Introduction to medial linearly distributive categories

Rose Kudzman-Blais

Supervised by Dr Richard Blute
University of Ottawa

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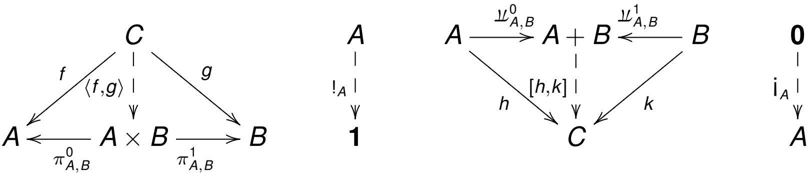
Cartesian and cocartesian categories

Definition

Consider a symmetric monoidal category, or SMC, $\mathcal{X}$.

- $\mathcal{X}$ is *cartesian* if its monoidal structure is given by the binary categorical product and the terminal object, and
- $\mathcal{X}$ *cocartesian* if its monoidal structure is given by the binary categorical coproduct and the initial object.



Comonoids

Definition

Consider a symmetric monoidal category, or SMC, $(\mathcal{X}, \oslash, I, \alpha, \rho, \gamma)$.

- A *cocommutative comonoid* is a triple $\langle A, \Delta_A, t_A \rangle$ of an object A in $\mathcal{X}$ equipped with two morphisms, the *diagonal* $\Delta_A : A \to A \oslash A$ and the *counit* $t_A : A \to I$ such that

$$\Delta_A; (1_A \oslash \Delta_A) = \Delta_A; (\Delta_A \oslash 1_A); \alpha_{A,A,A}$$

$$\Delta_A; (1_A \oslash t_A) = \rho_A \qquad \Delta_A; \gamma_{A,A} = \Delta_A$$

- A *comonoid morphism* $f : \langle A, \Delta_A, t_A \rangle \to \langle B, \Delta_B, t_B \rangle$ is a morphism $f : A \to B$ in $\mathcal{X}$ such that

$$f; \Delta_B = \Delta_A; (f \oslash f) \qquad f; t_B = t_A$$

Fox's theorem

Let $\mathbb{C}(\mathcal{X})$ denote the category of cocommutative comonoids and comonoid morphisms.

Proposition (Fox [7])

Given cocommutative comonoids $\langle A, \Delta_A, t_A \rangle$ and $\langle B, \Delta_B, t_B \rangle$, then $\langle A \oslash B, \Delta_{A \oslash B}, t_{A \oslash B} \rangle$ defined by

$$\Delta_{A \oslash B} = A \oslash B \xrightarrow{\Delta_A \oslash \Delta_B} (A \oslash A) \oslash (B \oslash B) \xrightarrow{s^{\oslash}_{A,A,B,B}} (A \oslash B) \oslash (A \oslash B)$$

$$t_{A \oslash B} = A \oslash B \xrightarrow{t_A \oslash t_B} I \oslash I \xrightarrow{\sim} I$$

is a cocommutative comonoid and $\mathbb{C}(\mathcal{X})$ is a cartesian category with this monoidal product.

$$s^{\oslash}_{A,B,C,D} \colon (A \oslash B) \oslash (C \oslash D) \xrightarrow{\sim} A \oslash (B \oslash (C \oslash D)) \xrightarrow{\sim} A \oslash ((B \oslash C) \oslash D) \xrightarrow{\sim} A \oslash ((C \oslash B) \oslash D) \xrightarrow{\sim} A \oslash (C \oslash (B \oslash D)) \xrightarrow{\sim} (A \oslash C) \oslash (B \oslash D)$$

Lemma (Fox [7])

Given a symmetric strong monoidal functor $F\colon \mathcal{X} \to \mathcal{Y}$ and a commutative comonoid $\langle A, \Delta_A, t_A\rangle$ in $\mathcal{X}$, the triple $\langle F(A), \Delta_{F(A)}, t_{F(A)}\rangle$ defined by

$$\Delta_{F(A)} = F(A) \xrightarrow{F(\Delta_A)} F(A \oslash A) \xrightarrow{m_{\oslash}^{-1}{}_{A,A}} F(A) \oslash F(A)$$

$$t_{F(A)} = F(A) \xrightarrow{F(t_A)} F(I) \xrightarrow{m_I^{-1}} I$$

is a cocommutative comonoid. As such, F canonically extends to a cartesian functor $\mathrm{C}(F)\colon \mathrm{C}(\mathcal{X}) \to \mathrm{C}(\mathcal{X})$.

Theorem (Fox [7])

The functor $\mathrm{C}(-)\colon \underline{\mathrm{SMON}} \to \underline{\mathrm{CART}}$ is right adjoint to the inclusion.

Fox's theorem

Corollary (Fox [7])

A SMC $\mathcal{X}$ is cartesian if and only if it is isomorphic to its category of cocommutative comonoids $\mathrm{C}(\mathcal{X})$.

