Additive Enrichment from Coderelictions

JS PL (he/him)
Good (very early!) morning from Tokyo

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Paris

Today's story comes from a chat with Thomas Ehrhard, when I was visiting l'IRIF in Paris this past May. So here's the only photo I have of myself from that visit.

Why do you need addition in differential categories?

- By additive enrichment I mean the Blute-Cockett-Seely definition to mean enriched over commutative monoids (written additively), so we are in category where we can sum parallel maps $f + g$ and have zero maps 0, such that composition preserves the additive structure.

- Differential categories need additive enrichment to help express the Leibniz rule and the constant rule:

$$(f \cdot g)'(x) = f'(x) \cdot g(x) + f(x) \cdot g'(x) \qquad\qquad c'(x) = 0$$

Differential Linear Categories Quickly

Differential linear categories provide the categorical semantics of Differential Linear Logic. Briefly, a differential linear category is:

- An **additive symmetric monoidal category**, that is, a symmetric monoidal category enriched over commutative monoids (so both composition and $\otimes$ preserve addition);

- Equipped with a **monoidal coalgebra modality** (or equivalently an **additive bialgebra modality**), which in particular is a comonad ! equipped with extra structural maps;

- Which comes with a **codereliction** which is a natural transformation $\eta_A : A \to !(A)$ satisfying the following five axioms[1]:
  - **[dc.1]** Constant rule: $\eta_A; \mathsf{e}_A = 0$
  - **[dc.2]** Product rule: $\eta_A; \Delta_A = (\eta_A \otimes \mathsf{u}_A) + (\mathsf{u}_A \otimes \eta_A)$
  - **[dc.3]** Linear rule: $\eta_A; \varepsilon_A = 1_A$
  - **[dc.4]** Chain rule: $\eta_A; \delta_A = (\mathsf{u}_A \otimes \eta_A); (\delta_A \otimes \eta_{!(A)}); \nabla_{!(A)}$
  - **[dc.m]** Monoidal rule: $(1_{!(B)} \otimes \eta_A); \mathsf{m}_{A,B} = (\varepsilon_B \otimes 1_A); \eta_{A \otimes B}$

  From the codereliction we can define the differential operator.

Blute, R., Cockett, R., Seely, R.A.G., Lemay, J.-S. P. **Differential categories revisited**. (2019)

[1] Composition is written is diagrammatic order using ;

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- Equipped with a **monoidal coalgebra modality** (or equivalently an **additive bialgebra modality**), which in particular is a comonad ! equipped with extra structural maps;

- Which comes with a **codereliction** which is a natural transformation $\eta_A : A \to !(A)$ satisfying the following five axioms[2]:

  - ~~**[dc.1]** Constant rule: $\eta_A; \mathsf{e}_A = 0$~~
  - ~~**[dc.2]** Product rule: $\eta_A; \Delta_A = (\eta_A \otimes \mathsf{u}_A) + (\mathsf{u}_A \otimes \eta_A)$~~
  - **[dc.3]** Linear rule: $\eta_A; \varepsilon_A = 1_A$
  - **[dc.4]** Chain rule: $\eta_A; \delta_A = (\mathsf{u}_A \otimes \eta_A); (\delta_A \otimes \eta_{!(A)}); \nabla_{!(A)}$
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  From the codereliction we can define the differential operator.

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In turns out **[dc.1]** and **[dc.2]** are redundant/can be proved. What do we notice?

ANSWER: The axioms of a codereliction don't need addition!

[2]Composition is written is diagrammatic order using ;

Today's Story

- Since addition does not seem necessary for the axioms of a codereliction, one can ask can we talk about differential linear categories without additive enrichment?
- Well, the other codereliction axioms still need the canonical bialgebra structural maps of the monoidal coalgebra modality...
- **BUT** the bialgebra structure no longer comes for free without additive enrichment! So we have to ask that our monoidal coalgebra modality comes with this extra bialgebra structure.
- **TODAY'S STORY:** It turns out that from a monoidal coalgebra modality with extra bialgebra structure (which we call a monoidal bialgebraic modality) which comes equipped with a codereliction, we obtain additive enrichment!

