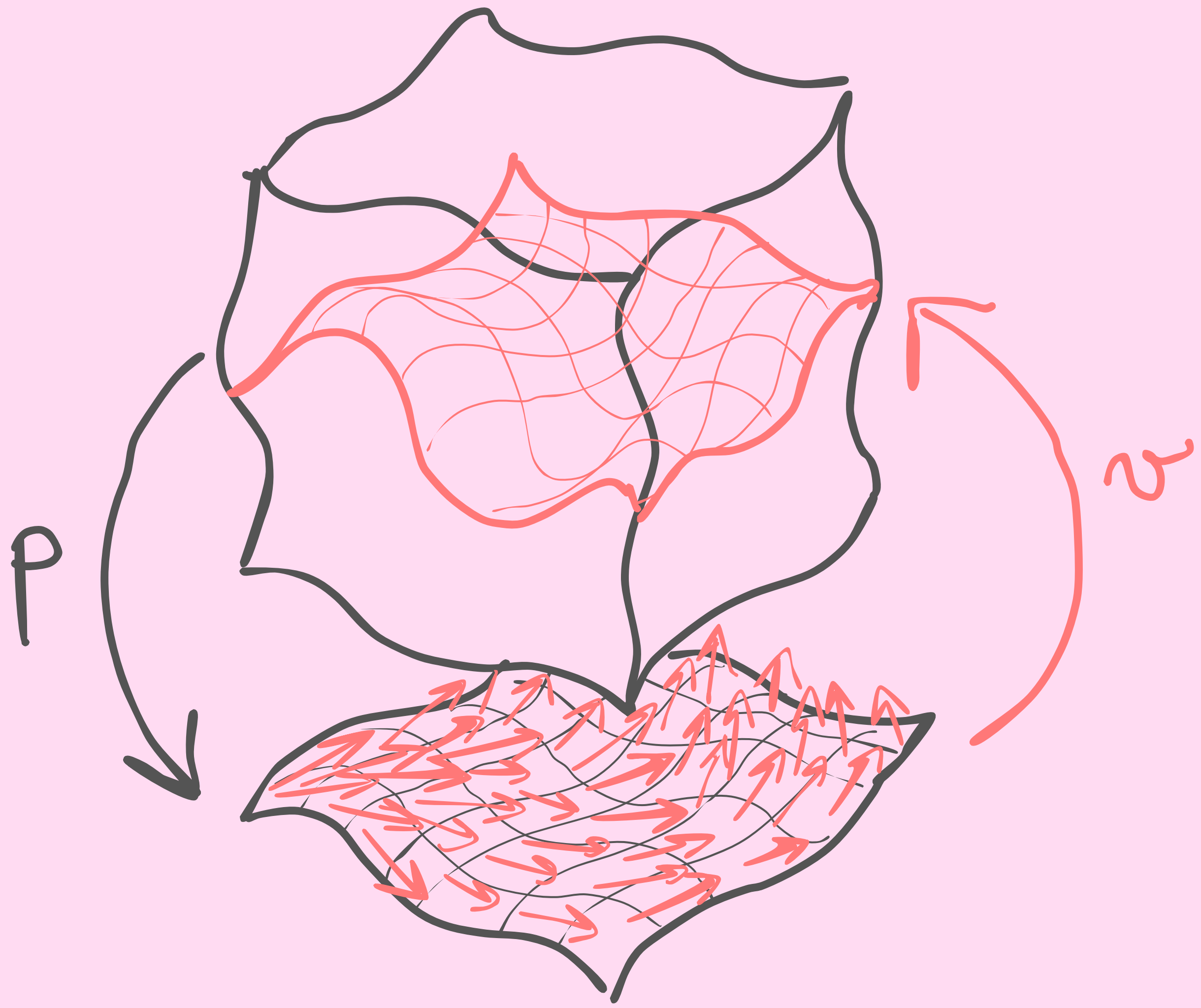


TANGENT CATEGORY OF VECTOR FIELDS

COCKETT
CRUTTWELL, LEMAY 2021

DEFINITION

A VECTOR FIELD IS A SECTION OF THE PROJECTION



VECTOR FIELDS FORM A TANGENT CATEGORY

$$(M, v : M \to TM)$$

$$f : (M, v) \to (N, u)$$

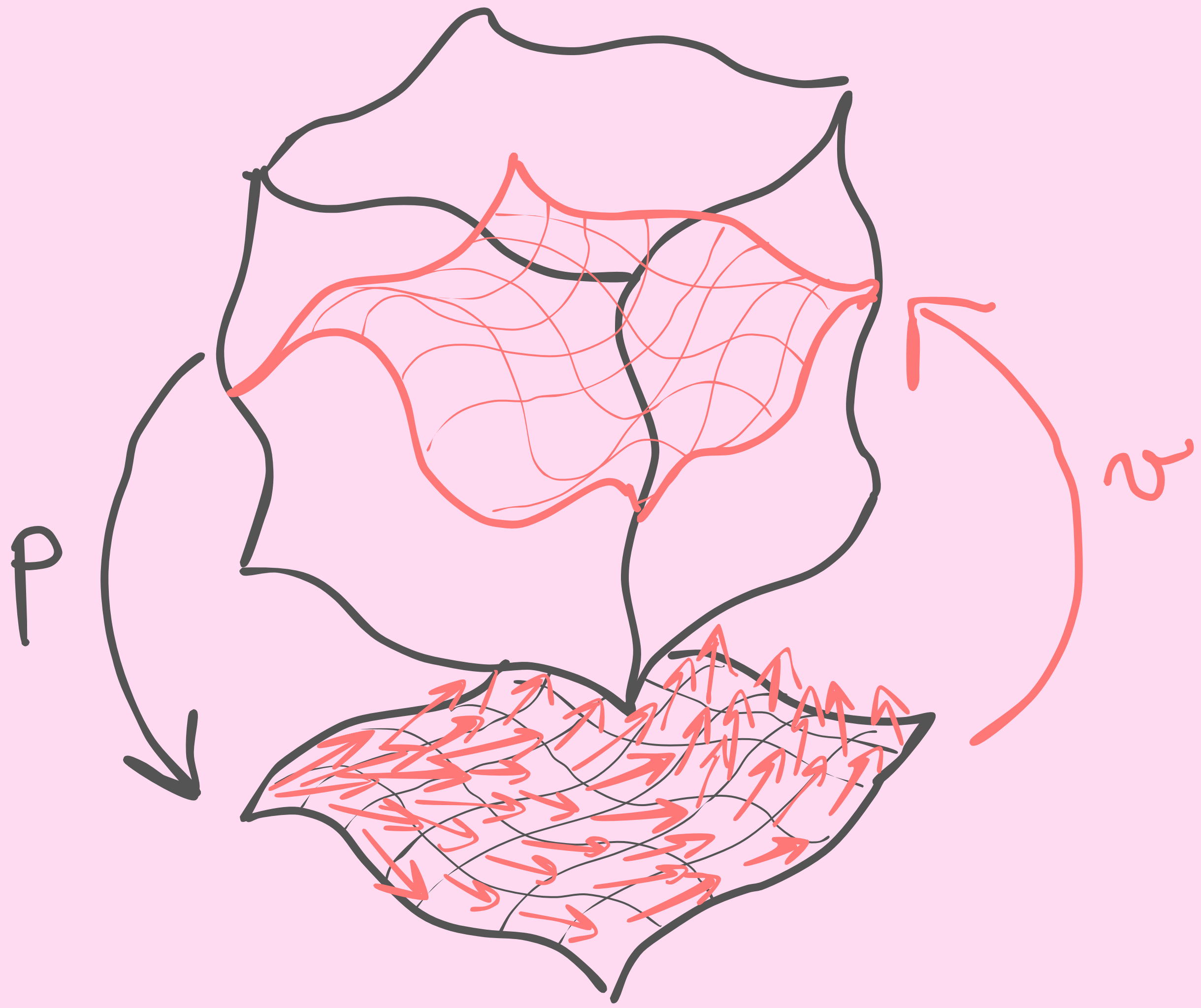
$$\begin{tikzcd} TM \arrow[r, "Tf"] & TN \\ M \arrow[u, "v"] \arrow[r, "f"'] & N \arrow[u, "u"'] \end{tikzcd}$$

DEFINITION

TANGENT CATEGORY OF VECTOR FIELDS

COCKETT
CRUTTWELL, LEMAY 2021

A VECTOR FIELD IS A SECTION OF THE PROJECTION



VECTOR FIELDS FORM A TANGENT CATEGORY

$$(M, v: M \to TM)$$

$$f: (M, v) \to (N, u)$$

$$T(M, v) = (TM, v_T)$$

$$v_T : TM \xrightarrow{Tv} T^2M \xrightarrow{c} T^2M$$

EXAMPLE

DIFFERENTIAL GEOMETRY

ROSICKY 1984

OBJECTS
SMOOTH MANIFOLDS

MORPHISMS
SMOOTH FUNCTIONS

TANGENT STRUCTURE
TANGENT BUNDLE OF DIFF GEOM

VECTOR FIELDS
USUAL NOTION OF VECTOR FIELDS

ALGEBRAIC GEOMETRY

COCKETT, CRUTTWELL 2014
CRUTTWELL, LEMAY 2023

EXAMPLE

OBJECTS
COMMUTATIVE AND UNITAL RINGS (AFFINE SCHEMES)

MORPHISMS
RING HOMOMORPHISMS OPPOSITE

TANGENT STRUCTURE

$$\mathcal{T}A = \mathrm{Sym}_A \Omega_A$$

VECTOR FIELDS
DERIVATIONS

EXAMPLE

ALGEBRAIC GEOMETRY

COCKETT, CRUTTWELL 2014
CRUTTWELL, LEMAY 2023

OBJECTS
COMMUTATIVE AND UNITAL RINGS (AFFINE SCHEMES)

MORPHISMS
RING HOMOMORPHISMS OPPOSITE

TANGENT STRUCTURE

$$\mathbb{T}A = \mathrm{Sym}_A \Omega_A$$

Module of Kähler differentials

symmetric algebra

VECTOR FIELDS
DERIVATIONS

CHAPTER 1

WHY WE NEED TANGENT OBJECTS

TANGENT OBJECTS ARE FORMAL TANGENT CATEGORIES.

