

Categorical Median Algebra

the study of "median algebras"

Advertisement for a paper by the same name
K, Mine, Sotirov. (near complete draft on my website)

seeking
feedback

① What is a median algebra?

↳ Forgetful Functors (The Setting)

↳ The Two-Element Boolean Algebra (Axioms by Example)

② Main Results

↳ Dual Adjunctions

← included in these notes but not enough time to talk about.

↳ Concrete Two-Variable Adjunctions

↳ Bool as a Category of Contravariant Adjunctions

↳ Isbell Duality

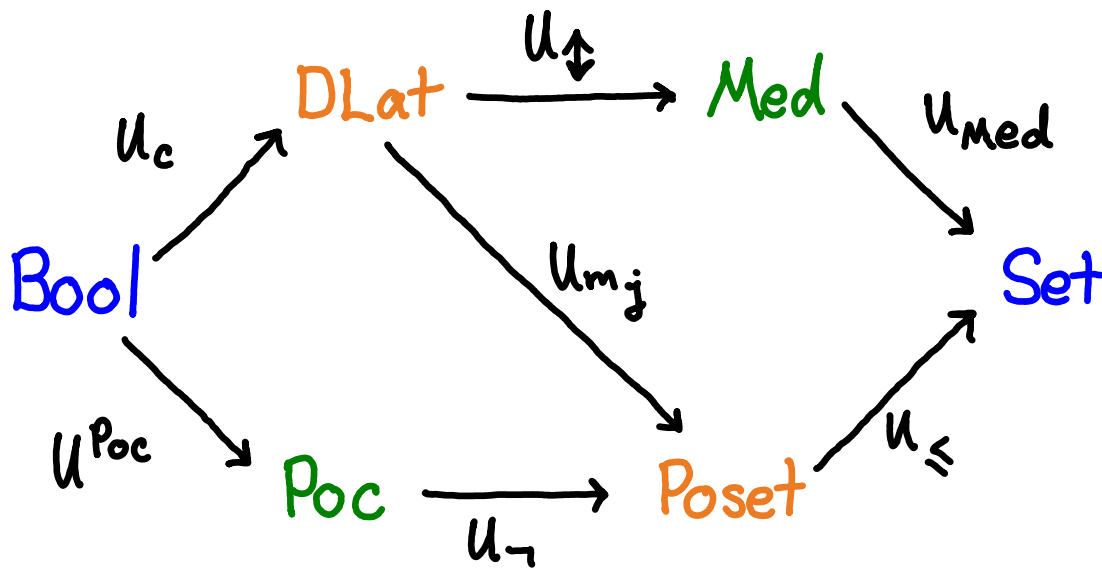
↳ Codensity Monads

③ Proof Techniques

↳ Concrete Bicompatibility (Tensor-Hom Adjunctions)

↳ Categories of Limit Preserving Functors (Sketches & Lex)

Forgetful Functors



$$U^{med} := U_d U_c$$

$$U_{Poc} := U_{\leq} U_{\neg}$$

$$U_d : \mathbf{DLat} \rightarrow \mathbf{Med}$$

$$D \mapsto (D, m_{Bool})$$

(self-dual) Boolean Majority Functions

$$m_{Bool}(x, y, z) = (x \wedge y) \vee (x \wedge z) \vee (y \wedge z)$$

$$= (x \vee y) \wedge (x \vee z) \wedge (y \vee z).$$

Claim: U_d "forgets only up-down" (Median Algebra - Isbell 1980)

$$U_c : \mathbf{Bool} \rightarrow \mathbf{DLat}$$

Boolean algebras are **Complemented** distributive lattices.

$$U_{mj} : \mathbf{DLat} \rightarrow \mathbf{Poset} \quad \forall D \in \mathbf{DLat} \quad \exists \text{ A compatible partial order}$$

$$x \wedge y = x \iff x \leq y \iff x \vee y = y.$$

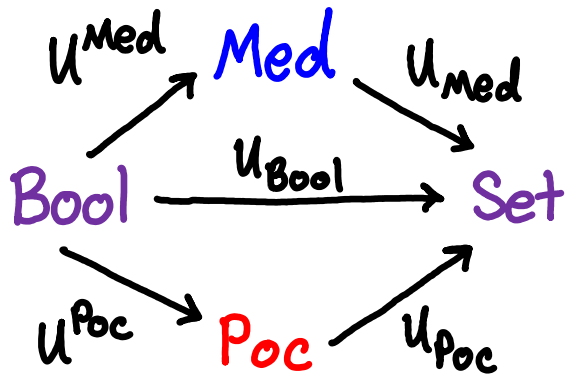
$$U_{\neg} : \mathbf{Poc} \rightarrow \mathbf{Poset}$$

$$(P, \perp, \neg, \Rightarrow) \mapsto (P, \Rightarrow)$$

The 'c' in Poc-set stands for **negation** ($\sim$ **Complementation** in Bool).

The Key takeaway:

$$U_{Med} U^{Med} = U_{Bool} = U_{Poc} U^{Poc}$$



Boolean algebras are examples of both poc-sets & median algebras. Surprisingly...

A simultaneous poc-set & median algebra is a Boolean algebra iff the two structures are suitably compatible.