Theorem

A differential linear category is equivalently a symmetric monoidal category with a monoidal bialgebraic modality equipped with a codereliction.

Coalgebra Modality

For a symmetric monoidal[3] category $\mathbb{X}$, with monoidal product $\otimes$ and monoidal unit I, a **coalgebra modality** $!$ on $\mathbb{X}$ consists of:

- An endofunctor $! : \mathbb{X} \to \mathbb{X}$
- Four natural transformations (the names come from Linear Logic):

$$\underbrace{\delta_A : !(A) \to !!(A)}_{\textit{Digging}} \qquad \underbrace{\varepsilon_A : !(A) \to A}_{\textit{Dereliction}} \qquad \underbrace{\Delta_A : !(A) \to !(A) \otimes !(A)}_{\textit{Contraction}} \qquad \underbrace{\mathsf{e}_A : !(A) \to I}_{\textit{Weakening}}$$

such that:

- $(!, \delta, \varepsilon)$ is a comonad
- $(!(A), \Delta_A, \mathsf{e}_A)$ is a cocommutative comonoid
- The digging $\delta_A : (!(A), \Delta_A, \mathsf{e}_A) \to (!!(A), \Delta_{!(A)}, \mathsf{e}_{!(A)})$ is a comonoid morphism.

Remark

It is worth mentioning that the naturality of Δ and e means that for every map $f : A \to B$, $!(f) : (!(A), \Delta_A, \mathsf{e}_A) \to (!(B), \Delta_B, \mathsf{e}_B)$ is a comonoid morphism.

[3] For simplicity, we work in the strict setting.

Monoidal Coalgebra Modality

A **monoidal coalgebra modality** on a symmetric monoidal $\mathbb{X}$ is a coalgebra modality ! further equipped with:

- A natural transformation and a map:

$$\mathsf{m}_{A,B} : !(A) \otimes !(B) \to !(A \otimes B) \qquad \mathsf{m}_I : I \to !(I)$$

such that:

- $(!, \mathsf{m}, \mathsf{m}_I)$ is a symmetric monoidal functor;
- δ and ε are monoidal transformations;
- Δ and e are monoidal transformations (or equivalently m and m_I are comonoid morphisms)
- Δ and e are !-coalgebra morphisms.

Remark

Monoidal coalgebra modalities are what capture the exponential modality from Linear Logic. There are some other names for monoidal coalgebra modalities in the literature including linear exponential modalities.

For all the commtuative diagrams (and string diagrams):

Blute, R., Cockett, R., Seely, R.A.G., Lemay, J.-S. P. **Differential categories revisited**. (2019)

Bialgebraic Modality **NEW**-ish

A **bialgebraic modality** on a symmetric monoidal $\mathbb{X}$ is a coalgebra modality ! further equipped with:

- Two natural transformations (the names come from Linear Logic):

$$\underbrace{\nabla_A : !(A) \otimes !(A) \to !(A)}_{\textit{Cocontraction}} \qquad \underbrace{\mathsf{u}_A : I \to !(A)}_{\textit{Coweakening}}$$

such that:

- $(!(A), \nabla_A, \mathsf{u}_A)$ is a commutative monoid;
- $(!(A), \nabla_A, \mathsf{u}_A, \Delta_A, \mathsf{e}_A)$ is a bimonoid.

Remark

It is worth mentioning that the naturality of ∇ and u means that for every map $f : A \to B$, $!(f) : (!(A), \nabla_A, \mathsf{u}_A) \to (!(B), \nabla_B, \mathsf{u}_B)$ is a monoid morphism. Thus $!(f)$ is a bimonoid morphism.

Remark

Why bialgebraic modality and not just bialgebra modality? Because the name bialgebra modality is already taken...