TANGENT OBJECT

DEFINITION

A TANGENT OBJECT IN A 2-CATEGORY CONSISTS OF:

OBJECT $\mathbb{X}$

TANGENT BUNDLE 1-MORPHISM

PROJECTION

ZERO MORPHISM

SUM MORPHISM

VERTICAL LIFT

CANONICAL FLIP

$$\begin{tikzcd} \mathbb{X} \arrow[r, "T"] \arrow[d, equal] \arrow[dr, phantom, "p\Downarrow\ \Uparrow z"] & \mathbb{X} \arrow[d, equal] \\ \mathbb{X} \arrow[r, equal] & \mathbb{X} \end{tikzcd}
\qquad
\begin{tikzcd} \mathbb{X} \arrow[r, "T_2"] \arrow[d, equal] \arrow[dr, phantom, "s\Downarrow"] & \mathbb{X} \arrow[d, equal] \\ \mathbb{X} \arrow[r, "T"'] & \mathbb{X} \end{tikzcd}$$

$$\begin{tikzcd} \mathbb{X} \arrow[rr, "T"] \arrow[d, equal] \arrow[drr, phantom, "e\Downarrow"] & & \mathbb{X} \arrow[d, equal] \\ \mathbb{X} \arrow[r, "T"'] & \mathbb{X} \arrow[r, "T"'] & \mathbb{X} \end{tikzcd}
\qquad
\begin{tikzcd} \mathbb{X} \arrow[r, "T"] \arrow[d, equal] & \mathbb{X} \arrow[r, "T"] \arrow[d, phantom, "\Downarrow c"] & \mathbb{X} \arrow[d, equal] \\ \mathbb{X} \arrow[r, "T"'] & \mathbb{X} \arrow[r, "T"'] & \mathbb{X} \end{tikzcd}$$

DEFINITION

TANGENT OBJECT

A TANGENT OBJECT IN A 2-CATEGORY CONSISTS OF:

OBJECT

TANGENT BUNDLE 1-MORPHISM

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CANONICAL FLIP

$\mathbb{X}$

$$\begin{tikzcd} \mathbb{X} \arrow[r, "T"] \arrow[d, equal] \arrow[dr, phantom, "p\Downarrow\ \Uparrow z"] & \mathbb{X} \arrow[d, equal] \\ \mathbb{X} \arrow[r, equal] & \mathbb{X} \end{tikzcd}$$

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$$\begin{tikzcd} \mathbb{X} \arrow[r, "T"] \arrow[d, equal] & \mathbb{X} \arrow[r, "T"] \arrow[d, phantom, "\Downarrow c"] & \mathbb{X} \arrow[d, equal] \\ \mathbb{X} \arrow[r, "T"'] & \mathbb{X} \arrow[r, "T"'] & \mathbb{X} \end{tikzcd}$$

TANGENT OBJECT

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$\mathbb{X}$

$$\begin{tikzcd}
\mathbb{X} \arrow[r, "\mathrm{T}"] \arrow[d, equal] \arrow[dr, phantom, "p\Downarrow\ \Uparrow z"] & \mathbb{X} \arrow[d, equal] \\
\mathbb{X} \arrow[r, equal] & \mathbb{X}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{X} \arrow[r, "\mathrm{T}_2"] \arrow[d, equal] \arrow[dr, phantom, "s\Downarrow"] & \mathbb{X} \arrow[d, equal] \\
\mathbb{X} \arrow[r, "\mathrm{T}"'] & \mathbb{X}
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{X} \arrow[rr, "\mathrm{T}"] \arrow[d, equal] \arrow[drr, phantom, "\ell\Downarrow"] & & \mathbb{X} \arrow[d, equal] \\
\mathbb{X} \arrow[r, "\mathrm{T}"'] & \mathbb{X} \arrow[r, "\mathrm{T}"'] & \mathbb{X}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{X} \arrow[r, "\mathrm{T}"] \arrow[d, equal] & \mathbb{X} \arrow[r, "\mathrm{T}"] \arrow[d, phantom, "\Downarrow c"] & \mathbb{X} \arrow[d, equal] \\
\mathbb{X} \arrow[r, "\mathrm{T}"'] & \mathbb{X} \arrow[r, "\mathrm{T}"'] & \mathbb{X}
\end{tikzcd}$$

DEFINITION

TANGENT OBJECT

A TANGENT OBJECT IN A 2-CATEGORY CONSISTS OF:

OBJECT

TANGENT BUNDLE 1-MORPHISM

PROJECTION

ZERO MORPHISM

SUM MORPHISM

VERTICAL LIFT

CANONICAL FLIP

$\mathbb{X}$

$$
\begin{tikzcd}
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\mathbb{X} \arrow[r, equal] & \mathbb{X}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{X} \arrow[r, "T_2"] \arrow[d, equal] \arrow[dr, phantom, "s\Downarrow"] & \mathbb{X} \arrow[d, equal] \\
\mathbb{X} \arrow[r, "T"'] & \mathbb{X}
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathbb{X} \arrow[rr, "T"] \arrow[d, equal] & \arrow[d, phantom, "e\Downarrow"] & \mathbb{X} \arrow[d, equal] \\
\mathbb{X} \arrow[r, "T"'] & \mathbb{X} \arrow[r, "T"'] & \mathbb{X}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{X} \arrow[r, "T"] \arrow[d, equal] & \mathbb{X} \arrow[r, "T"] \arrow[d, phantom, "\Downarrow c"] & \mathbb{X} \arrow[d, equal] \\
\mathbb{X} \arrow[r, "T"'] & \mathbb{X} \arrow[r, "T"'] & \mathbb{X}
\end{tikzcd}
$$

TANGENT OBJECT

DEFINITION

A TANGENT OBJECT IN A 2-CATEGORY CONSISTS OF:

OBJECT

$\mathbb{X}$

TANGENT BUNDLE 1-MORPHISM

PROJECTION

ZERO MORPHISM

$$\begin{tikzcd} \mathbb{X} \arrow[r, "T"] \arrow[d, equal] \arrow[dr, phantom, "p\Downarrow \Uparrow z"] & \mathbb{X} \arrow[d, equal] \\ \mathbb{X} \arrow[r, equal] & \mathbb{X} \end{tikzcd}$$

SUM MORPHISM

$$\begin{tikzcd} \mathbb{X} \arrow[r, "T_2"] \arrow[d, equal] \arrow[dr, phantom, "s\Downarrow"] & \mathbb{X} \arrow[d, equal] \\ \mathbb{X} \arrow[r, "T"'] & \mathbb{X} \end{tikzcd}$$

VERTICAL LIFT

$$\begin{tikzcd} \mathbb{X} \arrow[rr, "T"] \arrow[d, equal] \arrow[drr, phantom, "e\Downarrow"] & & \mathbb{X} \arrow[d, equal] \\ \mathbb{X} \arrow[r, "T"'] & \mathbb{X} \arrow[r, "T"'] & \mathbb{X} \end{tikzcd}$$

CANONICAL FLIP

$$\begin{tikzcd} \mathbb{X} \arrow[r, "T"] \arrow[d, equal] & \mathbb{X} \arrow[r, "T"] \arrow[d, phantom, "\Downarrow c"] & \mathbb{X} \arrow[d, equal] \\ \mathbb{X} \arrow[r, "T"'] & \mathbb{X} \arrow[r, "T"'] & \mathbb{X} \end{tikzcd}$$

TANGENT OBJECT

DEFINITION

A TANGENT OBJECT IN A 2-CATEGORY CONSISTS OF:

OBJECT

TANGENT BUNDLE 1-MORPHISM

PROJECTION

ZERO MORPHISM

SUM MORPHISM

VERTICAL LIFT

CANONICAL FLIP

$$\mathbb{X}$$

$$\begin{tikzcd}
\mathbb{X} \arrow[r, "T"] \arrow[d, equal] \arrow[dr, phantom, "p\Downarrow\ \Uparrow z"] & \mathbb{X} \arrow[d, equal] \\
\mathbb{X} \arrow[r, equal] & \mathbb{X}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{X} \arrow[r, "T_2"] \arrow[d, equal] \arrow[dr, phantom, "s\Downarrow"] & \mathbb{X} \arrow[d, equal] \\
\mathbb{X} \arrow[r, "T"'] & \mathbb{X}
\end{tikzcd}$$

$$\begin{tikzcd}
T_2 \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & T \arrow[d, "p"] \\
T \arrow[r, "p"'] & \mathrm{id}_{\mathbb{X}}
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{X} \arrow[rr, "T"] \arrow[d, equal] \arrow[drr, phantom, "e\Downarrow"] & & \mathbb{X} \arrow[d, equal] \\
\mathbb{X} \arrow[r, "T"'] & \mathbb{X} \arrow[r, "T"'] & \mathbb{X}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{X} \arrow[r, "T"] \arrow[d, equal] & \mathbb{X} \arrow[r, "T"] \arrow[d, phantom, "\Downarrow c"] & \mathbb{X} \arrow[d, equal] \\
\mathbb{X} \arrow[r, "T"'] & \mathbb{X} \arrow[r, "T"'] & \mathbb{X}
\end{tikzcd}$$