The Two-Element Boolean Algebra

Data: $(\mathcal{Q}, e_{\perp}, \neg, \Rightarrow, m)$

- (i) A set $\mathcal{Q} = \{\perp, \top\}$ containing "False" & "True"
- (ii) A distinguished element $e_{\perp} \in \mathcal{Q}$
- (iii) An unary operation $\neg: \mathcal{Q} \rightarrow \mathcal{Q}$ called negation
- (iv) A binary relation $\Rightarrow \subseteq \mathcal{Q} \times \mathcal{Q}$ called implication
- (v) A ternary operation $m: \mathcal{Q}^3 \rightarrow \mathcal{Q}$ "the median"

Axioms: ($\triangle$ we use $\rightarrow$ rather than $\Rightarrow$ in the paper)

(I) $(\mathcal{Q}, e_{\perp}, \neg, \Rightarrow)$ is a poc-set i.e.,

$\hookrightarrow (\mathcal{Q}, \Rightarrow)$ is a partial order

$\hookrightarrow \neg: (\mathcal{Q}, \Rightarrow) \rightarrow (\mathcal{Q}, \Rightarrow)$ is an antitone involution

$\hookrightarrow \forall p \in \mathcal{Q} \quad \perp \Rightarrow p$ bottom element

$\hookrightarrow \forall p \in \mathcal{Q} \quad p \Rightarrow \neg p$ implies $p = \perp$ non-contradiction

(II) $(\mathcal{Q}, m)$ is a median algebra i.e., $\forall w, x, y, z \in \mathcal{Q}$

$\hookrightarrow m(x, x, y) = x$ majority rule

$\hookrightarrow m(x, y, z) = m(y, x, z) = m(x, z, y)$ symmetry

$\hookrightarrow m(x, w, m(y, w, z)) = m(m(x, w, y), w, z)$ associativity

note: m is uniquely determined by majority rule &

Symmetry. m is associative because

$$m(-, \perp, -) = \wedge \quad \& \quad m(-, \top, -) = \vee$$

logical AND

logical OR

(III) $(\mathcal{Q}, e_{\perp}, \neg, \Rightarrow)$ and $(\mathcal{Q}, m)$ are compatible

$\hookrightarrow \neg \circ m = m \circ \neg^3$ negation is median preserving

proof:

De Morgan's Laws

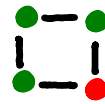
$$\neg m(p, \perp, q) = \neg(p \wedge q) = \neg p \vee \neg q = m(\neg p, \top, \neg q)$$

$$\neg m(p, \top, q) = \neg(p \vee q) = \neg p \wedge \neg q = m(\neg p, \perp, \neg q) \quad \square$$

$\hookrightarrow \Rightarrow$ is a **median subalgebra** of $(\mathcal{Q}^2, m^2) = (\mathcal{Q}^2, m_{\text{Bool}})$
i.e., $x \Rightarrow x', y \Rightarrow y', z \Rightarrow z'$ implies $m(x, y, z) \Rightarrow m(x', y', z')$

proof: As a relation,

$$\Rightarrow := \{(\perp, \perp), (\perp, \top), (\top, \top)\} \subseteq \mathcal{Q} \times \mathcal{Q}$$

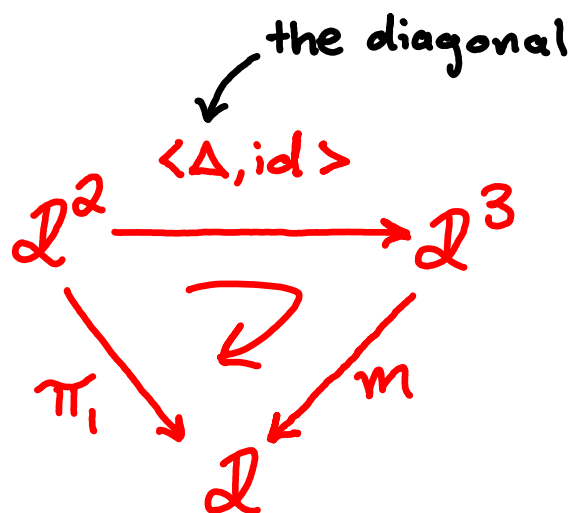


Given three elements of $\mathcal{Q} \times \mathcal{Q}$. $x \bullet \text{---} \underset{y=m(x,y,z)}{\bullet} \text{---} z \quad \square$

Lets take another look at these axioms using morphisms (rather than elements) to express ourselves.

majority rule $(\mathcal{Q}, m)$

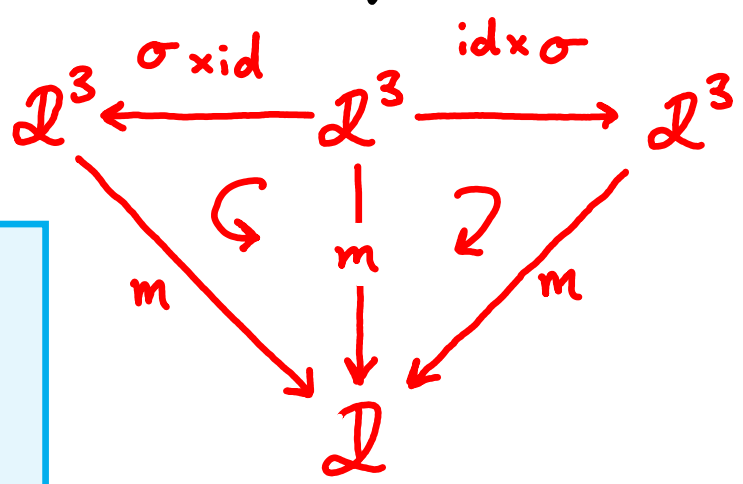
$$m(x, x, y) = x$$



Let $\sigma: \mathbb{Z}^2 \rightarrow \mathbb{Z}^2$ be the coordinate swapping function

