HOMOTOPY COCOMPLETIONS ENRICHED OVER A GENERAL BASE.

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Abstract. Starting from a 1-categorical base $\mathcal{V}$ which is *not* assumed endowed with a choice of model structure (or any kind of homotopical structure), we define homotopy colimits enriched in $\mathcal{V}$ in such a way that: (i) for $\mathcal{V} = \mathsf{Set}$, we retrieve the classical theory of homotopy colimits as presented in [1] and [3], and (ii) restricting to isomorphisms as weak equivalences, we retrieve ordinary and enriched 1-colimits. We construct the free homotopy $\mathcal{V}$-cocompletion of a small $\mathcal{V}$-category in such a way that it satisfies the expected universal property. For $\mathcal{V} = \mathsf{Set}$, we retrieve Dugger's construction of the universal homotopy theory on a small category $\mathcal{C}$. We define the homotopy theory of internal ∞-groupoids in $\mathcal{V}$ as the homotopy $\mathcal{V}$-enriched cocompletion of a point, and argue that $\mathcal{V}$-enriched homotopy colimits correspond to weighted colimits in ∞-categories enriched in internal ∞-groupoids in $\mathcal{V}$, thus providing a convenient model to perform computations. Again, taking $\mathcal{V} = \mathsf{Set}$, this retrieves the classical notions for ordinary $(\infty, 1)$-categories. We compare our approach with some previous definitions of enriched homotopy colimits, such as those in [4], [6] and [7], and we show that, when the latter are defined and well behaved, they coincide with ours up to Quillen homotopy. As an application, we show that, under our definitions, the so-called genuine (or fine) homotopy theory of G-spaces is the G-equivariant homotopy cocompletion of a point. This is a fact conjectured by Hill that, in the case of a finite group, was recently proven by completely different methods in [5].

References

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