TANGENT OBJECT

DEFINITION

A TANGENT OBJECT IN A 2-CATEGORY CONSISTS OF:

OBJECT

$\mathbb{X}$

TANGENT BUNDLE 1-MORPHISM

PROJECTION

ZERO MORPHISM

$$\begin{tikzcd}
\mathbb{X} \arrow[r, "T"] \arrow[d, equal] \arrow[dr, phantom, "p\Downarrow\;\Uparrow z"] & \mathbb{X} \arrow[d, equal] \\
\mathbb{X} \arrow[r, equal] & \mathbb{X}
\end{tikzcd}$$

SUM MORPHISM

$$\begin{tikzcd}
\mathbb{X} \arrow[r, "T_2"] \arrow[d, equal] \arrow[dr, phantom, "s\Downarrow"] & \mathbb{X} \arrow[d, equal] \\
\mathbb{X} \arrow[r, "T"'] & \mathbb{X}
\end{tikzcd}$$

VERTICAL LIFT

$$\begin{tikzcd}
\mathbb{X} \arrow[rr, "T"] \arrow[d, equal] \arrow[drr, phantom, "\ell\Downarrow"] & & \mathbb{X} \arrow[d, equal] \\
\mathbb{X} \arrow[r, "T"'] & \mathbb{X} \arrow[r, "T"'] & \mathbb{X}
\end{tikzcd}$$

CANONICAL FLIP

$$\begin{tikzcd}
\mathbb{X} \arrow[r, "T"] \arrow[d, equal] & \mathbb{X} \arrow[r, "T"] \arrow[d, phantom, "\Downarrow c"] & \mathbb{X} \arrow[d, equal] \\
\mathbb{X} \arrow[r, "T"'] & \mathbb{X} \arrow[r, "T"'] & \mathbb{X}
\end{tikzcd}$$

TANGENT OBJECT

DEFINITION

A TANGENT OBJECT IN A 2-CATEGORY CONSISTS OF:

OBJECT

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$$\begin{tikzcd} \mathbb{X} \arrow[r, "T"] \arrow[d, equal] \arrow[dr, phantom, "p\Downarrow\ \Uparrow z"] & \mathbb{X} \arrow[d, equal] \\ \mathbb{X} \arrow[r, equal] & \mathbb{X} \end{tikzcd}$$

$$\begin{tikzcd} \mathbb{X} \arrow[r, "T_2"] \arrow[d, equal] \arrow[dr, phantom, "s\Downarrow"] & \mathbb{X} \arrow[d, equal] \\ \mathbb{X} \arrow[r, "T"'] & \mathbb{X} \end{tikzcd}$$

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TANGENT OBJECT

EXAMPLE

TANGENT CATEGORIES ARE
TANGENT OBJECTS IN CAT

TANGENT MONADS ARE
TANGENT OBJECTS IN MND

TANGENT FIBRATIONS ARE
TANGENT OBJECTS IN FIB

TANGENT OBJECT

EXAMPLE

TANGENT CATEGORIES ARE
TANGENT OBJECTS IN CAT

TANGENT MONADS ARE
TANGENT OBJECTS IN MND

TANGENT FIBRATIONS ARE
TANGENT OBJECTS IN FIB

MND:

Objects: $(\mathbb{X}, S)$

Category

Monad on $\mathbb{X}$

Morphisms:

$(F, \alpha) : (\mathbb{X}, S) \to (\mathbb{X}', S')$

$$\begin{tikzcd} \mathbb{X} \arrow[r, "F"] \arrow[d, "S"'] \arrow[dr, phantom, "\alpha \swArrow"] & \mathbb{X}' \arrow[d, "S'"] \\ \mathbb{X} \arrow[r, "F"'] & \mathbb{X}' \end{tikzcd}$$

2-Morphisms:

$$\begin{tikzcd} (\mathbb{X}, S) \arrow[r, bend left, "{(F,\alpha)}"] \arrow[r, bend right, "{(G,\beta)}"'] \arrow[r, phantom, "\Downarrow \varphi"] & (\mathbb{X}', S') \end{tikzcd}$$

$\varphi : F \Rightarrow G$
+ compatibility

TANGENT OBJECT

COCKETT, LEMAY
LUCYSHYN-WRIGHT 2019

EXAMPLE

TANGENT CATEGORIES ARE
TANGENT OBJECTS IN CAT

TANGENT MONADS ARE
TANGENT OBJECTS IN MND

TANGENT FIBRATIONS ARE
TANGENT OBJECTS IN FIB

Tangent monads
are monads in TNGCAT

$$\mathrm{MND}(\mathrm{TNG}(\mathbb{K})) \cong \mathrm{TNG}(\mathrm{MND}(\mathbb{K}))$$

So tangent monads
are tangent objects in MND

TANGENT OBJECT

EXAMPLE

TANGENT CATEGORIES ARE TANGENT OBJECTS IN CAT

TANGENT MONADS ARE TANGENT OBJECTS IN MND

TANGENT FIBRATIONS ARE TANGENT OBJECTS IN FIB

FIB:

Categories

Objects: $(\mathbb{X}, \mathbb{X}', \Pi : \mathbb{X}' \to \mathbb{X})$

Fibration

Morphisms:

$(F, G) : \Pi_o \to \Pi_\bullet$

$$\begin{tikzcd} \mathbb{X}_o' \arrow[r, "G"] \arrow[d, "\Pi_o"'] & \mathbb{X}_\bullet' \arrow[d, "\Pi_\bullet"] \\ \mathbb{X}_o \arrow[r, "F"'] & \mathbb{X}_\bullet \end{tikzcd}$$

2-Morphisms:

$$\begin{tikzcd} \Pi_o \arrow[r, bend left, "(F,G)"] \arrow[r, bend right, "(F',G')"'] \arrow[r, phantom, "\Downarrow(\varphi,\psi)"] & \Pi_\bullet \end{tikzcd}$$

$\varphi : F \Rightarrow F'$ $\psi : G \Rightarrow G'$

+ compatibility

TANGENT OBJECT

COCKETT, CRUTTWELL 2017

EXAMPLE

TANGENT CATEGORIES ARE
TANGENT OBJECTS IN CAT

TANGENT MONADS ARE
TANGENT OBJECTS IN MND

TANGENT FIBRATIONS ARE
TANGENT OBJECTS IN FIB

Tangent fibrations are fibrations between tangent cats s.t. tang. bundle functors preserve cartesian lifts.

$\mathrm{TNG}(\mathrm{FIB}) \cong \mathrm{TNGFIB}$

RESTRICTION TANGENT CATEGORIES

COCKETT, CRUTTWELL 2014

NON-EXAMPLE

A RESTRICTION TANGENT CATEGORY CONSISTS OF:

RESTRICTION CATEGORY

TANGENT BUNDLE RESTRICTION FUNCTOR

PROJECTION

ZERO MORPHISM

SUM MORPHISM

VERTICAL LIFT

CANONICAL FLIP

HOWEVER PULLBACKS ARE REPLACED WITH RESTRICTION PULLBACKS!

CHAPTER 2

VECTOR FIELDS BUT FORMALLY

VECTOR FIELDS FORM THE UNIVERSAL VECTOR FIELD OF THE GLOBAL TANGENT CATEGORY.

THE GLOBAL TANGENT CATEGORY

SET UP

GIVEN A TANGENT CATEGORY

$(\mathbb{X}, \mathbb{T})$

THE SLICE 2-CATEGORY

$\mathrm{TNGCAT}/(\mathbb{X}, \mathbb{T})$

COMES EQUIPPED WITH A
TANGENT STRUCTURE

SET UP

THE GLOBAL TANGENT CATEGORY

GIVEN A TANGENT CATEGORY

$(\mathbb{X}, \mathbb{T})$

THE SLICE 2-CATEGORY

$\mathrm{TNGCAT}/(\mathbb{X}, \mathbb{T})$

COMES EQUIPPED WITH A TANGENT STRUCTURE

Objects: $(F, \alpha) : (\mathbb{X}', \mathbb{T}') \longrightarrow (\mathbb{X}, \mathbb{T})$

$F : \mathbb{X}' \longrightarrow \mathbb{X} \quad \alpha : F\mathrm{T}' \Rightarrow \mathrm{T}F$

1-Morphisms: $(H, \gamma; \varphi) : (F, \alpha) \longrightarrow (G, \beta)$

SET UP

THE GLOBAL TANGENT CATEGORY

GIVEN A TANGENT CATEGORY

$(\mathbb{X},\mathbb{T})$

THE SLICE 2-CATEGORY

$\mathsf{TNGCAT}/(\mathbb{X},\mathbb{T})$

COMES EQUIPPED WITH A TANGENT STRUCTURE

Objects: $(F,\alpha):(\mathbb{X}',\mathbb{T}')\longrightarrow(\mathbb{X},\mathbb{T})$