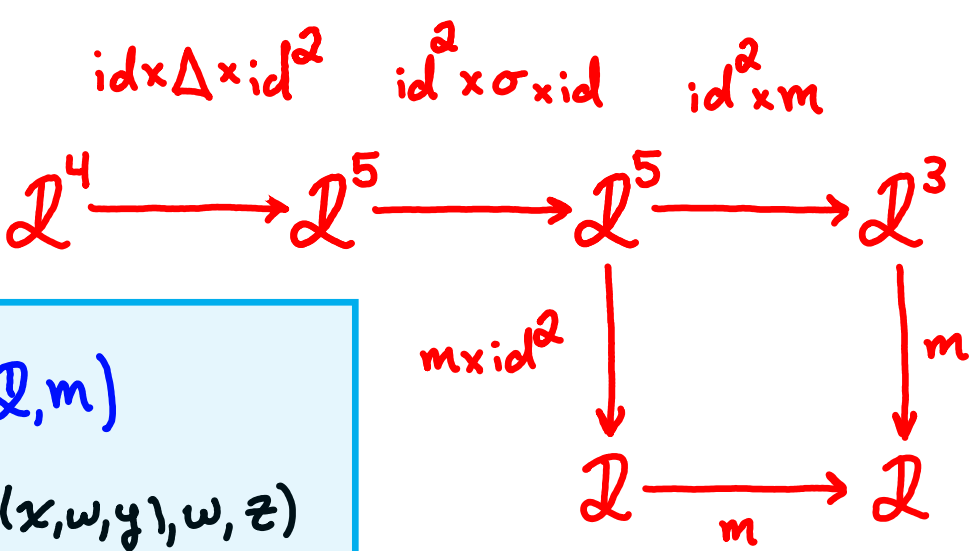
Symmetry $(\mathbb{Z}, m)$

$$m(x, y, z) = m(y, x, z) = m(x, z, y)$$



Associativity $(\mathbb{Z}, m)$

$$m(x, w, m(y, w, z)) = m(m(x, w, y), w, z)$$



The punch line: $(\text{Poc}, \times, \{\cdot\}, \sigma)$ is a symmetric monoidal category. In particular, these diagrams are internal to Poc !

Lemma: $\exists$ subcategories $\mathcal{M} \subseteq \text{Med}, \mathcal{P} \subseteq \text{Poc}$

Underlying a pair of normal limit sketches $\mathcal{I}, \mathcal{M}$

$\mathcal{I} := (\mathcal{M}, \mathbb{L}_{\mathcal{M}}, \Sigma_{\mathcal{M}}) \quad \mathcal{M} := (\mathcal{P}, \mathbb{L}_{\mathcal{P}}, \Sigma_{\mathcal{P}}) \quad \text{s.t.}$

$\text{Mod}_{\text{set}}(\mathcal{I}) \cong \text{Poc} \quad \& \quad \text{Mod}_{\text{set}}(\mathcal{M}) \cong \text{Med}.$

proof sketch: $\text{Ob}(\mathbb{P}) := \{\mathcal{Q}^k\}_{k=1}^5$

$\text{Mor}(\mathbb{P})$ is generated under composition by the projections

$$\bigcup_{k=1}^5 \{\pi_i: \mathcal{Q}^k \longrightarrow \mathcal{Q}\}_{i=1}^k \quad \& \quad \text{the 8 morphisms used in}$$

diagrams above.

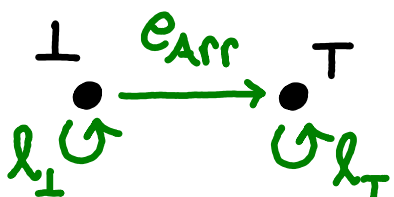
$\mathbb{L}_{\mathbb{P}}$ is the set of discrete diagrams with 2, 3, 4 and 5 copies of $\mathcal{Q}$. Normality of $\mathcal{M}$ determines $\Sigma_{\mathbb{P}}$.

Then models preserve products, projections and commutativity of the diagrams above.

Constructing $\mathcal{P}$ requires more effort because while Med is a variety, Poc is not even a quasi-variety.

Lets revisit the median algebra $\mathcal{B} := (\Rightarrow, m^2 | \Rightarrow)$.