Corollary (Heunen, Vicary [10])

A SMC $\mathcal{X}$ is cartesian if and only if there are natural transformations

$$t_A \colon A \to I \qquad \Delta_A \colon A \to A \oslash A$$

such that (A, Δ_A, t_A) is cocommutative comonoid and

$$t_{A \oslash B} = (t_A \oslash t_B); \rho_I^{-1} \qquad t_I = 1_I$$

$$\Delta_{A \oslash B} = (\Delta_A \oslash \Delta_B); s^{\oslash}_{A,A,B,B} \qquad \Delta_I = \rho_I.$$

Linearly distributive categories

Definition (Cockett, Seely [5])

A *linearly distributive category*, or LDC, $(\mathbb{X}, \otimes, \top, \oplus, \bot)$ consists of:

- a category $(\mathbb{X}, ;, 1_A)$,
- a *tensor* monoidal structure $(\mathbb{X}, \otimes, \top, \alpha_\otimes, u^R_\otimes, u^L_\oplus)$,
- a *par* monoidal structure $(\mathbb{X}, \oplus, \bot, \alpha_\oplus, u^R_\oplus, u^L_\oplus)$, and
- left and right *linear distributivity* natural transformations

$$\delta^R_{A,B,C} \colon (A \oplus B) \otimes C \to A \oplus (B \otimes C)$$

$$\delta^L_{A,B,C} \colon A \otimes (B \oplus C) \to (A \otimes B) \oplus C$$

satisfying coherence conditons.

Symmetric and cartesian linearly distributive categories

Definition (Cockett, Seely [5])

A linearly distributive category $(\mathbb{X}, \otimes, \top, \oplus, \bot)$ is *symmetric* if

- $(\mathbb{X}, \otimes, \top)$ is a symmetric monoidal category with $\otimes$-braiding $\gamma_\otimes$,
- $(\mathbb{X}, \oplus, \bot)$ is a symmetric monoidal category with $\oplus$-braiding $\gamma_\oplus$,

such that

$$\delta^R_{A,B,C} = \gamma_{\otimes_{A\oplus B,C}}; (1_C \otimes \gamma_{\oplus_{A,B}}); \delta^L_{C,B,A}; (\gamma_{\otimes_{C,B}} \oplus 1_A); \gamma_{\oplus_{B\otimes C,A}}$$

A linearly distributive category $(\mathbb{X}, \otimes, \top, \oplus, \bot)$ is *cartesian* if

- the tensor monoidal structure is cartesian $(\mathbb{X}, \times, \mathbf{1})$, and
- the par monoidal structure is cocartesian $(\mathbb{X}, +, \mathbf{0})$.

Example (Cockett, Seely [5])

A category with finite biproducts is a compact cartesian linearly distributive category.

Definition (Cockett, Seely [5])

A linearly distributive category has *negation* if there are object functions ${}^{\perp}(-)$ and $(-)^{\perp}$, together with the following parametrized family of maps

$$ {}^{\perp}A \otimes A \to \bot \qquad A \otimes A^{\perp} \to \bot \qquad \top \to A^{\perp} \oplus A \qquad \top \to A \oplus {}^{\perp}A $$

satisfying coherence conditions.

Proposition (Joyal's Paradox, Girard [9])

A cartesian linearly distributive category has negation and only if it is a poset.

1. $Hom(\mathbf{0} \times A, B) \cong Hom(\mathbf{0}, B + A^{\perp}) \Rightarrow \mathbf{0} \times A$ *is initial*
2. *If* $Hom(A, \mathbf{0}) \neq \emptyset$, *then consider* $f \colon A \to \mathbf{0}$ *and any* $g \colon A \to B$. *Now,* $g = \langle 1_A, f\rangle ; \pi^1_{A,\mathbf{0}} ; g$ *and* $\pi^1_{A,\mathbf{0}} = !_B^{A\times\mathbf{0}} \colon A \times \mathbf{0} \to B$. *So,* $Hom(A, B) = \{\langle 1_A, f\rangle ; !_B^{A\times\mathbf{0}}\}$
3. $Hom(A, B) \cong Hom(\mathbf{1} \times A, B) \cong Hom(\mathbf{1}, B + A^{\perp}) \cong Hom((B + A^{\perp})^{\perp}, \mathbf{0})$ *If* $Hom(A, B) \neq \emptyset$, *then* $(B + A^{\perp})^{\perp}$ *initial, and* $Hom(A, B)$ *contains one element.*

Distributive cartesian linearly distributive categories

Definition (Cockett [2])

A *distributive category* is a category with finite products and coproducts such that the canonical natural transformations

$$d^R_{A,B,C} = [\amalg^0_{A,B} \times 1_C, \amalg^1_{A,B} \times 1_C] \colon (A \times C) + (B \times C) \to (A + B) \times C$$

$$\rho_A = \langle 1_{\mathbf{0}}, \mathrm{i}_A \rangle \colon \mathbf{0} \to \mathbf{0} \times A$$

are isomorphisms.

Proposition (Cockett, Seely [5])

A distributive category is a cartesian linearly distributive category if and only if it is a preorder.