$F:\mathbb{X}'\longrightarrow\mathbb{X}$ $\quad\alpha:F\mathrm{T}'\Rightarrow\mathrm{T}F$

1-Morphisms: $(H,\gamma;\varphi):(F,\alpha)\longrightarrow(G,\beta)$

$$
\begin{tikzcd}
(\mathbb{X}'',\mathbb{T}'') \arrow[drr, "{(G,\beta)}"] & & \\
(\mathbb{X}',\mathbb{T}') \arrow[u, bend left, "{(H,\gamma)}"] \arrow[rr, "{(F,\alpha)}"'] & \Uparrow\varphi & (\mathbb{X},\mathbb{T})
\end{tikzcd}
$$

THE GLOBAL TANGENT CATEGORY

SET UP

GIVEN A TANGENT CATEGORY

$(\mathbb{X},\mathbb{T})$

THE SLICE 2-CATEGORY

$\mathrm{TNGCAT}/(\mathbb{X},\mathbb{T})$

COMES EQUIPPED WITH A TANGENT STRUCTURE

Objects: $(F,\alpha):(\mathbb{X}',\mathbb{T}')\longrightarrow(\mathbb{X},\mathbb{T})$

$F:\mathbb{X}'\longrightarrow\mathbb{X}\quad \alpha:F\mathbb{T}'\Rightarrow\mathbb{T}F$

1-Morphisms: $(H,\gamma;\varphi):(F,\alpha)\longrightarrow(G,\beta)$

$$\begin{tikzcd}
(\mathbb{X}'',\mathbb{T}'') \arrow[dr, "{(G,\beta)}"] \arrow[d, phantom, shift left=8ex, "\Uparrow\varphi' \ \Uparrow\varphi"] & \\
(\mathbb{X}',\mathbb{T}') \arrow[u, bend left, "{(H',\gamma')}"] \arrow[u, bend right, "{(H,\gamma)}"'] \arrow[r, "{(F,\alpha)}"'] & (\mathbb{X},\mathbb{T})
\end{tikzcd}$$

THE GLOBAL TANGENT CATEGORY

SET UP

GIVEN A TANGENT CATEGORY

$(\mathbb{X},\mathbb{T})$

THE SLICE 2-CATEGORY

$\mathrm{TNGCAT}/(\mathbb{X},\mathbb{T})$

COMES EQUIPPED WITH A TANGENT STRUCTURE

Objects: $(F,\alpha):(\mathbb{X}',\mathbb{T}')\longrightarrow(\mathbb{X},\mathbb{T})$

$F:\mathbb{X}'\longrightarrow\mathbb{X}\quad \alpha:F\mathrm{T}'\Rightarrow\mathrm{T}F$

1-Morphisms: $(H,\gamma;\varphi):(F,\alpha)\longrightarrow(G,\beta)$

2-morphisms:

$\psi:(H,\gamma;\varphi)\Rightarrow(H',\gamma';\varphi')$

$$
\begin{tikzcd}
(\mathbb{X}'',\mathbb{T}'') \arrow[drr, "{(G,\beta)}"] & & \\
(\mathbb{X}',\mathbb{T}') \arrow[u, bend left=60, "{(H',\gamma')}"] \arrow[u, bend right=60, "{(H,\gamma)}"'] \arrow[u, phantom, "\overset{\psi}{\Leftarrow}"] \arrow[rr, "{(F,\alpha)}"'] \arrow[urr, phantom, "\Uparrow\varphi' \;\; \Uparrow\varphi"] & & (\mathbb{X},\mathbb{T})
\end{tikzcd}
$$

THE GLOBAL TANGENT CATEGORY

SET UP

GIVEN A TANGENT CATEGORY

$(\mathbb{X}, \mathbb{T})$

THE SLICE 2-CATEGORY

$\mathrm{TNGCAT}/(\mathbb{X}, \mathbb{T})$

COMES EQUIPPED WITH A TANGENT STRUCTURE

$$\overline{\mathbb{T}}[(F,\alpha)] := (\mathbb{X}', \mathbb{T}') \xrightarrow{(F,\alpha)} (\mathbb{X}, \mathbb{T}) \xrightarrow{(\mathrm{T}, c)} (\mathbb{X}, \mathbb{T})$$

THE GLOBAL TANGENT CATEGORY

SET UP

LET'S CONSIDER THE FORGETFUL FUNCTOR:

$$U : VF(\mathbb{X}, \mathbb{T}) \longrightarrow (\mathbb{X}, \mathbb{T})$$

Tangent category of vector fields of $(\mathbb{X}, \mathbb{T})$

THE GLOBAL TANGENT CATEGORY

SET UP

LET'S CONSIDER THE
FORGETFUL FUNCTOR:

$$U : VF(\mathbb{X}, \mathbb{T}) \to (\mathbb{X}, \mathbb{T})$$

$$\begin{array}{ccc} (M, v) & & M \\ f \downarrow & & f \downarrow \\ (N, u) & & N \end{array}$$

U STRICTLY PRESERVES
THE TANGENT STRUCTURE

THE GLOBAL TANGENT CATEGORY

SET UP

LET'S CONSIDER THE
FORGETFUL FUNCTOR:

$$U : VF(\mathbb{X}, \mathbb{T}) \longrightarrow (\mathbb{X}, \mathbb{T})$$

$$\begin{tikzcd} (M, v) \arrow[d, "f"'] & & M \arrow[d, "f"'] \\ (N, u) & & N \end{tikzcd}$$

U STRICTLY PRESERVES
THE TANGENT STRUCTURE

SO U BECOMES AN OBJECT OF

$$\mathrm{TNGCAT}/(\mathbb{X}, \mathbb{T})$$

THE GLOBAL TANGENT CATEGORY

SET UP

LET'S CONSIDER THE FORGETFUL FUNCTOR:

$$U : \mathsf{VF}(\mathbb{X},\mathbb{T}) \to (\mathbb{X},\mathbb{T})$$

$$\begin{tikzcd} (M,v) \arrow[d, "f"'] \\ (N,u) \end{tikzcd} \qquad \qquad \begin{tikzcd} M \arrow[d, "f"'] \\ N \end{tikzcd}$$

U STRICTLY PRESERVES THE TANGENT STRUCTURE

SO U BECOMES AN OBJECT OF

$$\mathsf{TNGCAT}/(\mathbb{X},\mathbb{T})$$

U HAS A CANONICAL VECTOR FIELD

$$\hat{v} : U \to \mathbb{T}U$$

$$\begin{tikzcd} \mathsf{VF}(\mathbb{X},\mathbb{T}) \arrow[r, "U"] \arrow[d, equal] \arrow[dr, phantom, "\hat{v}\Nearrow"] & (\mathbb{X},\mathbb{T}) \arrow[d, "(\mathrm{T},c)"] \\ \mathsf{VF}(\mathbb{X},\mathbb{T}) \arrow[r, "U"'] & (\mathbb{X},\mathbb{T}) \end{tikzcd}$$

THE GLOBAL TANGENT CATEGORY

SET UP

LET'S CONSIDER THE FORGETFUL FUNCTOR:

$$\mathcal{U} : \mathsf{VF}(\mathbb{X}, \mathbb{T}) \to (\mathbb{X}, \mathbb{T})$$

$$\begin{tikzcd} (M, v) \arrow[d, "f"'] \\ (N, u) \end{tikzcd} \qquad \qquad \begin{tikzcd} M \arrow[d, "f"'] \\ N \end{tikzcd}$$

U STRICTLY PRESERVES THE TANGENT STRUCTURE

SO U BECOMES AN OBJECT OF

$$\mathsf{TNGCAT}/(\mathbb{X}, \mathbb{T})$$

U HAS A CANONICAL VECTOR FIELD

$$\hat{v} : \mathcal{U} \to \bar{\mathrm{T}}\mathcal{U}$$

$$\begin{tikzcd} \mathsf{VF}(\mathbb{X}, \mathbb{T}) \arrow[r, "\mathcal{U}"] \arrow[d, equal] \arrow[dr, phantom, "\hat{v} \Nearrow"] & (\mathbb{X}, \mathbb{T}) \arrow[d, "(\mathrm{T}, c)"] \\ \mathsf{VF}(\mathbb{X}, \mathbb{T}) \arrow[r, "\mathcal{U}"'] & (\mathbb{X}, \mathbb{T}) \end{tikzcd}$$

$$\hat{v} : \underbrace{\mathcal{U}(M, v)}_{M} \to \underbrace{\bar{\mathrm{T}}\mathcal{U}(M, v)}_{\mathrm{T}M}$$

THE GLOBAL TANGENT CATEGORY

SET UP

LET'S CONSIDER THE FORGETFUL FUNCTOR:

$$U : \mathrm{VF}(\mathbb{X},\mathbb{T}) \longrightarrow (\mathbb{X},\mathbb{T})$$

$$\begin{array}{ccc} (M,v) & \qquad & M \\ f\big\downarrow & & f\big\downarrow \\ (N,u) & & N \end{array}$$