As a graph, we may draw $\Rightarrow$ as

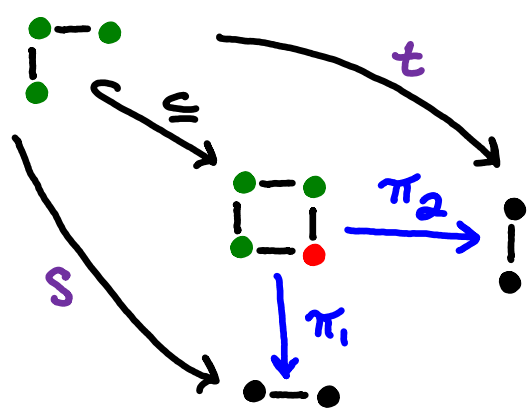


its source & target maps $\{l_{\perp}, e_{\text{Arr}}, l_T\} \xrightleftharpoons[s]{t} \mathcal{Q}$

are obtained by post-composing

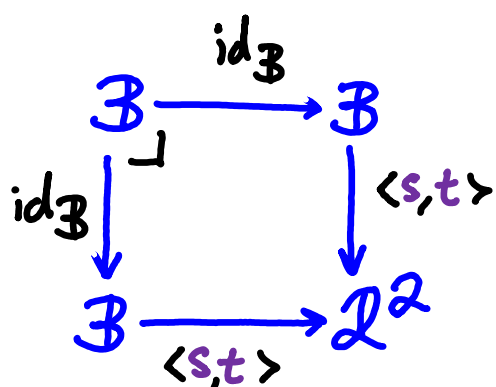
the projections $\pi_1, \pi_2: \mathcal{Q}^2 \rightarrow \mathcal{Q}$

with the inclusion $\Rightarrow \hookrightarrow \mathcal{Q}^2$.



Claim: $\Rightarrow$ is a partial order (there is a standard approach)

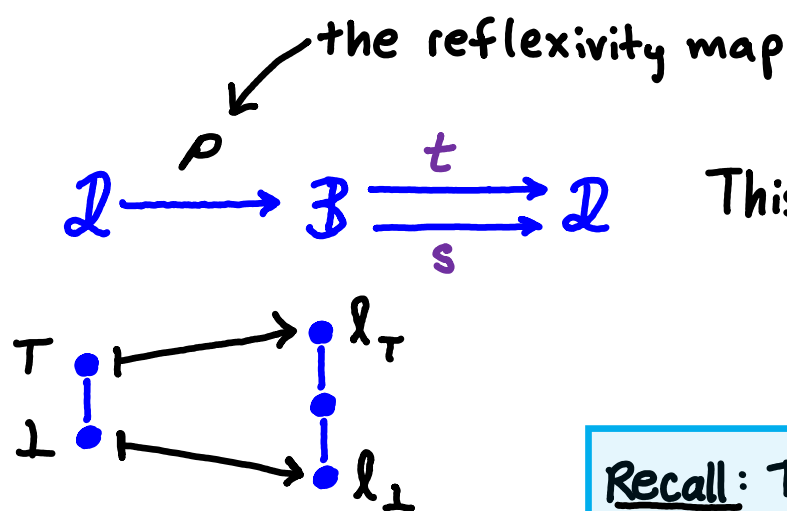
1) $\Rightarrow$ is a subobject of $\mathcal{Q}^2$



is a pullback in *Med*

because $\langle s, t \rangle$ is injective.

2) $(\mathcal{Q}, \Rightarrow)$ is reflexive

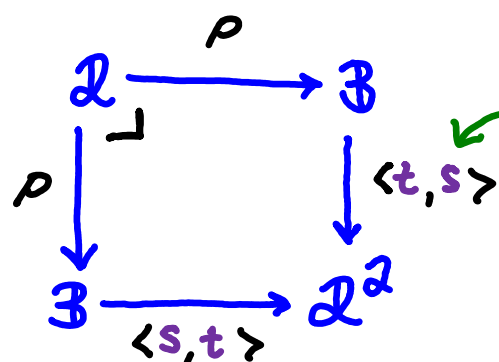


This is an equalizer in *Med*.

$$t \circ \rho = \text{id}_{\mathcal{Q}} = s \circ \rho$$

Recall: The equalizer of a graph is its set of vertex loops

3) $(\mathcal{Q}, \Rightarrow)$ is antisymmetric

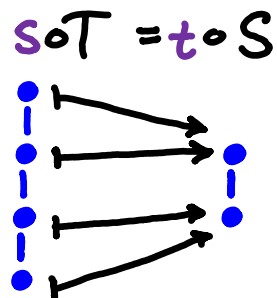
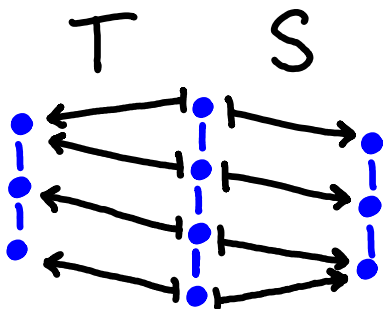
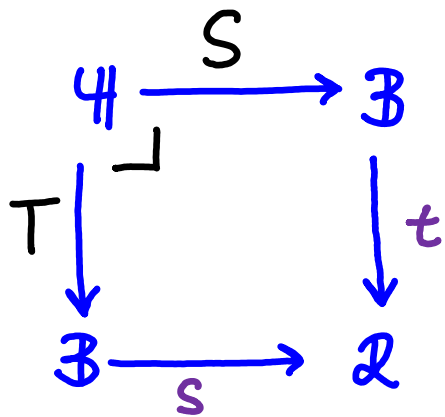