Exception monad cartesian linearly distributive categories

Definition

Given a distributive category $\mathcal{X}$, define the exception monad by

$$(\ \ ) + \mathbf{1} \colon \mathcal{X} \to \mathcal{X} \qquad \eta_A = \amalg^0_{A,\mathbf{1}} \qquad \mu_A = [1_{A+\mathbf{1}}, \amalg^1_{A,\mathbf{1}}]$$

Proposition (Cockett, Seely [5])

The Kleisli category $\mathcal{X}_{+\mathbf{1}}$ of the exception monad is an isomix cartesian linearly distributive category, with products, coproducts and zero object defined by

- *Initial object and binary coproducts in $\mathcal{X}_{+\mathbf{1}}$ are the same as in $\mathcal{X}$,*
- *Terminal object in $\mathcal{X}_{+\mathbf{1}}$ is the initial object $\mathbf{0}$ in $\mathcal{X}$, with unique arrows*

$$!_A; \amalg^1{}_{\mathbf{0},\mathbf{1}} \colon A \to \mathbf{0} + \mathbf{1}$$

- *Binary products in $\mathcal{X}_{+\mathbf{1}}$ are $A \curlyvee B = (A + B) + (A \times B)$, with projections*

$$\overline{\pi}^0_{A,B} = [[\eta_A, !_B; \amalg^1_{A,\mathbf{1}}], \pi^0_{A,B}; \eta_A] \colon (A + B) + (A \times B) \to A + \mathbf{1}$$
$$\overline{\pi}^1_{A,B} = [[!_A; \amalg^1_{B,\mathbf{1}}, \eta_B], \pi^1_{A,B}; \eta_A] \colon (A + B) + (A \times B) \to B + \mathbf{1}$$

Cartesian Linearly Distributive Categories

The left linearly distributivity in $\mathcal{X}_{+\mathbf{1}}$ is the map in $\mathcal{X}$

$$\delta^L_{A,B,C} \colon (A + (B + C)) + (A \times (B + C)) \to (((A + B) + (A \times B)) + C) + \mathbf{1}$$

defined as the map induced by coproduct universal properties:

$$A \xrightarrow{\amalg^0_{A,B}} A + B \xrightarrow{\amalg^0_{A+B,A\times B}} A \parr B \xrightarrow{\amalg^0_{A\parr B,C}} (A \parr B) + C \xrightarrow{\eta_{(A\parr B)+C}} ((A \parr B) + C) + \mathbf{1}$$

$$B \xrightarrow{\amalg^1_{A,B}} A + B \xrightarrow{\amalg^0_{A+B,A\times B}} A \parr B \xrightarrow{\amalg^0_{A\parr B,C}} (A \parr B) + C \xrightarrow{\eta_{(A\parr B)+C}} ((A \parr B) + C) + \mathbf{1}$$

$$C \xrightarrow{\amalg^1_{A\parr B,C}} (A \parr B) + C \xrightarrow{\eta_{(A\parr B)+C}} ((A \parr B) + C) + \mathbf{1}$$

$$A \times (B + C) \xrightarrow{{d^L}^{-1}_{A,B,C}} (A \times B) + (A \times C) \xrightarrow{1_{A\times B} + !_{A\times C}} (A \times B) + \mathbf{1}$$
$$\xrightarrow{\pi^1_{A+B,A\times B} + 1_{\mathbf{1}}} (A \parr B) + \mathbf{1} \xrightarrow{\pi^0_{A\parr B,C} + 1_{\mathbf{1}}} ((A \parr B) + C) + \mathbf{1}$$

Distributive restriction categories

Definition (Cockett, Lemay [4])

- A *restriction category* is a category with a morphism functor $\overline{(\ \ )}$, known as the restriction operator, satisfying 4 axioms.

$$\overline{f}; f = f \qquad \overline{f}; \overline{g} = \overline{g}; \overline{f} \qquad \overline{\overline{g}; f} = \overline{g}; \overline{f} \qquad f; \overline{g} = \overline{f; g}; f$$

- A morphism $f \colon A \to B$ in a restriction category is *total* if $\overline{f} = 1_A$.
- A restriction category is *cartesian* if it has a *restriction terminal object* $\mathbf{1}_R$ and binary *restriction product* $A \times_R B$
  - there exists a unique total map $t_A \colon A \to \mathbf{1}_R$ such that $f; t_B = \overline{f}; t_A$,
  - there are total maps $p^0_{A,B} \colon A \times_R B \to A$ and $p^1_{A,B} \colon A \times_R B \to A$ and there exist a unique maps $\langle f, g \rangle_R$ such that $\langle f, g \rangle_R; p^0_{A,B} = \overline{g}; f$ and $\langle f, g \rangle_R; p^1_{A,B} = \overline{f}; g$
- A restriction category is *cocartesian* if it has finite coproducts and all injection maps are total.
- A *distributive restriction category* is a cartesian and cocartesian restriction category such that the restriction products distribute over the coproducts.

Join restriction categories

Definition (Cockett, Lemay [4])

Consider a restriction category with zero maps such that $\overline{0} = 0$.

- Let $f, g\colon A \to B$ be parallel maps, then

$$f \leq g \iff \overline{f}; g = f \qquad f \smile g \iff \overline{f}; g = \overline{g}; f$$

$$f \perp g \iff \overline{f}; g = 0$$

- The *join* of a finite family pair-wise compatible maps $f_i\colon A \to B$ is a map $\vee_i f_i\colon A \to B$ such that

$$f_j \leq \vee_i f_i \qquad \text{and} \qquad \text{if } f_i \leq g \Rightarrow \vee_i f_i \leq g$$

- A *join restriction category* is a restriction category with all joins of finite families of pairwise compatible maps and they are stable under pre-composition.

Classical restriction categories

Definition (Cockett, Lemay [4])

Consider a restriction category with zero maps such that $\overline{0} = 0$.

- Let $f \leq g \colon A \to B$, then the *relative complement* of f in g is a map $f \backslash g \colon A \to B$ such that

$$g \backslash f \perp f \qquad \text{and} \qquad g \backslash f \vee f = g$$

- A *classical restriction category* is a join restriction category such that all relative complements exists.