U STRICTLY PRESERVES THE TANGENT STRUCTURE

SO U BECOMES AN OBJECT OF

$$\mathrm{TNGCAT}/(\mathbb{X},\mathbb{T})$$

U HAS A CANONICAL VECTOR FIELD

$$\hat{v} : U \longrightarrow \mathrm{T}U$$

$$\begin{tikzcd} \mathrm{VF}(\mathbb{X},\mathbb{T}) \arrow[r, "U"] \arrow[d, equal] \arrow[dr, phantom, "\hat{v}\,\Nearrow"] & (\mathbb{X},\mathbb{T}) \arrow[d, "(\mathrm{T}\text{,}c)"] \\ \mathrm{VF}(\mathbb{X},\mathbb{T}) \arrow[r, "U"'] & (\mathbb{X},\mathbb{T}) \end{tikzcd}$$

$$\hat{v} : \underbrace{U(M,v)}_{M} \longrightarrow \underbrace{\mathrm{T}U(M,v)}_{\mathrm{T}M}$$

$$M \overset{v}{\dashrightarrow} \mathrm{T}M$$

THE UNIVERSAL PROPERTY OF VECTOR FIELDS

THEOREM

$(U, \hat{v})$

IS THE UNIVERSAL VECTOR FIELD OF

$\mathrm{TNGCAT}/(\mathbb{X}, \mathbb{T})$

THEOREM

THE UNIVERSAL PROPERTY OF VECTOR FIELDS

CONSIDER AN OBJECT

$$(F, \alpha) : (\mathbb{X}', \mathbb{T}') \longrightarrow (\mathbb{X}, \mathbb{T})$$

OF

$$\mathrm{TNGCAT}/(\mathbb{X}, \mathbb{T})$$

TOGETHER WITH A VECTOR FIELD

$$u : (F, \alpha) \to \mathbb{T}(F, \alpha)$$

$$\begin{tikzcd}
(\mathbb{X}', \mathbb{T}') \arrow[r, "{(F,\alpha)}"] \arrow[d, equal] \arrow[dr, phantom, "u \Nearrow"] & (\mathbb{X}, \mathbb{T}) \arrow[d, "{(\mathbb{T}, c)}"] \\
(\mathbb{X}', \mathbb{T}') \arrow[r, "{(F,\alpha)}"'] & (\mathbb{X}, \mathbb{T})
\end{tikzcd}$$

THEOREM

THE UNIVERSAL PROPERTY OF VECTOR FIELDS

CONSIDER AN OBJECT

$$(F,\alpha):(\mathbb{X}',\mathbb{T}')\longrightarrow(\mathbb{X},\mathbb{T})$$

OF

$$\mathsf{TNGCAT}/(\mathbb{X},\mathbb{T})$$

TOGETHER WITH A VECTOR FIELD

$$u:(F,\alpha)\to\bar{\mathbb{T}}(F,\alpha)$$

$$\begin{tikzcd}
(\mathbb{X}',\mathbb{T}') \arrow[r, "{(F,\alpha)}"] \arrow[d, equal] & (\mathbb{X},\mathbb{T}) \arrow[d, "{(\mathbb{T},c)}"] \\
(\mathbb{X}',\mathbb{T}') \arrow[r, "{(F,\alpha)}"'] \arrow[ur, phantom, "u \Nearrow"] & (\mathbb{X},\mathbb{T})
\end{tikzcd}$$

THERE EXISTS A UNIQUE STRICT MORPHISM OF VECTOR FIELDS

$$\exists!\, u \overset{!}{\longrightarrow} \hat{v}$$

$$\begin{tikzcd}
(\mathbb{X}',\mathbb{T}') \arrow[r, dashed, "!"] \arrow[d, "{(F,\alpha)}"] \arrow[d, bend right=60, "{\mathbb{T}(F,\alpha)}"'] \arrow[d, phantom, "\overset{u}{\Leftarrow}", near start, shift right=20] & \mathrm{VF}(\mathbb{X},\mathbb{T}) \arrow[d, "U"'] \arrow[d, bend left=60, "\bar{\mathbb{T}}U"] \arrow[d, phantom, "\overset{\hat{v}}{\Rightarrow}", shift left=20] \\
(\mathbb{X},\mathbb{T}) \arrow[r, equal] \arrow[ur, phantom, "//"] & (\mathbb{X},\mathbb{T})
\end{tikzcd}$$

THE UNIVERSAL PROPERTY OF VECTOR FIELDS

THEOREM

S.T. FOR ANY LAX MORPHISM OF VECTOR FIELDS

$$(H,\gamma;\varphi) : \mathcal{U} \longrightarrow \hat{\mathcal{V}}$$

$$\begin{tikzcd}
(\mathbb{X}',\mathbb{T}') \arrow[r, "{(H,\gamma)}"] \arrow[d, "{(F,\alpha)}"] \arrow[d, bend right=70, "{T(F,\alpha)}"'] \arrow[d, phantom, shift right=4ex, "\overset{\mathcal{U}}{\Leftarrow}"] \arrow[dr, phantom, "\varphi \Nearrow"] & VF(\mathbb{X},\mathbb{T}) \arrow[d, "U"'] \arrow[d, bend left=70, "TU"] \arrow[d, phantom, shift left=4ex, "\overset{\hat{\mathcal{V}}}{\Rightarrow}"] \\
(\mathbb{X},\mathbb{T}) \arrow[r, equal] & (\mathbb{X},\mathbb{T})
\end{tikzcd}$$

THERE EXISTS A UNIQUE STRICT MORPHISM OF VECTOR FIELDS

$$\exists!\ \mathcal{U} \xrightarrow{!} \hat{\mathcal{V}}$$

$$\begin{tikzcd}
(\mathbb{X}',\mathbb{T}') \arrow[r, dashed, "!"] \arrow[d, "{(F,\alpha)}"] \arrow[d, bend right=70, "{T(F,\alpha)}"'] \arrow[d, phantom, shift right=4ex, "\overset{\mathcal{U}}{\Leftarrow}"] \arrow[dr, phantom, "//"] & VF(\mathbb{X},\mathbb{T}) \arrow[d, "U"'] \arrow[d, bend left=70, "TU"] \arrow[d, phantom, shift left=4ex, "\overset{\hat{\mathcal{V}}}{\Rightarrow}"] \\
(\mathbb{X},\mathbb{T}) \arrow[r, equal] & (\mathbb{X},\mathbb{T})
\end{tikzcd}$$

THE UNIVERSAL PROPERTY OF VECTOR FIELDS

THEOREM

S.T. FOR ANY LAX MORPHISM OF VECTOR FIELDS

$$(H,\gamma;\varphi) : u \to \hat{v}$$

$$\begin{tikzcd}
(\mathbb{X}',\mathbb{I}') \arrow[r, "{(H,\gamma)}"] \arrow[d, "{(F,\alpha)}"] \arrow[d, bend right=60, "{\mathbb{T}(F,\alpha)}"'] \arrow[d, phantom, shift right=6, "\overset{u}{\Leftarrow}"] \arrow[dr, phantom, "\varphi \Nearrow"] & \mathrm{VF}(\mathbb{X},\mathbb{I}) \arrow[d, "U"'] \arrow[d, bend left=60, "\mathbb{T}U"] \arrow[d, phantom, shift left=6, "\overset{\hat{v}}{\Rightarrow}"] \\
(\mathbb{X},\mathbb{I}) \arrow[r, equal] & (\mathbb{X},\mathbb{I})
\end{tikzcd}$$

THERE EXISTS A UNIQUE STRICT MORPHISM OF VECTOR FIELDS

$$\exists!\ u \xrightarrow{!} \hat{v}$$

$$\begin{tikzcd}
(\mathbb{X}',\mathbb{I}') \arrow[r, dashed, "!"] \arrow[d, "{(F,\alpha)}"] \arrow[d, bend right=60, "{\mathbb{T}(F,\alpha)}"'] \arrow[d, phantom, shift right=6, "\overset{u}{\Leftarrow}"] \arrow[dr, phantom, "//"] & \mathrm{VF}(\mathbb{X},\mathbb{I}) \arrow[d, "U"'] \arrow[d, bend left=60, "\mathbb{T}U"] \arrow[d, phantom, shift left=6, "\overset{\hat{v}}{\Rightarrow}"] \\
(\mathbb{X},\mathbb{I}) \arrow[r, equal] & (\mathbb{X},\mathbb{I})
\end{tikzcd}$$

THERE EXISTS A UNIQUE 2-MORPHISM

$$! \Rrightarrow (H,\gamma;\varphi)$$

THEOREM

THE UNIVERSAL PROPERTY OF VECTOR FIELDS

AND FOR ANY COLAX MORPHISM OF VECTOR FIELDS

$$(H,\gamma;\varphi) : u \nrightarrow \hat{v}$$

$$\begin{tikzcd}
(\mathbb{X}',\mathbb{I}') \arrow[r, "{(H,\gamma)}"] \arrow[d, "{(F,\alpha)}"] \arrow[d, bend right=60, "{\mathbb{T}(F,\alpha)}"'] \arrow[d, phantom, shift right=20, "\overset{u}{\Leftarrow}"] \arrow[dr, phantom, "\varphi \Swarrow"] & \mathrm{VF}(\mathbb{X},\mathbb{I}) \arrow[d, "U"'] \arrow[d, bend left=60, "\mathbb{T}U"] \arrow[d, phantom, shift left=20, "\overset{\hat{v}}{\Rightarrow}"] \\
(\mathbb{X},\mathbb{I}) \arrow[r, equal] & (\mathbb{X},\mathbb{I})
\end{tikzcd}$$