the inclusion of the converse $\Leftarrow$

$$\langle t, s \rangle \circ \rho = \Delta = \langle s, t \rangle \circ \rho$$

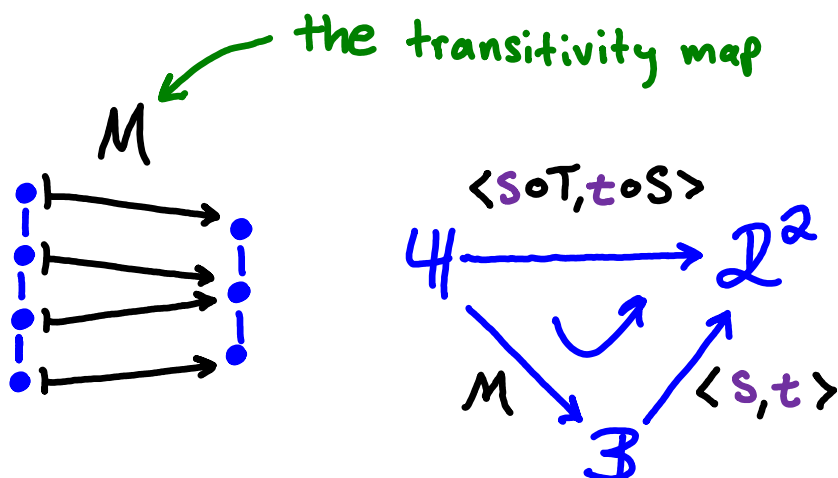
is a pullback square in *Med*.

4) $(\mathcal{Q}, \Rightarrow)$ is transitive

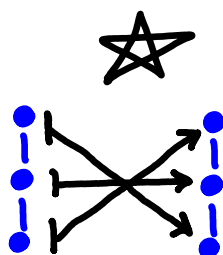
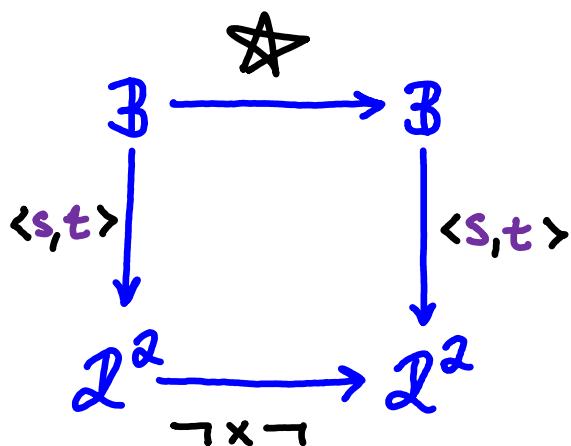


is a pullback square, &

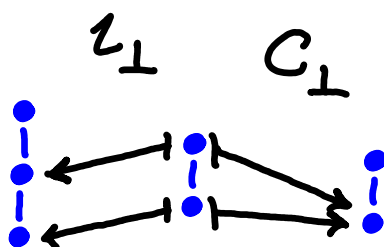
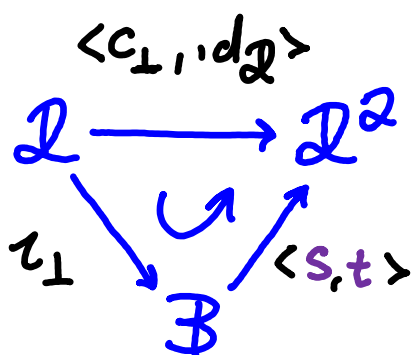
$$s \circ M = s \circ T, \quad t \circ M = t \circ S$$



5) $\neg \cdot (\mathcal{Q}, \Rightarrow) \rightarrow (\mathcal{Q}, \Rightarrow)$ is antitone



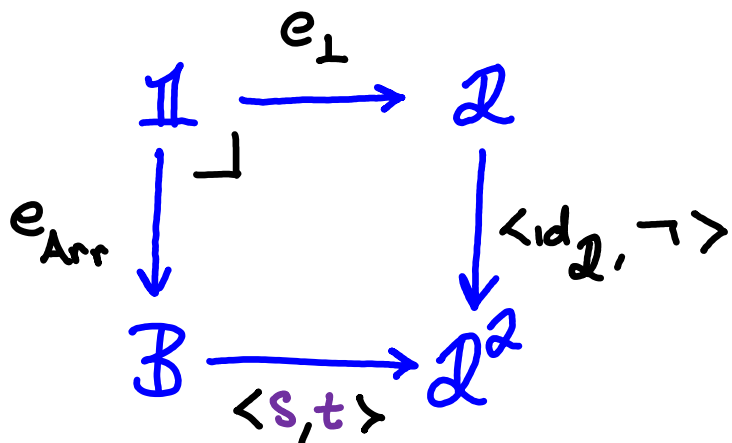
6) $\perp$ is the bottom element of $(\mathcal{Q}, \Rightarrow)$



7) $(\mathcal{Q}, e_1, \neg, \Rightarrow)$ satisfies non-contradiction

$\forall p \in \mathcal{Q}, p \Rightarrow \neg p$ implies $p = \perp$

Can be expressed as a pullback.