Remark

We will come back to something interesting about bialgebraic modalities at the end, as why they are interesting on their own...

Monoidal Bialgebraic Modality **NEW**

A **monoidal bialgebraic modality** on a symmetric monoidal $\mathbb{X}$ is a bialgebra modality ! which is also a monoidal coalgebra modality, and such that:

- ∇ and u are !-coalgebra morphisms, that is, the following diagrams commute:

$$
\begin{array}{ccccc}
!(A) \otimes !(A) & & \xrightarrow{\nabla_A} & & !(A) \\
\downarrow{\scriptstyle \delta_A \otimes \delta_A} & & & & \downarrow{\scriptstyle \delta_A} \\
!!(A) \otimes !!(A) & \xrightarrow[\mathsf{m}_{!(A),!(A)}]{} & !(!(A) \otimes !(A)) & \xrightarrow[!(\nabla_A)]{} & !!(A)
\end{array}
\qquad
\begin{array}{ccc}
I & \xrightarrow{\mathsf{u}_A} & !(A) \\
\downarrow{\scriptstyle \mathsf{m}_I} & & \downarrow{\scriptstyle \delta_A} \\
!(I) & \xrightarrow[!(\mathsf{u}_A)]{} & !!(A)
\end{array}
$$

- m is compatible with ∇ and u in the sense that the following diagrams commute:

$$
\begin{array}{ccccc}
!(A) \otimes !(B) \otimes !(B) & & \xrightarrow{1_{!(A)} \otimes \nabla_B} & & !(A) \otimes !(B) \\
\downarrow{\scriptstyle \Delta_A \otimes 1_{!(B)} \otimes 1_{!(B)}} & & & & \\
!(A) \otimes !(A) \otimes !(B) \otimes !(B) & & & & \downarrow{\scriptstyle \mathsf{m}_{A,B}} \\
\downarrow{\scriptstyle 1_{!(A)} \otimes \sigma_{!(A),!(B)} \otimes 1_{!(A)}} & & & & \\
!(A) \otimes !(B) \otimes !(A) \otimes !(B) & \xrightarrow[\mathsf{m}_{A,B} \otimes \mathsf{m}_{A,B}]{} & !(A \otimes B) \otimes !(A \otimes B) & \xrightarrow[\nabla_{A,B}]{} & !(A \otimes B)
\end{array}
$$

$$
\begin{array}{ccc}
!(A) & \xrightarrow{1_{!(A)} \otimes \mathsf{u}_B} & !(A) \otimes !(B) \\
\downarrow{\scriptstyle \mathsf{e}_A} & & \downarrow{\scriptstyle \mathsf{m}_{A,B}} \\
I & \xrightarrow[\mathsf{u}_{A \otimes B}]{} & !(A \otimes B)
\end{array}
$$

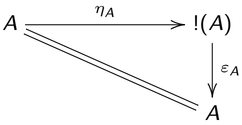
Codereliction **NEW**-ish

For a monoidal bialgebraic modality !, a **codereliction** is a natural transformation:

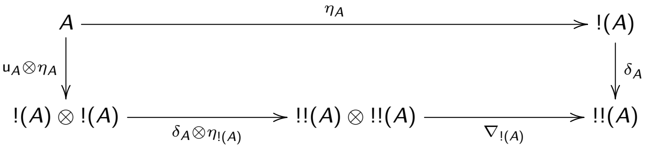
$$\eta_A : A \to !(A)$$

such that the following diagrams commute:

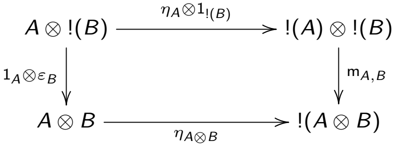
- **[dc.3]** Linear rule:



- **[dc.4]** Chain rule:



- **[dc.m]** Monoidal rule:



Uniqueness of Coderelictions

Lemma

For a monoidal bialgebraic modality, if a codereliction exists, then it is unique.