Classical distributive restriction categories

Proposition

Every classical distributive restriction category has classical products, i.e. binary products of the form $A \& B = (A + B) + (A \times_R B)$, with projections

$$\pi^0_{A,B} = [[1_A, 0_{A,B}], p^0_{A,B}] \qquad \pi^1_{A,B} = [[0_{A,B}, 1_B], p^1_{A,B}]$$

Remark

$f \& g = \langle \pi^0_{A,A'}; f, \pi^1_{B,B'}; g \rangle \neq (f + g) + (f \times_R g)$ for $f \colon A \to B$ and $g \colon A' \to B'$

Lemma (Cockett, Lemay [4])

- *Given a distributive restriction category $\mathcal{X}$, the subcategory of total maps $T[\mathcal{X}]$ is a distributive category.*
- *Given a distributive category $\mathbb{X}$, the Kleisli category $\mathbb{X}_{+1}$ of its exception monad is a classical distributive restriction category.*

Classical distributive restriction categories

Proposition (Cockett, Lemay [4])

Given a classical distributive restriction category $\mathcal{X}$, then we have a restriction isomorphism $\mathcal{X} \cong T[\mathcal{X}]_{+1}$.

Theorem

Every classical distributive restriction category is an isomix cartesian LDC.

Example (Cockett, Lemay [4])

- PAR, the category of sets and partial functions
- k-$\mathsf{CALG}_{\bullet}^{op}$, the opposite category of commutative k-algebras and the non-unital k-algebra morphisms
- $\mathsf{TOP}_{\bullet}^{clopen}$, the category of topological spaces and partial continuous maps defined on clopen sets
- $\mathsf{STONE}_{\bullet}^{clopen}$, the category of Stone spaces and partial continuous functions defined on clopen sets $\Rightarrow$ opposite category of Boolean algebras and maps which preserve meets, joins and the bottom.

Motivation

Is there a Fox-like theorem for cartesian LDCs?

By Fox's theorem and its dual statement:

Proposition

A SLDC $\mathbb{X}$ is cartesian if and only if there are natural transformations

$$\Delta_A : A \to A \otimes A \qquad t_A : A \to \top \qquad \nabla_A : A \oplus A \to A \qquad s_A : \bot \to A$$

such that, $\forall A, B \in \mathbb{X}$,

- $\langle A, \Delta_A, t_A \rangle$ *determines a $\otimes$-cocommutative comonoid,*
- $\langle A, \nabla_A, s_A \rangle$ *determines a $\oplus$-commutative monoid, and*

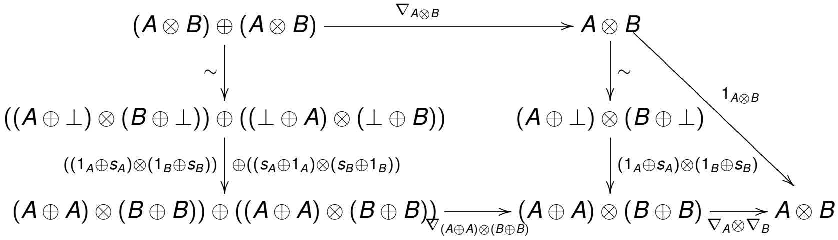
$$\Delta_{A\otimes B} = (\Delta_A \otimes \Delta_B); s^{\otimes}_{A,A,B,B} \qquad t_{A\otimes B} = (t_A \otimes t_B); {u^R_{\otimes\top}}^{-1}$$

$$\nabla_{A\oplus B} = s^{\oplus}_{A,B,A,B}; (\nabla_A \oplus \nabla_B) \qquad s_{A\oplus B} = {u^R_{\oplus\bot}}^{-1}; (s_A \oplus s_B)$$

$$\Delta_\top = u^R_{\otimes\top} \qquad t_\top = 1_\top \qquad \nabla_\bot = u^R_{\oplus\bot} \qquad s_\bot = 1_\bot$$

Given a cartesian linearly distributive category $\mathbb{X}$, there is a map

$$\nabla_{A\otimes B} : (A \otimes B) \oplus (A \otimes B) \to A \otimes B$$



$$\Rightarrow \nabla_{A\otimes B} : (A \otimes B) \oplus (A \otimes B) \xrightarrow{\mu_{A,B,A,B}} (A \oplus A) \otimes (B \oplus B) \xrightarrow{\nabla_A \otimes \nabla_B} A \otimes B$$

Similarly for $s_{A\otimes B}$, $t_{A\oplus B}$ and $\Delta_{A\oplus B}$

Duoidal categories

We need a symmetric linearly distributive category $\mathbb{X}$ with arrows

$$(A \otimes B) \oplus (C \otimes D) \to (A \oplus C) \otimes (B \oplus D)$$

$$\bot \to \bot \otimes \bot \qquad\qquad \top \oplus \top \to \top$$

Answer: **duoidal categories**

Definition (Aguiar, Mahajan [1])

A *duoidal category* $(\mathcal{X}, \diamond, I, \star, J,)$ is category $\mathcal{X}$ with two monoidal structures $(\mathcal{X}, \star, J)$ and $(\mathcal{X}, \diamond, I)$ equipped with morphisms

$$\Delta_I : I \to I \star I \qquad \mu_J : J \diamond J \to J \qquad \iota : I \to J$$

and an *interchange* natural transformation

$$\zeta_{A,B,C,D} : (A \star B) \diamond (C \star D) \to (A \diamond C) \star (B \diamond D)$$

satisfying some coherence conditions.