THERE EXISTS A UNIQUE STRICT MORPHISM OF VECTOR FIELDS

$$\exists!\; u \xrightarrow{\;!\;} \hat{v}$$

$$\begin{tikzcd}
(\mathbb{X}',\mathbb{I}') \arrow[r, dashed, "!"] \arrow[d, "{(F,\alpha)}"] \arrow[d, bend right=60, "{\mathbb{T}(F,\alpha)}"'] \arrow[d, phantom, shift right=20, "\overset{u}{\Leftarrow}"] \arrow[dr, phantom, "//"] & \mathrm{VF}(\mathbb{X},\mathbb{I}) \arrow[d, "U"'] \arrow[d, bend left=60, "\mathbb{T}U"] \arrow[d, phantom, shift left=20, "\overset{\hat{v}}{\Rightarrow}"] \\
(\mathbb{X},\mathbb{I}) \arrow[r, equal] & (\mathbb{X},\mathbb{I})
\end{tikzcd}$$

THE UNIVERSAL PROPERTY OF VECTOR FIELDS

THEOREM

AND FOR ANY COLAX MORPHISM OF VECTOR FIELDS

$$(H,\gamma;\varphi) : u \nrightarrow \hat{v}$$

$$\begin{tikzcd}
(\mathbb{X}',\mathbb{I}') \arrow[r, "{(H,\gamma)}"] \arrow[d, "{(F,\alpha)}"] \arrow[d, bend right=60, "{T(F,\alpha)}"'] \arrow[d, phantom, shift right=4ex, "\overset{u}{\Leftarrow}"] \arrow[dr, phantom, "\varphi\ \Swarrow"] & VF(\mathbb{X},\mathbb{I}) \arrow[d, "U"'] \arrow[d, bend left=60, "TU"] \arrow[d, phantom, shift left=4ex, "\overset{\hat{v}}{\Rrightarrow}"] \\
(\mathbb{X},\mathbb{I}) \arrow[r, equal] & (\mathbb{X},\mathbb{I})
\end{tikzcd}$$

THERE EXISTS A UNIQUE STRICT MORPHISM OF VECTOR FIELDS

$$\exists!\ u \xrightarrow{!} \hat{v}$$

$$\begin{tikzcd}
(\mathbb{X}',\mathbb{I}') \arrow[r, dashed, "!"] \arrow[d, "{(F,\alpha)}"] \arrow[d, bend right=60, "{T(F,\alpha)}"'] \arrow[d, phantom, shift right=4ex, "\overset{u}{\Leftarrow}"] \arrow[dr, phantom, "//"] & VF(\mathbb{X},\mathbb{I}) \arrow[d, "U"'] \arrow[d, bend left=60, "TU"] \arrow[d, phantom, shift left=4ex, "\overset{\hat{v}}{\Rrightarrow}"] \\
(\mathbb{X},\mathbb{I}) \arrow[r, equal] & (\mathbb{X},\mathbb{I})
\end{tikzcd}$$

THERE EXISTS A UNIQUE 2-MORPHISM

$$! \Leftarrow (H,\gamma;\varphi)$$

FORMAL VECTOR FIELDS

DEFINITION

A TANGENT OBJECT

$$(\mathbb{X},\mathbb{T})$$

ADMITS THE CONSTRUCTION OF VECTOR FIELDS IF THE TANGENT CATEGORY

$$\mathsf{TNG}(\mathbb{K})/(\mathbb{X},\mathbb{T})$$

ADMITS A UNIVERSAL VECTOR FIELD

$$(U:\mathsf{VF}(\mathbb{X},\mathbb{T})\to(\mathbb{X},\mathbb{T}),\hat{v})$$

TANGENT MONADS

EXAMPLE

CONSIDER A TANGENT MONAD:

$(S, \alpha) : (\mathbb{X}, \mathbb{T}) \to (\mathbb{X}, \mathbb{T})$

$S : \mathbb{X} \to \mathbb{X}$ Monad

$\alpha : ST \Rightarrow TS$ Distr. law

TANGENT MONADS

EXAMPLE

CONSIDER A TANGENT MONAD:

$(S,\alpha) : (\mathbb{X},\mathbb{T}) \to (\mathbb{X},\mathbb{T})$

$S : \mathbb{X} \to \mathbb{X}$ Monad

$\alpha : ST \Rightarrow TS$ Distr. law

$VF(S,\alpha) : VF(\mathbb{X},\mathbb{T}) \to VF(\mathbb{X},\mathbb{T})$

$$\begin{array}{ccc} (M,v) & & (SM, SM \xrightarrow{Sv} STM \xrightarrow{\alpha} TSM) \\ f\downarrow & & Sf\downarrow \\ (N,u) & & (SN, SN \xrightarrow{Su} STN \xrightarrow{\alpha} TSN) \end{array}$$

TANGENT MONADS

EXAMPLE

CONSIDER A TANGENT MONAD:

$$(S,\alpha):(\mathbb{X},\mathbb{T})\longrightarrow(\mathbb{X},\mathbb{T})$$

$$S:\mathbb{X}\to\mathbb{X} \quad \text{Monad}$$

$$\alpha: S\mathrm{T}\Rightarrow \mathrm{T}S \quad \text{Distr. law}$$

THE UNIVERSAL VECTOR FIELD IS THEN:

$$\begin{tikzcd}
(\mathsf{VF}(\mathbb{X},\mathbb{T});\mathsf{VF}(S,\alpha)) \arrow[r, "U"] \arrow[d, equal] \arrow[dr, phantom, "\hat{v}\Uparrow"] & (\mathbb{X},\mathbb{T};S,\alpha) \arrow[d, "(\mathrm{T}{,}c)"] \\
(\mathsf{VF}(\mathbb{X},\mathbb{T});\mathsf{VF}(S,\alpha)) \arrow[r, "U"] & (\mathbb{X},\mathbb{T};S,\alpha)
\end{tikzcd}$$

$$\mathsf{VF}(S,\alpha):\mathsf{VF}(\mathbb{X},\mathbb{T})\longrightarrow\mathsf{VF}(\mathbb{X},\mathbb{T})$$

$$\begin{tikzcd}
(M,v) \arrow[d, "f"] & (SM, SM \xrightarrow{Sv} S\mathrm{T}M \xrightarrow{\alpha} \mathrm{T}SM) \arrow[d, "Sf"] \\
(N,u) & (SN, SN \xrightarrow{Su} S\mathrm{T}N \xrightarrow{\alpha} \mathrm{T}SN)
\end{tikzcd}$$

TANGENT FIBRATIONS

EXAMPLE

CONSIDER A TANGENT FIBRATION:

$\Pi : (\mathbb{X}', \mathbb{T}') \longrightarrow (\mathbb{X}, \mathbb{T})$

$\Pi : \mathbb{X}' \longrightarrow \mathbb{X}$ Fibration

$(\mathbb{T}, \mathbb{T}') : \Pi \to \Pi$ Preserves cartesian lifts

TANGENT FIBRATIONS

EXAMPLE

CONSIDER A TANGENT FIBRATION:

$\pi : (\mathbb{X}', \mathbb{T}') \longrightarrow (\mathbb{X}, \mathbb{T})$

$\pi : \mathbb{X}' \longrightarrow \mathbb{X}$ Fibration

$(\mathbb{T}, \mathbb{T}') : \pi \to \pi$ Preserves cartesian lifts

THE UNIVERSAL VECTOR FIELD IS THEN:

$$\begin{array}{cc}
\mathsf{VF}(\mathbb{X}', \mathbb{T}') & \begin{tikzcd} \mathrm{T}E \\ E \arrow[u, "w"] \end{tikzcd} \\
| & \\
\mathsf{VF}(\pi) & \\
\downarrow & \\
\mathsf{VF}(\mathbb{X}, \mathbb{T}) & \begin{tikzcd} \mathrm{T}N \arrow[r, "\mathrm{T}f"] & \mathrm{T}M \\ N \arrow[u, "u"] \arrow[r, "f"'] & M \arrow[u, "v"'] \end{tikzcd}
\end{array}$$

EXAMPLE

TANGENT FIBRATIONS

CONSIDER A TANGENT FIBRATION:

$\Pi : (\mathbb{X}', \mathbb{T}') \longrightarrow (\mathbb{X}, \mathbb{T})$

$\Pi : \mathbb{X}' \longrightarrow \mathbb{X}$ Fibration

$(\mathbb{T}, \mathbb{T}') : \Pi \to \Pi$ Preserves cartesian lifts

THE UNIVERSAL VECTOR FIELD IS THEN:

$$\mathrm{VF}(\mathbb{X}', \mathbb{T}') \qquad \begin{tikzcd} \mathrm{T}' f^* E \arrow[r, "\mathrm{T}'\varphi_f"] & \mathrm{T} E \\ f^* E \arrow[r, "\varphi_f"'] & E \arrow[u, "w"'] \end{tikzcd}$$

$$\mathrm{VF}(\Pi)$$

$$\mathrm{VF}(\mathbb{X}, \mathbb{T}) \qquad \begin{tikzcd} \mathrm{T} N \arrow[r, "\mathrm{T} f"] & \mathrm{T} M \\ N \arrow[u, "u"] \arrow[r, "f"'] & M \arrow[u, "v"'] \end{tikzcd}$$

TANGENT FIBRATIONS

EXAMPLE

CONSIDER A TANGENT FIBRATION:

$\Pi : (\mathbb{X}', \mathbb{T}') \longrightarrow (\mathbb{X}, \mathbb{T})$

$\Pi : \mathbb{X}' \longrightarrow \mathbb{X}$ Fibration

$(\mathbb{T}, \mathbb{T}') : \Pi \longrightarrow \Pi$ Preserves cartesian lifts

$$\begin{tikzcd} \mathsf{VF}(\mathbb{X}', \mathbb{T}') \arrow[d, dash] \\ \mathsf{VF}(\Pi) \arrow[d] \\ \mathsf{VF}(\mathbb{X}, \mathbb{T}) \end{tikzcd}$$

THE UNIVERSAL VECTOR FIELD IS THEN:

$$\begin{tikzcd} \mathbb{T}' f^* E \arrow[r, "\mathbb{T}'\varphi_f"] & \mathbb{T}E \\ f^* E \arrow[u, dashed, "{f[w]}"] \arrow[r, "\varphi_f"'] & E \arrow[u, "w"'] \end{tikzcd}$$

$$\begin{tikzcd} \mathbb{T}N \arrow[r, "\mathbb{T}f"] & \mathbb{T}M \\ N \arrow[u, "u"] \arrow[r, "f"'] & M \arrow[u, "v"'] \end{tikzcd}$$

CHAPTER 3

STRUCTURES OF FORMAL VECTOR FIELDS

STRUCTURES ARE INDUCED BY THE UNIVERSALITY AND BY THE GLOBAL TANGENT STRUCTURE.