Remark

This was essentially my talk for Octoberfest 2022.

Objective: Additive Enrichment

GOAL: Show that from a monoidal bialgebraic modality with a codereliction we get additive enrichment. Explicitly we want to show that the underlying category is an additive symmetric monoidal category.

An **additive symmetric monoidal category** is a symmetric monoidal category $\mathbb{X}$ such that each homset $\mathbb{X}(A, B)$ is a commutative monoid, with binary operation $+ : \mathbb{X}(A, B) \times \mathbb{X}(A, B) \to \mathbb{X}(A, B)$ and neutral element $0 \in \mathbb{X}(A, B)$, such that:

- Composition preserves the additive structure, that is:

$$f; (g + h) = (f; g) + (f; h) \qquad f; 0 = 0$$
$$(g + h); k = (g; k) + (h; k) \qquad 0; k = 0$$

- The monoidal product preserves the additive structure, that is:

$$f \otimes (g + h) = (f \otimes g) + (f \otimes h) \qquad f \otimes 0 = 0$$
$$(g + h) \otimes k = (g \otimes k) + (h \otimes k) \qquad 0 \otimes k = 0$$

Main Result

Theorem

Let $\mathbb{X}$ be a symmetric monoidal category with a monoidal bialgebraic modality $!$ which comes equipped with a codereliction η. Then $\mathbb{X}$ is an additive symmetric monoidal category where:

- *For parallel maps $f : A \to B$ and $g : A \to B$, their sum $f + g : A \to B$ is defined as the following composite:*

$$A \xrightarrow{\eta_A} !(A) \xrightarrow{\Delta_A} !(A) \otimes !(A) \xrightarrow{!(f) \otimes !(g)} !(B) \otimes !(B) \xrightarrow{\nabla_B} !(B) \xrightarrow{\varepsilon_B} B$$

- *The zero map $0 : A \to B$ is defined as the following composite:*

$$A \xrightarrow{\eta_A} !(A) \xrightarrow{\mathsf{e}_A} I \xrightarrow{\mathsf{u}_B} !(B) \xrightarrow{\varepsilon_B} B$$

Remark

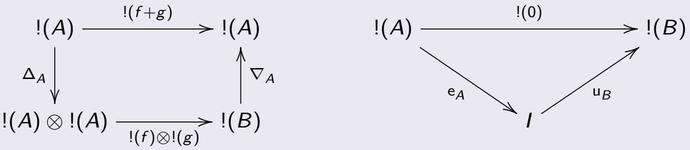
Note that we've used the dereliction and codereliction to take the canonical bialgebra convolution operation on $\mathbb{X}(!(A), !(B))$ down to $\mathbb{X}(A, B)$.

Some words on the proof

The key to the proof is first proving the following lemma:

Lemma

The following diagrams commute:



where $+$ *and* 0 *are defined as on the previous slide.*

Proving this does require a bit of work. But this identity is crucial for associativity $+$ and that 0 is its unit. Commutativity of $+$ is also straightforward.

Proving that composition preserves the monoid structure is mostly automatic by naturality.

Proving that the monoidal product preserves the monoid structure follows from the monoidal rule **[dc.m]** (and also involves some work).

You didn't need the chain rule

For both uniqueness of the coderelicition and the additive enrichment theorem, we only needed η to satisfy **[dc.3]** and **[dc.m]**. The chain rule was not required... Name suggestions for such an η?

- Weak codereliction
- Quasi-codereliction
- Pseudo-codereliction
- Chainless codereliciton

But for today's story, let's assume the chain rule as well. Because you do need the chain rule for...

We want this:

Theorem

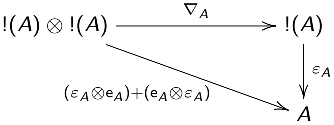
A differential linear category is equivalently a symmetric monoidal category with a monoidal bialgebraic modality equipped with a codereliction.