Definition

A duoidal category $(\mathcal{X}, \diamond, I, \star, J)$ is *symmetric* if

- $(\mathcal{X}, \diamond, I)$ is a symmetric monoidal category with $\diamond$-braiding $\gamma_\diamond$,
- $(\mathcal{X}, \star, J)$ is a symmetric monoidal category with $\oplus$-braiding $\gamma_\star$,

satisfying coherence conditions.

Example (Aguiar, Mahajan [1])

Consider a monoidal category $(\mathcal{X}, \oslash, I)$ with finite coproducts, then $(\mathcal{X}, +, \mathbf{0}, \oslash, I)$ is a duoidal category with structure maps

$$\Delta_\mathbf{0} = \mathrm{i}_{\mathbf{0}\oslash\mathbf{0}} \colon \mathbf{0} \to \mathbf{0} \oslash \mathbf{0} \qquad \mu_I = [1_I, 1_I] \colon I + I \to I \qquad \iota = \mathrm{i}_I \colon \mathbf{0} \to I$$

and interchange natural transformation

$$\zeta_{A,B,C,D} = [\amalg^0_{A,C} \oslash \amalg^0_{B,D}, \amalg^1_{A,C} \oslash \amalg^1_{B,D}] \colon (A \oslash B) + (C \oslash D) \to (A + C) \oslash (B + D)$$

Duoidal bimonoids

Definition (Aguiar, Mahajan [1])

A *bicommutative duoidal bimonoid* in $\mathcal{X}$ is a quintuple $\langle A, \mu, \eta, \Delta, \epsilon \rangle$ where

- $\langle A, \mu, \eta \rangle$ is a commutative $\diamond$-monoid, and
- $\langle A, \Delta, \epsilon \rangle$ is a cocommutative $\star$-comonoid

such that

$$A \diamond A \xrightarrow{\mu} A \xrightarrow{\Delta} A \star A = A \diamond A \xrightarrow{\Delta \diamond \Delta} (A \star A) \diamond (A \star A) \xrightarrow{\zeta_{A,A,A,A}} (A \diamond A) \star (A \diamond A) \xrightarrow{\mu \star \mu} A \star A$$

$$A \diamond A \xrightarrow{\mu} A \xrightarrow{\epsilon} J = A \diamond A \xrightarrow{\epsilon \diamond \epsilon} J \diamond J \xrightarrow{\mu_J} J$$

$$I \xrightarrow{\eta} A \xrightarrow{\Delta} A \star A = I \xrightarrow{\Delta_I} I \star I \xrightarrow{\eta \star \eta} A \star A$$

$$I \xrightarrow{\eta} A \xrightarrow{\epsilon} J = I \xrightarrow{\iota} J$$

A *morphism of duoidal bimonoids* is a morphism of the underlying monoids and comonoids.

Medial linearly distributive categories

Definition

A *medial linearly distributive category*, or MLDC, $(\mathbb{X}, \otimes, \top, \oplus, \bot)$ consists of:

- a category $(\mathbb{X}, ;, 1_A)$,
- a *tensor* monoidal structure $(\mathbb{X}, \otimes, \top)$.
- a *par* monoidal structure $(\mathbb{X}, \oplus, \bot)$,
- *nullary medial*, *nullary comedial* and *mix* morphisms,

$$\nabla_\top \colon \top \oplus \top \to \top \qquad \Delta_\bot \colon \bot \to \bot \otimes \bot \qquad m \colon \bot \to \top$$

- a *medial* natural transformation,

$$\mu_{A,B,C,D} \colon (A \otimes B) \oplus (C \otimes D) \to (A \oplus C) \otimes (B \oplus D)$$

- left and right *linear distributivity* natural transformations

$$\delta^R_{A,B,C} \colon (A \oplus B) \otimes C \to A \oplus (B \otimes C) \quad \delta^L_{A,B,C} \colon A \otimes (B \oplus C) \to (A \otimes B) \oplus C$$

satisfying the coherence conditions on the next slide.

Medial linearly distributive categories

Definition

- $(\mathbb{X}, \otimes, \top, \oplus, \bot)$ is a linearly distributive category,
- $(\mathbb{X}, \oplus, \bot, \otimes, \top)$ is a duoidal category, and
- the *medial maps* interact coherently with the linear distributivities:

$$\begin{array}{ccccc}
((A \otimes B) \oplus (C \otimes D)) \otimes X & \xrightarrow{\mu \otimes 1} & ((A \oplus C) \otimes (B \oplus D)) \otimes X & \xrightarrow{\alpha_\otimes} & (A \oplus C) \otimes ((B \oplus D) \otimes X) \\
\downarrow{\scriptstyle \delta^R} & & & & \downarrow{\scriptstyle 1 \otimes \delta^R} \\
(A \otimes B) \oplus ((C \otimes D) \otimes X) & \xrightarrow[1 \oplus \alpha_\otimes]{} & (A \otimes B) \oplus (C \otimes (D \otimes X)) & \xrightarrow[\mu]{} & (A \oplus C) \otimes (B \oplus (D \otimes X))
\end{array}$$

and 3 other variations.