THE COMMUTATIVE MONOID OF VECTOR FIELDS

SET UP

LET'S START WITH THE ZERO

$$0 : (\mathbb{X}, \mathbb{T}) \longrightarrow VF(\mathbb{X}, \mathbb{T})$$

$$\qquad M \qquad\qquad\qquad (M, z : M \to TM)$$

THE COMMUTATIVE MONOID OF VECTOR FIELDS

SET UP

LET'S START WITH THE ZERO

$$0 : (\mathfrak{X}, \mathbb{T}) \longrightarrow VF(\mathfrak{X}, \mathbb{T})$$

$$M \qquad\qquad (M, z : M \to TM)$$

THE FIRST STEP IS TO DEFINE A NEW VECTOR FIELD

$$\begin{tikzcd}
(\mathfrak{X}, \mathbb{T}) \arrow[r, equal] & (\mathfrak{X}, \mathbb{T}) \arrow[d, "{(T, c)}"] \\
(\mathfrak{X}, \mathbb{T}) \arrow[u, equal] \arrow[r, equal] \arrow[ur, phantom, "z \Nearrow"] & (\mathfrak{X}, \mathbb{T})
\end{tikzcd}$$

THE COMMUTATIVE MONOID OF VECTOR FIELDS

SET UP

LET'S START WITH THE ZERO

$$0 : (\mathbb{X}, \mathbb{T}) \longrightarrow VF(\mathbb{X}, \mathbb{T})$$

$$M \qquad\qquad (M, z : M \to TM)$$

THE FIRST STEP IS TO DEFINE A NEW VECTOR FIELD

$$\begin{tikzcd}
(\mathbb{X}, \mathbb{T}) \arrow[r, equal] & (\mathbb{X}, \mathbb{T}) \arrow[d, "{(T, c)}"] \\
(\mathbb{X}, \mathbb{T}) \arrow[u, equal] \arrow[r, equal] \arrow[ur, phantom, "z \Nearrow"] & (\mathbb{X}, \mathbb{T})
\end{tikzcd}$$

BY THE UNIV. PROP. OF V.F. THERE EXISTS A UNIQUE STRICT MORPHISM

$$0 : z \xrightarrow{!} \hat{v}$$

THE COMMUTATIVE MONOID OF VECTOR FIELDS

SET UP

LET'S START WITH THE ZERO

$$0 : (\mathbb{X}, \mathbb{T}) \longrightarrow VF(\mathbb{X}, \mathbb{T})$$
$$M \qquad\qquad (M, z : M \to TM)$$

THE FIRST STEP IS TO DEFINE A NEW VECTOR FIELD

$$\begin{tikzcd}
(\mathbb{X}, \mathbb{T}) \arrow[r, equal] & (\mathbb{X}, \mathbb{T}) \arrow[d, "{(T,c)}"] \\
(\mathbb{X}, \mathbb{T}) \arrow[u, equal] \arrow[r, equal] \arrow[ur, phantom, "z \Nearrow"] & (\mathbb{X}, \mathbb{T})
\end{tikzcd}$$

BY THE UNIV. PROP. OF V.F. THERE EXISTS A UNIQUE STRICT MORPHISM

$$0 : z \xrightarrow{\ !\ } \hat{v}$$

$$\begin{tikzcd}
(\mathbb{X}, \mathbb{T}) \arrow[d, bend right, "{(T,c)}"'] \arrow[d, bend left, equal] \arrow[d, phantom, "\overset{z}{\Leftarrow}"] \arrow[r, dashed, "0"] & VF(\mathbb{X}, \mathbb{T}) \arrow[d, bend right, "U"'] \arrow[d, bend left, "\mathbb{T}U"] \arrow[d, phantom, "\overset{\hat{v}}{\Rightarrow}"] \\
(\mathbb{X}, \mathbb{T}) \arrow[r, equal] & (\mathbb{X}, \mathbb{T})
\end{tikzcd}$$

THE COMMUTATIVE MONOID OF VECTOR FIELDS

SET UP

LET'S NOW DEFINE THE SUM

$$+ : VF_2(\mathbb{X}, \mathbb{T}) \longrightarrow VF(\mathbb{X}, \mathbb{T})$$

$$(M, u, v) \qquad\qquad (M, u+v)$$

SET UP

THE COMMUTATIVE MONOID OF VECTOR FIELDS

LET'S NOW DEFINE THE SUM

$$+ : VF_2(\mathbb{X},\mathbb{T}) \longrightarrow VF(\mathbb{X},\mathbb{T})$$
$$(M; u, v) \qquad\qquad (M, u+v)$$

LET'S FIRST DEFINE THE TWO VECTOR FIELDS

$$\hat{v}_K := \begin{tikzcd} VF_2(\mathbb{X},\mathbb{T}) \arrow[r, "\pi_K"] \arrow[d, equal] & VF(\mathbb{X},\mathbb{T}) \arrow[r, "U"] \arrow[d, equal] \arrow[dr, phantom, "\hat{v}\ \Nearrow"] & (\mathbb{X},\mathbb{T}) \arrow[d, "{(T,c)}"] \\ VF_2(\mathbb{X},\mathbb{T}) \arrow[r, "\pi_K"'] & VF(\mathbb{X},\mathbb{T}) \arrow[r, "U"'] & (\mathbb{X},\mathbb{T}) \end{tikzcd}$$

THE COMMUTATIVE MONOID OF VECTOR FIELDS

SET UP

LET'S NOW DEFINE THE SUM

$$+ : VF_2(\mathbb{X}, \mathbb{T}) \longrightarrow VF(\mathbb{X}, \mathbb{T})$$
$$(M; u, v) \qquad (M, u+v)$$

CONCRETELY:

$$\hat{v}_1 : \underbrace{U\pi_1(M; u, v)}_{M} \longrightarrow \underbrace{TU\pi_1(M, u, v)}_{TM}$$
$$M \xrightarrow{u} TM$$

$$\hat{v}_2 : \underbrace{U\pi_2(M; u, v)}_{M} \longrightarrow \underbrace{TU\pi_2(M, u, v)}_{TM}$$
$$M \xrightarrow{v} TM$$

THE COMMUTATIVE MONOID OF VECTOR FIELDS

SET UP

LET'S NOW DEFINE THE SUM

$$+ : VF_2(\mathfrak{X}, \mathbb{T}) \longrightarrow VF(\mathfrak{X}, \mathbb{T})$$

$$(M; u, v) \qquad (M, u+v)$$

CONCRETELY:

$$\hat{v}_1 : \underbrace{U\pi_1(M; u, v)}_{M} \longrightarrow \underbrace{TU\pi_1(M, u, v)}_{TM}$$

$$M \xrightarrow{u} TM$$

$$\hat{v}_2 : \underbrace{U\pi_2(M; u, v)}_{M} \longrightarrow \underbrace{TU\pi_2(M, u, v)}_{TM}$$

$$M \xrightarrow{v} TM$$

LET'S SUM THEM UP:

$$\hat{v}_1 + \hat{v}_2 : \underbrace{U\pi_k(M; u, v)}_{M} \longrightarrow \underbrace{TU\pi_k(M; u, v)}_{TM}$$

$$M \xrightarrow{u+v} TM$$

SET UP

THE COMMUTATIVE MONOID OF VECTOR FIELDS

LET'S NOW DEFINE THE SUM

$$+ : \mathrm{VF}_2(\mathbb{X},\mathbb{T}) \longrightarrow \mathrm{VF}(\mathbb{X},\mathbb{T})$$

$$(M; u, v) \qquad (M, u+v)$$

CONCRETELY:

$$\hat{v}_1 : \underbrace{U\pi_1(M;u,v)}_{M} \xrightarrow{u} \underbrace{TU\pi_1(M,u,v)}_{TM}$$

$$\hat{v}_2 : \underbrace{U\pi_2(M;u,v)}_{M} \xrightarrow{v} \underbrace{TU\pi_2(M,u,v)}_{TM}$$