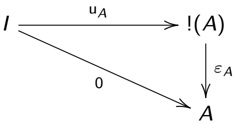
To see why this is true, it comes from the fact that for an additive symmetric monoidal category, monoidal coalgebra modality = additive bialgebra modality.

Additive Bialgebra Modality

For an additive symmetric monoidal category $\mathbb{X}$, an **additive bialgebra modality** is a bialgebraic modality ! such that:

- The dereliction ε is compatible with the monoid structure in the sense that the following diagrams commute:

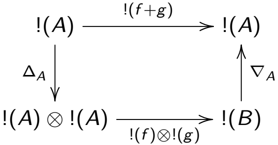


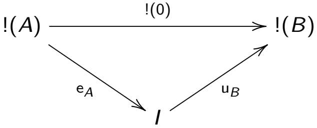


Remark

Stopping here is what's called a bialgebra modality.

- The following diagrams commute:





Remark

Hopefully these two diagrams look familiar!

Additive Bialgebra Modality = Monoidal Coalgebra Modality

Theorem

For an additive symmetric monoidal category, the following are in bijective correspondence:

- *Monoidal Coalgebra Modalities*
- *Additive Bialgebra Modalities*

So in the additive setting, from m and m_I we can build ∇ and u, and vice-verca. For full details see:

Blute, R., Cockett, R., Seely, R.A.G., Lemay, J.-S. P. **Differential categories revisited**. (2019)

Additive Bialgebra Modality = Monoidal Coalgebra Modality = Monoidal Bialgebraic Modality

Theorem

For an additive symmetric monoidal category, the following are in bijective correspondence:

- *Monoidal Coalgebra Modalities*
- *Additive Bialgebra Modalities*
- *Monoidal Bialgebraic Modalities*

- Starting from a monoidal bialgebraic modality !.
  - Step 1: By definition ! is a monoidal coalgebra modality.
  - Step 2: ! is also an additive bialgebra modality.
  - The construction for Step 2 gives us back exactly ∇ and u we started with.

- Starting from a monoidal coalgebra modality !.
  - Step 1: ! is also an additive bialgebra modality.
  - Step 2: So by definition we get ! is a bialgebraic modality as well.
  - Step 3: That ∇ and u are !-coalgebra morphisms follows from the fact that they are constructed from !-coalgebra morphisms.
  - Step 4: The other required diagrams were shown to be true for monoidal coalgebra modalities/additive bialgebra modalities:

Fiore, M. **Differential structure in models of multiplicative biadditive intuitionistic linear logic** (2007)

Blute, R., Cockett, R., Seely, R.A.G., Lemay, J.-S. P. **Differential categories revisited**. (2019)

It really is all the same!

Theorem

For an additive symmetric monoidal category $\mathbb{X}$, if $!$ is a monoidal coalgebra modality with a codereliction, then the induced additive enrichment from the previous slide is the same as the starting enrichment on $\mathbb{X}$.

It really is all the same!

Theorem

A differential linear category is equivalently a symmetric monoidal category with a monoidal bialgebraic modality equipped with a codereliction.

Quick word about negatives

If you also want enrichment over abelian groups, then you need an **antipode**, so you need Hopf structure!

A **(monoidal) Hopf algebraic modality** on a symmetric monoidal $\mathbb{X}$ is a monoidal bialgebraic modality ! further equipped with a natural transformation:

$$\mathsf{S}_A : !(A) \to !(A)$$

such that:

- $(!(A), \nabla_A, \mathsf{u}_A, \Delta_A, \mathsf{e}_A, \mathsf{S}_A)$ is a Hopf monoid.