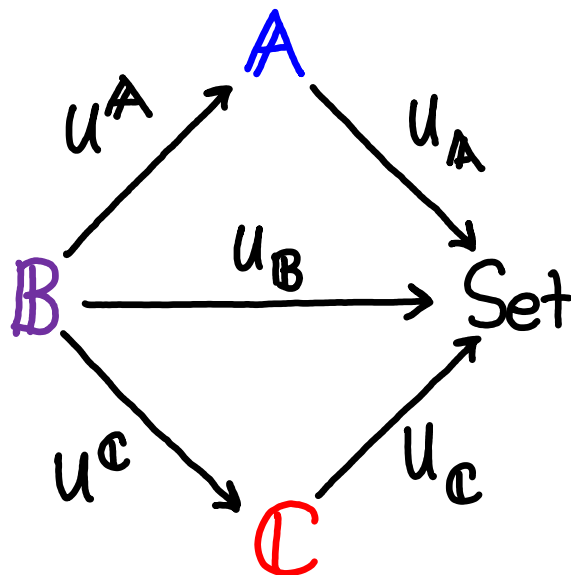


Commutativity: $1 \Rightarrow \neg 1$

Universality: 1 is the unique element implying its own negation

This suggests the definition of $\mathcal{P} = (\mathcal{M}, \llbracket _ \rrbracket_{\mathcal{M}}, \Sigma_{\mathcal{M}})$.

Concrete Bicompatibility



A commutative diagram of faithful functors as above is a nice general setting in which "Bic" makes sense.

Bicompatibility of poc-sets & median algebras

Definition: $\psi \in \text{Bic}(M \times P, B) \subseteq \text{Set}(U_{\text{Med}} M \times U_{\text{Poc}} P, U_{\text{Bool}} B)$
iff

- 1.) $\forall x \in U_{\text{Med}} M \quad \exists \psi_x: P \rightarrow U^{\text{Poc}} B \quad \text{s.t.} \quad U_{\text{Poc}} \psi_x = \psi(x, -)$
- 2.) $\forall p \in U_{\text{Poc}} P \quad \exists \psi_p: M \rightarrow U^{\text{Med}} B \quad \text{s.t.} \quad U_{\text{Med}} \psi_p = \psi(-, p)$

note: $\forall f: M' \rightarrow M, g: P' \rightarrow P, h: B \rightarrow B', \psi \in \text{Bic}(M \times P, B)$

$$\psi' := U_{\text{Bool}} h \circ \psi \circ (U_{\text{Med}} f \times U_{\text{Poc}} g) \in \text{Bic}(M' \times P', B')$$

$$\begin{array}{ccc} M \times P & \xrightarrow{\psi} & B \\ \uparrow f \times g & \searrow \quad \downarrow h & \\ M' \times P' & \xrightarrow{\psi'} & B' \end{array} \quad \begin{array}{l} \text{Therefore, } \exists \text{ Functors} \\ \text{Bic}(M \times P, -): \text{Bool} \rightarrow \text{Set} \\ \text{Bic}(- \times -, B): \text{Med}^{\text{op}} \times \text{Poc}^{\text{op}} \rightarrow \text{Set} \end{array}$$

$\forall M \in \text{Med}, P \in \text{Poc}, B \in \text{Bool} \quad \exists \text{ natural isomorphisms}$

$$(I) \text{Bic}(M \times P, -) \cong \text{Bool}(M \otimes P, -) \quad M \otimes P := F_{\text{Bool}}(M \times P) / \sim$$

$$(II) \text{Bic}(M \times -, B) \cong \text{Poc}(-, B^M) \quad B^M \cong \text{Med}(M, U^{\text{Med}} B)$$

$$(III) \text{Bic}(- \times P, B) \cong \text{Med}(-, B^P) \quad B^P \cong \text{Poc}(P, U^{\text{Poc}} B)$$

The data of (I) extends to a functor $\otimes: \text{Med} \times \text{Poc} \rightarrow \text{Bool}$

The universal property of (I) implies naturality of

$$\text{Bic}(M \times P, B) \cong \text{Bool}(M \otimes P, B) \quad \text{in all three variables.}$$

$\exists!$ extensions $(-)^M: \text{Bool} \rightarrow \text{Poc}$, $(-)^P: \text{Bool} \rightarrow \text{Med}$ s.t.

$$(M \otimes -) \dashv (-)^M \quad \text{and} \quad (- \otimes P) \dashv (-)^P,$$

by Yoneda's lemma. Therefore, by the parameter theorem,

$\exists!$ extensions $(-)^{(-)}: \text{Med}^{\text{op}} \times \text{Bool} \rightarrow \text{Poc}$, $(-)^{(-)}: \text{Poc}^{\text{op}} \times \text{Bool} \rightarrow \text{Med}$

s.t. there is a two-variable adjunction

$$\text{Med}(M, B^P) \cong \text{Bool}(M \otimes P, B) \cong \text{Poc}(P, B^M)$$

i.e., both isomorphisms are natural in all three variables.