LET'S SUM THEM UP:

$$\hat{v}_1 + \hat{v}_2 : \underbrace{U\pi_K(M;u,v)}_{M} \xrightarrow{u+v} \underbrace{TU\pi_K(M;u,v)}_{TM}$$

BY THE UNIV. PROP. OF .V.F. THERE EXISTS A UNIQUE STRICT MORPHISM

$$+ : \hat{v}_1 + \hat{v}_2 \longrightarrow \hat{v}$$

THE COMMUTATIVE MONOID OF VECTOR FIELDS

SET UP

LET'S NOW DEFINE THE SUM

$$+ : VF_2(\mathbb{X},\mathbb{T}) \longrightarrow VF(\mathbb{X},\mathbb{T})$$

$$(M; u, v) \qquad\qquad (M, u+v)$$

CONCRETELY:

$$\hat{v}_1 : \underbrace{U\pi_1(M;u,v)}_{M} \longrightarrow \underbrace{\mathbb{T}U\pi_1(M,u,v)}_{\mathbb{T}M} \qquad M \xrightarrow{\;u\;} \mathbb{T}M$$

$$\hat{v}_2 : \underbrace{U\pi_2(M;u,v)}_{M} \longrightarrow \underbrace{\mathbb{T}U\pi_2(M,u,v)}_{\mathbb{T}M} \qquad M \xrightarrow{\;v\;} \mathbb{T}M$$

LET'S SUM THEM UP:

$$\hat{v}_1 + \hat{v}_2 : \underbrace{U\pi_K(M;u,v)}_{M} \longrightarrow \underbrace{\mathbb{T}U\pi_K(M;u,v)}_{\mathbb{T}M} \qquad M \xrightarrow{\;u+v\;} \mathbb{T}M$$

BY THE UNIV. PROP. OF .V.F. THERE EXISTS A UNIQUE STRICT MORPHISM

$$+ : \hat{v}_1 + \hat{v}_2 \longrightarrow \hat{v}$$

$$\begin{tikzcd}
VF_2(\mathbb{X},\mathbb{T}) \arrow[r, dashed, "+"] \arrow[d, bend right=60, "\mathbb{T}U\pi_K"'] \arrow[d, bend left=60, "U\pi_K"] \arrow[d, phantom, "\overset{\hat{v}_1+\hat{v}_2}{\Leftarrow}"] & VF(\mathbb{X},\mathbb{T}) \arrow[d, bend right=60, "U"'] \arrow[d, bend left=60, "\mathbb{T}U"] \arrow[d, phantom, "\overset{\hat{v}}{\Rightarrow}"] \\
(\mathbb{X},\mathbb{T}) \arrow[r, equal] & (\mathbb{X},\mathbb{T})
\end{tikzcd}$$

THE COMMUTATIVE MONOID OF VECTOR FIELDS

THEOREM

CONSIDER A TANGENT OBJECT

$$(\mathbb{X},\mathbb{T})$$

WHICH ADMITS THE CONSTRUCION OF VECTOR FIELDS.

$$VF_2(\mathbb{X},\mathbb{T}) \xrightarrow{+} VF(\mathbb{X},\mathbb{T}) \underset{0}{\overset{U}{\rightleftarrows}} (\mathbb{X},\mathbb{T})$$

CONSTITUTES A COMMUTATIVE MONOID IN

$$\mathsf{TNG}(\mathbb{K})/(\mathbb{X},\mathbb{T})$$

THE LIE ALGEBRA OF VECTOR FIELDS

SET UP

IF A TANGENT OBJECT

$(\mathbb{X},\mathbb{T})$

HAS NEGATIVES, SO DOES

$\mathrm{TNG}(\mathbb{K})/(\mathbb{X},\mathbb{T})$

THE LIE ALGEBRA OF VECTOR FIELDS

SET UP

IF A TANGENT OBJECT

$(\mathbb{X}, \mathbb{T})$

HAS NEGATIVES, SO DOES

$\mathrm{TNG}(\mathbb{K})/(\mathbb{X}, \mathbb{T})$

WE CAN TAKE THE LIE BRACKET

$$[\hat{v}_1, \hat{v}_2] : \underbrace{U\pi_K(M; u, v)}_{} \to \underbrace{\mathbb{T}U\pi_K(M; u, v)}_{}$$

$$M \xrightarrow{[u,v]} \mathbb{T}M$$

THE LIE ALGEBRA OF VECTOR FIELDS

SET UP

IF A TANGENT OBJECT

$(\mathbb{X},\mathbb{T})$

HAS NEGATIVES, SO DOES

$\mathsf{TNG}(\mathbb{K})/(\mathbb{X},\mathbb{T})$

WE CAN TAKE THE LIE BRACKET

$$[\hat{v}_1,\hat{v}_2] : \underbrace{U\pi_K(M;u,v)} \to \underbrace{\mathrm{T}U\pi_K(M;u,v)}$$

$$M \xrightarrow{[u,v]} \mathrm{T}M$$

BY THE UNIV. PROP. OF V.F. THERE EXISTS A UNIQUE STRICT MORPHISM

$$[,] : [\hat{v}_1,\hat{v}_2] \longrightarrow \hat{v}$$

THE LIE ALGEBRA OF VECTOR FIELDS

SET UP

IF A TANGENT OBJECT

$(\mathbb{X}, \mathbb{T})$

HAS NEGATIVES, SO DOES

$\mathsf{TNG}(\mathbb{K})/(\mathbb{X}, \mathbb{T})$

WE CAN TAKE THE LIE BRACKET

$$[\hat{v}_1, \hat{v}_2] : \underbrace{U\pi_K(M; \mu, v)} \to \underbrace{\mathbb{T}U\pi_K(M; \mu, v)}$$

$$M \xrightarrow{[\mu, v]} \mathbb{T}M$$

BY THE UNIV. PROP. OF V.F. THERE EXISTS A UNIQUE STRICT MORPHISM

$$[,] : [\hat{v}_1, \hat{v}_2] \longrightarrow \hat{v}$$

$$\begin{tikzcd}
\mathsf{VF}_2(\mathbb{X}, \mathbb{T}) \arrow[rr, dashed, "{[,]}"] \arrow[d, bend right=60, "\mathbb{T}U\pi_K"'] \arrow[d, bend left=60, "U\pi_K"] \arrow[d, phantom, "{[\hat{v}_1, \hat{v}_2]} \Leftarrow"] & & \mathsf{VF}(\mathbb{X}, \mathbb{T}) \arrow[d, bend right=60, "U"'] \arrow[d, bend left=60, "\mathbb{T}U"] \arrow[d, phantom, "\hat{v} \Rightarrow"] \\
(\mathbb{X}, \mathbb{T}) \arrow[rr, equal] & & (\mathbb{X}, \mathbb{T})
\end{tikzcd}$$

THE LIE ALGREBRA OF VECTOR FIELDS

THEOREM

CONSIDER A TANGENT OBJECT WITH NEGATIVES

$$(\mathbb{X}, \mathbb{T})$$

WHICH ADMITS THE CONSTRUCION OF VECTOR FIELDS.

$$(U : VF(\mathbb{X}, \mathbb{T}) \to (\mathbb{X}, \mathbb{T}), 0, +, [,])$$

CONSTITUTES A LIE ALGEBRA OBJECT IN

$$TNG(\mathbb{K})/(\mathbb{X}, \mathbb{T})$$

CHAPTER 4

NEW IDEAS AND FUTURE WORK

DIFFERENTIAL BUNDLES CONNECTIONS AND MANY MORE.

FUTURE WORK AND CONJECTURES

SET UP

IN TANGENT CATEGORY THEORY THERE ARE A FEW OTHER CONSTRUCTIONS:

DIFFERENTIAL OBJECTS

DIFFERENTIAL BUNDLES

CONNECTIONS

DYNAMICAL SYSTEMS

DIFFERENTIAL EQUATIONS

INVOLUTION ALGEBROIDS

FUTURE WORK AND CONJECTURES

SET UP

IN TANGENT CATEGORY THEORY THERE ARE A FEW OTHER CONSTRUCTIONS:

DIFFERENTIAL OBJECTS

DIFFERENTIAL BUNDLES

CONNECTIONS

DYNAMICAL SYSTEMS

DIFFERENTIAL EQUATIONS

INVOLUTION ALGEBROIDS

WE CONJECTURE THAT THEIR FORMAL COUNTERPART CAN BE DEFINED AS THE UNIVERSAL «CONSTRUCTION» IN

$\mathrm{TNG}(\mathbb{K})/(\mathbb{X},\mathbb{I})$

SET UP

FUTURE WORK AND CONJECTURES

IN TANGENT CATEGORY THEORY THERE ARE A FEW OTHER CONSTRUCTIONS:

DIFFERENTIAL OBJECTS

DIFFERENTIAL BUNDLES

CONNECTIONS

DYNAMICAL SYSTEMS

DIFFERENTIAL EQUATIONS

INVOLUTION ALGEBROIDS

WE CONJECTURE THAT THEIR FORMAL COUNTERPART CAN BE DEFINED AS THE UNIVERSAL «CONSTRUCTION» IN

$$\mathrm{TNG}(\mathbb{K})/(\mathbb{X},\mathbb{I})$$

E.G. DIFFERENTIAL BUNDLES FORM THE UNIVERSAL DIFFERENTIAL BUNDLE IN

$$\mathrm{TNG}(\mathbb{K})/(\mathbb{X},\mathbb{I})$$

THE END

THANKS FOR LISTENING

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