Theorem

Let $\mathbb{X}$ be a symmetric monoidal category with a monoidal Hopf bialgebraic modality ! which comes equipped with a codereliction η. Then $\mathbb{X}$ is also enriched over abelian groups where:

- *For a map $f : A \to B$, its negative $-f : A \to B$ is defined as the following composite:*

$$A \xrightarrow{\eta_A} !(A) \xrightarrow{\mathsf{S}_A} !(A) \xrightarrow{!(f)} !(B) \xrightarrow{\varepsilon_B} B$$

Lemay, J.-S. P. **Lifting Coalgebra Modalities and MELL Model Structure to Eilenberg-Moore Categories** (2019)

Quick word about bialgebraic modalities again:

An interesting fact about bialgebraic modalities is that their coKleisli category is a **left additive category** (no codereliction required!).

An **left additve category** is a category $\mathbb{X}$ such that each homset $\mathbb{X}(A, B)$ is a commutative monoid, with binary operation $+ : \mathbb{X}(A, B) \times \mathbb{X}(A, B) \to \mathbb{X}(A, B)$ and neutral element $0 \in \mathbb{X}(A, B)$, such that:

- Pre-Composition preserves the additive structure, that is:

$$(g + h); k = (g; k) + (h; k) \qquad 0; k = 0$$

Theorem

Let $\mathbb{X}$ be a symmetric monoidal category with a bialgebraic modality $!$. Then $\mathbb{X}_!$ is a left additive category where:

- *For parallel coKleisli maps $f : !(A) \to B$ and $g : !(A) \to B$, their sum $f + g : !(A) \to B$ is defined as the following composite:*

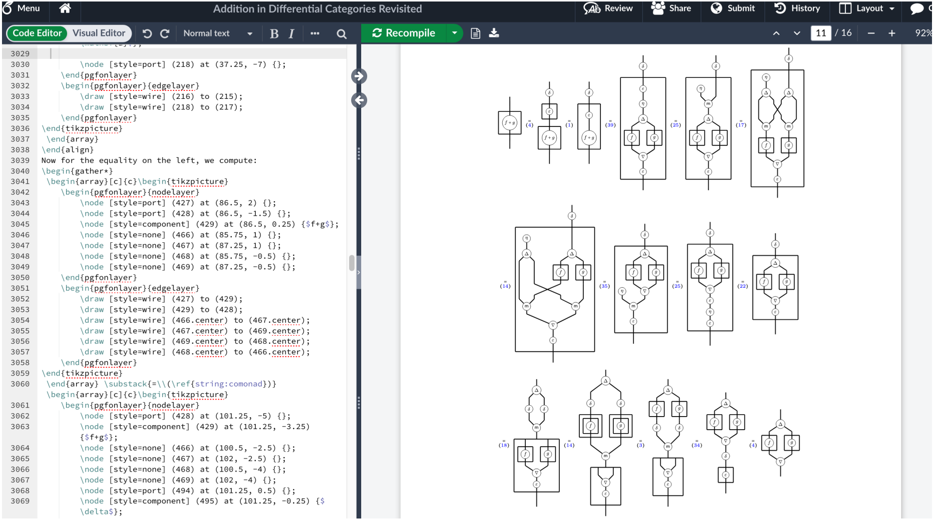
$$!(A) \xrightarrow{\Delta_A} !(A) \otimes !(A) \xrightarrow{\delta_A \otimes \delta_A} !!(A) \otimes !!(A) \xrightarrow{!(f) \otimes !(g)} !(B) \otimes !(B) \xrightarrow{\nabla_B} !(B) \xrightarrow{\varepsilon_B} B$$

- *The zero map $0 : !(A) \to B$ is defined as the following composite:*

$$!(A) \xrightarrow{\mathsf{e}_A} I \xrightarrow{\mathsf{u}_B} !(B) \xrightarrow{\varepsilon_B} B$$

If $\mathbb{X}$ has products, then $\mathbb{X}_!$ is also a **Cartesian left additive category**.

Preprint coming soon (with lots of nice string diagrams!)



That's all folks! Here's a photo of me from a hike in the Japanese Alps

HOPE YOU ENJOYED MY TALK! THANK YOU FOR LISTENING! MERCI!

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https://sites.google.com/view/jspl-personal-webpage/