In particular, $\forall B \in \text{Bool} \exists$ a contravariant (right) adjunction

$$\text{Med}^{\text{op}} \begin{array}{c} \xrightarrow{B^{(-)}} \\ \xleftarrow{B^{(-)}} \end{array} \text{Poc}$$

Recall, we claim

$$B^M \cong \text{Med}(M, u^{\text{Med}} B)$$

$$B^P \cong \text{Poc}(P, u^{\text{Poc}} B)$$

Thus, $\text{Med}(M, \text{Poc}(P, u^{\text{Poc}} B)) \cong \text{Poc}(P, \text{Med}(M, u^{\text{Med}} B))$.

Conversely, $\text{Bool} \cong$ The category of contravariant adjunctions

restricted Yoneda embedding

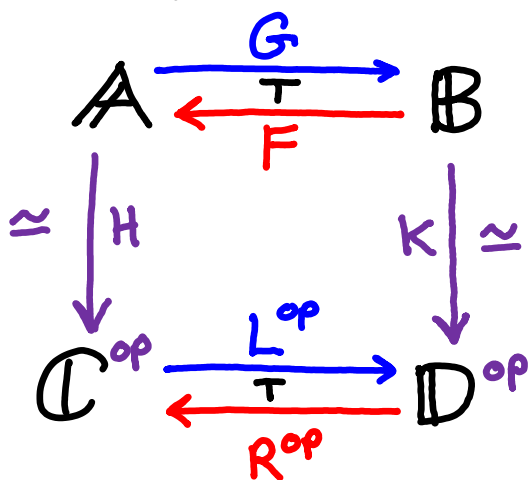
$$\text{Med}^{\text{op}} \begin{array}{c} \xrightarrow{\tau} \\ \xleftarrow{\tau} \end{array} \text{Poc}.$$

Idea: Consider Bool as $\text{Mod}_{\text{Med}}(\mathbb{Z})$ or $\text{Mod}_{\text{Poc}}(\mathcal{M})$.

(Both median algebra objects in Poc & poc-set objects in Med are Boolean algebras.)

Dual Adjunctions

We say $(F \dashv G, \eta, \varepsilon)$ is "dual" to $(L \dashv R, \eta', \varepsilon')$ if $\exists (H, K)$ s.t.



$$(i) \quad KG = L^{\text{op}} H$$

$$(ii) \quad HF = R^{\text{op}} K$$

$$(iii) \quad K\eta = \varepsilon' K$$

$$(iv) \quad H\varepsilon = \eta' H$$

both anti-equivalences

(H, K) is a morphism of adjunctions

(i) Says the square of right adjoints commutes.

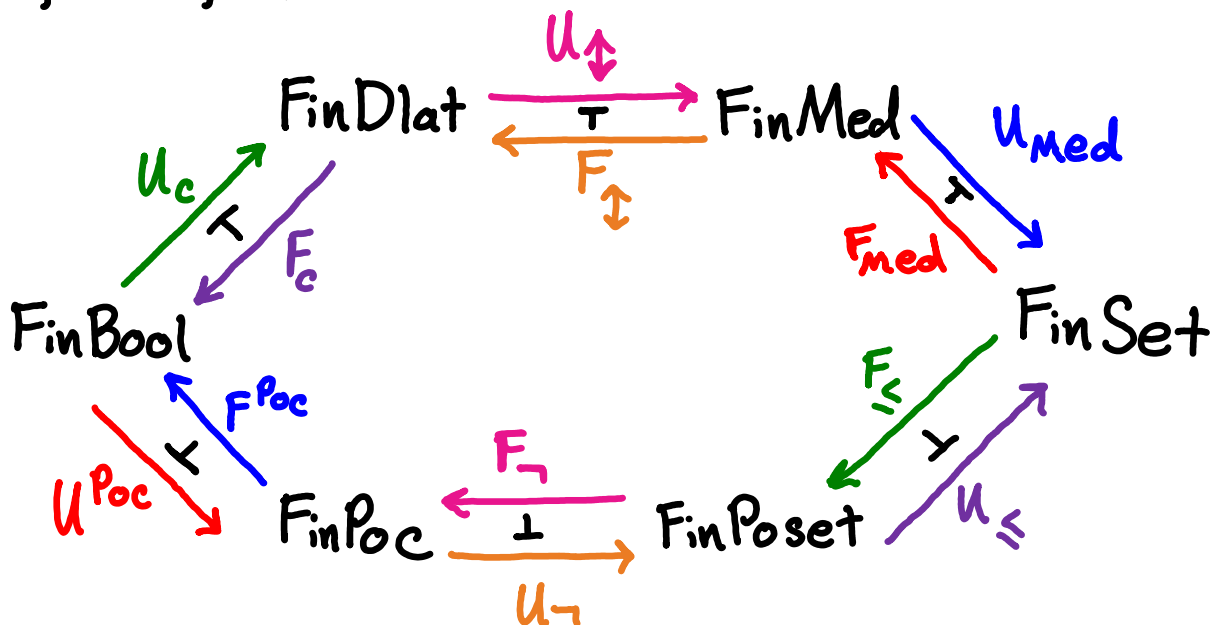
(ii) Says the square of left adjoints commutes.

(iii) the unit of $F \dashv G$ dualizes to the counit of $L \dashv R$ @ the dual

(iv) the counit of $F \dashv G$ dualizes to the counit of $L^{\text{op}} \dashv R^{\text{op}}$ @ the dual.

IV.7 Proposition 1 Mac Lane · (i) & (ii) imply (iii) iff (iv).