Definition

A MLDC $(\mathbb{X}, \otimes, \top, \oplus, \bot)$ is *symmetric* if

- $(\mathbb{X}, \otimes, \top, \oplus, \bot)$ is a symmetric LDC and
- $(\mathbb{X}, \oplus, \bot, \otimes, \top)$ is a symmetric duoidal category.

Examples of medial linearly distributive categories

Example

1. Symmetric monoidal categories $(\mathcal{X}, \oslash, I, \oslash, I)$

$$\nabla_I = \rho_I^{-1} = \lambda_I^{-1} \colon I \oslash I \to I \qquad \Delta_I = \rho_I = \lambda_I \colon I \to I \oslash I \qquad m = 1_I \colon I \to I$$

$$\mu_{A,B,C,D} = s^{\oslash}_{A,B,C,D} \colon (A \oslash B) \oslash (C \oslash D) \to (A \oslash C) \oslash (B \oslash D)$$
$$\delta^R_{A,B,C} = \alpha_{A,B,C} \colon (A \oslash B) \oslash C \to A \oslash (B \oslash C)$$

2. Cartesian linearly distributive category $(\mathbb{X}, \times, \mathbf{1}, +, \mathbf{0})$

$$\nabla_{\mathbf{1}} = !_{\mathbf{1}+\mathbf{1}} \colon \mathbf{1} + \mathbf{1} \to \mathbf{1} \qquad \Delta_{\mathbf{0}} = \mathrm{i}_{\mathbf{0}\times\mathbf{0}} \colon \mathbf{0} \to \mathbf{0} \times \mathbf{0} \qquad m = !_{\mathbf{0}} = \mathrm{i}_{\mathbf{1}} \colon \mathbf{0} \to \mathbf{1}$$

$$\mu_{A,B,C,D} = [\iota^0_{A,C} \times \iota^0_{B,D}, \iota^1_{A,C} \times \iota^1_{B,D}] = \langle \pi^0_{A,B} + \pi^0_{C,D}, \pi^1_{A,B} + \pi^1_{C,D} \rangle \colon$$
$$(A \times B) + (C \times D) \to (A + C) \oslash (B + D)$$

Normal duoidal categories

Definition (Garner, Franco [8])

A duoidal category $(\mathcal{X}, \diamond, I, \star, J)$ is *normal* if $\iota\colon I \to J$ is invertible.

Proposition (Spivak, Srinivasan [12])

A normal duoidal category $(\mathcal{X}, \diamond, I, \star, J)$ determines an isomix linearly distributive category $(\mathcal{X}, \diamond, I, \star, J)$, with

$$\begin{aligned}\delta^L_{A,B,C} = & A \diamond (B \star C) \xrightarrow{\sim} (A \star J) \diamond (B \star C) \xrightarrow{\zeta_{A,J,B,C}} (A \diamond B) \star (J \diamond C) \\ & \xrightarrow{\sim} (A \diamond B) \star (I \diamond C) \xrightarrow{\sim} (A \diamond B) \star C\end{aligned}$$

Warning! Normal duoidal categories do not provide examples of medial linearly distributive categories.

Normal duoidal categories

Consider an isomix medial LDC $(\mathbb{X}, \otimes, \top, \oplus, \bot)$, there are maps

$$\alpha^L_{A,B,C}\colon A \oplus (B \otimes C) \to (A \oplus B) \otimes C \qquad \alpha^R_{A,B,C}\colon (A \otimes B) \oplus C \to A \otimes (B \oplus C)$$

Are these inverses of the linearly distributivities δ^R and δ^L?

Proposition (Cockett, Lack [3])

In any restriction category, every monomorphism (so in particular every isomorphism) is total.

Consider linear distributivities in PAR

$$\delta^L_{A,B,C}\colon (A + (B + C)) + (A \times (B + C)) \to ((A + B) + (A \times B)) + C$$

$$a \mapsto a \qquad b \mapsto b \qquad c \mapsto c$$

$$(a, b) \mapsto (a, b) \qquad (a, c) \mapsto \uparrow$$

$\Rightarrow \delta^L_{A,B,C}$ is not a total map!

Distributive symmetric monoidal categories

Definition (Yau [11])

A symmetric monoidal category $(\mathcal{X}, \oslash, I)$ is *distributive* if

- $\mathcal{X}$ has finite coproducts, and
- the following canonical natural transformations

$$d^L_{A,B,C} = [1_A \oslash \iota^0_{B,C}, 1_A \oslash \iota^1{}_{B,C}]\colon (A \oslash B) + (C \oslash D) \to A \oslash (B + C)$$

$$\lambda^\bullet_A = \mathrm{i}_{\mathbf{0} \oslash A}\colon \mathbf{0} \to \mathbf{0} \oslash A$$

Lemma (Folklore)

Given a distributive symmetric monoidal category $(\mathcal{X}, \oslash, I)$, the functor, known as the "either-or-both" product,

$$- \curlyvee - = (- + -) + (- \oslash -)\colon \mathcal{X} \times \mathcal{X} \to \mathcal{X}$$

defines a symmetric monoidal category $(\mathcal{X}, \curlyvee, \mathbf{0})$.

Examples of medial linearly distributive categories

Theorem

Given a distributive symmetric monoidal category $(\mathcal{X}, \oslash, I)$ *with a zero object* $\emptyset$*, then* $(\mathcal{X}, \curlyvee, \emptyset, +, \emptyset)$ *is an isomix symmetric medial linearly distributive category.*

$$\delta^L_{A,B,C} \colon (A + (B + C)) + (A \times (B + C)) \to ((A + B) + (A \oslash B)) + C$$

is determined uniquely by the following maps and the universal properties of coproducts:

$$A \xrightarrow{\amalg^0_{A,B}} A + B \xrightarrow{\amalg^0_{A+B,A\oslash B}} A \curlyvee B \xrightarrow{\amalg^0_{A\curlyvee B,C}} (A \curlyvee B) + C$$

$$B \xrightarrow{\amalg^1_{A,B}} A + B \xrightarrow{\amalg^0_{A+B,A\oslash B}} A \curlyvee B \xrightarrow{\amalg^0_{A\curlyvee B,C}} (A \curlyvee B) + C$$

$$C \xrightarrow{\amalg^1_{A\curlyvee B,C}} (A \curlyvee B) + C$$

$$A \oslash (B + C) \xrightarrow{1_A \oslash (1_B + !_C)} A \oslash (B + \emptyset) \xrightarrow{1_A \oslash u^R_{+B}} A \oslash B \xrightarrow{\amalg^1_{A+B,A\oslash B}} A \curlyvee B \xrightarrow{\amalg^0_{A\curlyvee B,C}} (A \curlyvee B) + C$$

Example

3. Distributive restriction category with zero object

$\Rightarrow$ Restriction product $\times$ and restriction terminal object $\mathbf{1}$ determine a distributive symmetric monoidal category $(\mathcal{X}, \times, \mathbf{1})$.

- Classical distributive restriction categories, e.g. PAR, k-CALG$_{\bullet}^{op}$, TOP$_{\bullet}^{clopen}$ and STONE

  Warning! these are not cartesian LDCs as $\curlyvee$ is not the categorical product & of classical distributive restriction categories

- TOP$_{\bullet}$, the category of topological spaces and partial continuous maps $\rightarrow$ distributive restriction category, which is not classical (Cockett, Lemay [4])

Example

4. Symmetric monoidal closed categories $\mathcal{X}$ with finite coproducts and a zero object (Elgueta [6])

   $\Rightarrow$ Closure ensures the monoidal product distributes over the coproducts

$$Hom((A+B)\oslash C, D) \cong Hom(A+B,[C,D]) \cong Hom(A,[C,D]) \times Hom(B,[C,D])$$
$$\cong Hom(A\oslash C, D) \times Hom(B\oslash C, D) \cong Hom((A\oslash C)+(B\oslash C), D)$$

   Then, Yoneda Lemma implies $(A+B)\oslash C \cong (A\oslash C)+(B\oslash C)$.

   - Mod(R), the category of R-modules and module homomorphisms

     $\rightarrow$ $\oslash$ is usual tensor product of R-modules and $+$ is the direct sum
   - Q-Rel, the category of sets and Q-relations

     $\rightarrow$ $\oslash$ is the cartesian product and $+$ is the disjoint union

   $\Rightarrow$ in both Mod(R) and Q-Rel, the coproduct is a biproduct

Medial bimonoids

Let $\mathbb{X}$ be a symmetric medial linearly distributive category.

Definition

- A *bicommutative medial bimonoid* in $\mathbb{X}$ is a bicommutative duoidal bimonoid in $(\mathbb{X}, \oplus, \bot, \otimes, \top)$, i.e. a quintuple $\langle A, \Delta_A, t_A, \nabla_A, s_A \rangle$ consisting of an object A and four morphisms

$$\Delta_A : A \to A \otimes A \qquad t_A : A \to \top \qquad \nabla_A : A \oplus A \to A \qquad s_A : \bot \to A$$

  satisfying coherence conditions.
- A *medial bimonoid morphism* in $\mathbb{X}$ is a morphism of duoidal bimonoids in $(\mathbb{X}, \oplus, \bot, \otimes, \top)$.

Proposition

$\langle \top, u^R_{\otimes_\top}, 1_\top, \nabla_\top, m \rangle$ and $\langle \bot, \Delta_\bot, m, u^R_{\oplus_\bot}, 1_\bot \rangle$ are bicommutative medial bimonoids.

Medial bimonoids

Proposition

Given two bicommutative medial bimonoids $\langle A, \Delta_A, t_A, \nabla_A, s_A \rangle$ *and* $\langle B, \Delta_B, t_B, \nabla_B, s_B \rangle$ *in* $\mathbb{X}$*, then* $\langle A \otimes B, \Delta_{A\otimes B}, t_{A\otimes B}, \nabla_{A\otimes B}, s_{A\otimes B} \rangle$ *defined by*

$$\Delta_{A\otimes B} = (\Delta_A \otimes \Delta_B); s^{\otimes}_{A,A,B,B} \qquad t_{A\otimes B} = (t_A \otimes t_B); {u^R_{\otimes\top}}^{-1}$$
$$\nabla_{A\otimes B} = \mu_{A,B,A,B}; (\nabla_A \otimes \nabla_B) \qquad s_{A\otimes B} = \Delta_\bot; (s_A \otimes s_B)\ ,$$

and $\langle A \oplus B, \Delta_{A\oplus B}, t_{A\oplus B}, \nabla_{A\oplus B}, s_{A\oplus B} \rangle$ *defined by*

$$\Delta_{A\oplus B} = (\Delta_A \oplus \Delta_B); \mu_{A,A,B,B} \qquad t_{A\oplus B} = (t_A \oplus t_B); \nabla_\top$$
$$\nabla_{A\oplus B} = s^{\oplus}_{A,B,A,B}; (\nabla_A \oplus \nabla_B) \qquad s_{A\oplus B} = {u^R_{\oplus\bot}}^{-1}; (s_A \oplus s_B)\ ,$$

are bicommutative medial bimonoids.

Linearly distributive category of medial bimonoids

Definition

Define $\mathrm{B}(\mathbb{X})$ to be the category of bicommutative medial bimonoids and bimonoid morphisms in $\mathbb{X}$.

Lemma

$\mathrm{B}(\mathbb{X})$ is a linearly distributive category.

By our earlier characterization of cartesian LDCs:

Theorem

$\mathrm{B}(\mathbb{X})$ is a cartesian linearly distributive category.

Proposition (Aguiar, Mahajan [1])

Let $(\mathcal{X}, +, \mathbf{0}, \star, J)$ *be a cocartesian duoidal category, then the category of duoidal bimonoids of* $\mathcal{X}$ *is isomorphic to the category of* $\star$*-comonoids in* $\mathcal{X}$*.*

Proposition

Consider a symmetric cocartesian medial LDC $(\mathbb{X}, \otimes, \top, +, \mathbf{0})$*, then* $\mathrm{B}(\mathbb{X})$ *is isomorphic to the cartesian LDC of cocommutative* $\otimes$*-comonoids* $\mathrm{C}_\otimes(\mathbb{X})$*.*

Proposition (Folklore)

Given a distributive symmetric monoidal category $(\mathcal{X}, \oslash, I)$ *with biproducts, the category of cocommutative* $\curlyvee$*-comonoids* $\mathrm{C}_\curlyvee(\mathcal{X})$ *is isomorphic to the category of cocommutative* $\oslash$*-cosemigroups* $\mathrm{CSGp}_\oslash(\mathcal{X})$*.*

Corollary

Consider a distributive symmetric monoidal categories $(\mathcal{X}, \oslash, I)$ *with biproducts, then* $\mathrm{B}(\mathcal{X}) \cong \mathrm{CSGp}_\oslash(\mathcal{X})$*.*

Linearly distributive Fox theorem

- Given a symmetric strong medial linear functor $F\colon \mathbb{X} \to \mathbb{Y}$, F canonically extends to a cartesian linear functor $\mathrm{B}(F)\colon \mathrm{B}(\mathbb{X}) \to \mathrm{B}(\mathbb{Y})$.
- Given a medial linear transformation $\alpha\colon F \Rightarrow G$, α canonically extends to a cartesian linear transformation $\mathrm{B}(\alpha)\colon \mathrm{B}(F) \Rightarrow \mathrm{B}(G)$.

Therefore, we get:

Theorem

$\mathrm{B}(-)\colon \mathbf{MLDC_s} \to \mathbf{CLDC}$ *is right adjoint to the inclusion 2-functor.*

Corollary

A symmetric medial linearly distributive category is cartesian if and only if it is isomorphic to its category of bicommutative medial bimonoids.

References I

[1] M. Aguiar, S. Mahajan, *Monoidal functors, species and Hopf algebras*, CRM Monograph Series, vol. 29, American Mathematical Society, Providence, RI, 2010.

[2] J.R.B. Cockett, *Introduction to distributive categories*, Math. Structures Comput. Sci. **3** (1993), no, 3, 277–307.

[3] J.R.B. Cockett, S. Lack, *Restriction categories. I. Categories of partial maps*, Theoret. Comput. Sci. **270** (2002), no.1-2, 223–259.

[4] J.R.B. Cockett, J.-S. Lemay, *Classical distributive restriction categories*, Theory Appl. Categ. **42** (2024), no. 6, 102–144.

[5] J.R.B. Cockett, R.A.G. Seely, *Weakly distributive categories*, J. Pure Appl. Algebra **114** (1997), no. 2, 133–173.

[6] J. Elgueta, *The groupoid of finite sets is biinitial in the 2-category of rig categories*, J. Pure Appl. Algebra **225** (2021), no. 11, 39 pp.

[7] T. Fox, *Coalgebras and Cartesian categories*, Comm. Algebra **4** (1976), no. 7, 665–667.

[8] R. Garner, I. López Franco, *Commutativity*, J. Pure Appl. Algebra **220**, no.5, 1707–1751.

[9] J.-Y. Girard, *A new constructive logic: classical logic*, Math. Structures Comput. Sci. **1** (1991), no. 3, 255–296

References II

[10] C. Heunen, J. Vicary, *Categories for quantum theory*, Oxford Graduate Texts in Mathematics, vol. 28, Oxford University Press, Oxford, 2019.

[11] D. Yau, *Bimonoidal Categories, E_n-Monoidal Categories, and Algebraic K-Theory (Volume I: Symmetric Bimonoidal Categories and Monoidal Bicategories)*, Mathematical Surveys and Monographs, vol. 283, American Mathematical Society, Rhode Island, 2024.

[12] D.I. Spivak, P.V. Srinivasan, *What kind of linearly distributive category do polynomial functors form?*, preprint, 2024.