Examples: In our examples, the morphisms/dualities are given by $(\text{Hom}(-, \mathcal{Q}), \text{Hom}(-, \mathcal{Q}))$



$F_c + U_c$ is dual to $F_{\leq} + U_{\leq}$

Recall:

- Finite Boolean algebras are powers of $\mathcal{Q}$
- Ultrafilters in finite Boolean algebras are principal
- $\mathcal{Q}$ represents the contravariant ultrafilter functor $\text{Bool}^{\text{op}} \rightarrow \text{Set}$

& the representation sends principal ultrafilters to projections

Therefore $\text{Bool}(\mathcal{Q}^n, \mathcal{Q}) = \{\pi_1, \dots, \pi_n\}$.

Note: U_c is full, so $\text{Dlat}(U_c \mathcal{Q}^n, U_c \mathcal{Q}) = F_{\leq} \{U_c \pi_1, \dots, U_c \pi_n\}$.

Claim: The unit $\eta_D : D \rightarrow U_c F_c D$ is given by tupling all elements of $\text{Dlat}(D, U_c \mathcal{Q})$.

proof: By adjunction, pre-composition with η_D gives a bijection between a set of projections and $\text{Dlat}(D, U_c \mathcal{Q})$.

$$\begin{array}{ccc}
 \text{Bool}(F_c D, \mathcal{Q}) & \xrightarrow{\sim} & U_{\leq} \text{Dlat}(D, U_c \mathcal{Q}) \\
 \searrow U_c & & \nearrow (\eta_D)^* \\
 & U_{\leq} \text{Dlat}(U_c F_c D, U_c \mathcal{Q}) \\
 & \text{"} & \\
 & U_{\leq} \{U_c \pi_1, \dots, U_c \pi_n\}
 \end{array}$$

thus $(\eta_D)^*(\pi_i) = \phi_i$, or equivalently $\eta_D = \langle \phi_i \rangle_{1 \leq i \leq n}$. $\square$

This proves the Stone representation theorem for distributive lattices
 Since maps $D \rightarrow U_c \mathcal{Q}$ separate elements.

We identify $U_{\leq} \text{Dlat}(U_c F_c D, U_c \mathcal{Q})$ with $U_{\leq} \text{Dlat}(D, U_c \mathcal{Q})$ via $(\eta_D)^*$

Then by adjunction $(\eta_D)^* = \text{id}_{U_{\leq} \text{Dlat}(D, U_c \mathcal{Q})}$ induces

$$F_{\leq} U_{\leq} \text{Dlat}(D, U_c \mathcal{Q}) \longrightarrow \text{Dlat}(D, U_c \mathcal{Q})$$

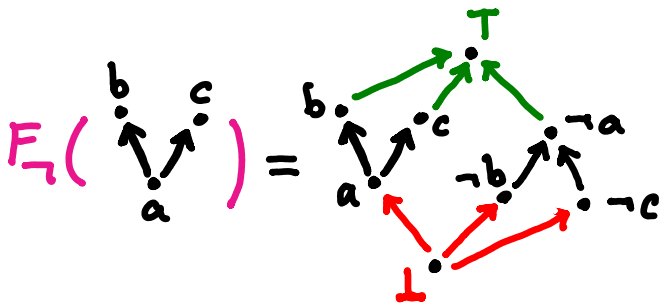
which is the counit of $F_{\leq} \dashv U_{\leq}$ @ the dual of D .

This proves (i), (ii), and (iii).

$$\underline{F_{\uparrow} \dashv U_{\uparrow}} \text{ is dual to } \underline{F_{\neg} \dashv U_{\neg}}$$

This duality formalizes the claim that $U_{\uparrow}$ "forgets only up-down."

Given $(P, \leq) \in \text{Poset}$, $F_{\neg}(P, \leq)$ is computed by taking the disjoint union of $(P, \leq)$ & $(P, \leq^{\text{op}})$ and freely adjoining a **top** & **bottom** element. Negation is $\text{id}_P: (P, \leq) \rightarrow (P, \leq^{\text{op}})$.



distributive lattices have an orientation, median algebras do not.

$$\text{Poc}(F_{\neg}(P, \leq), \mathcal{Q}) \stackrel{(i)}{=} U_{\uparrow} \text{Poset}(P, \leq, \mathcal{Q}) = U_{\uparrow} \left(\begin{array}{c} \{a, b, c\} \\ \vdots \\ \{b, c\} \\ \swarrow \quad \searrow \\ \{b\} \quad \{c\} \\ \vdots \\ \emptyset \end{array} \right)$$

In this example, (i) translates to

$$\text{Med}(U \downarrow D, \mathcal{Q}) = F_{\neg} \text{Dlat}(D, \mathcal{Q})$$

We prove this as follows. First, note $\text{Dlat}(\mathcal{Q}, \mathcal{Q}) = \{\text{id}_{\mathcal{Q}}\}$

Also observe $\text{Med}(\mathcal{Q}, \mathcal{Q}) = \{\perp, \text{id}_{\mathcal{Q}}, \neg, \top\} = \text{Set}(\mathcal{Q}, \mathcal{Q})$

Lastly, we prove that every morphism in the image of

$\text{Med}(U \downarrow D, U \downarrow \mathcal{Q})$ is obtained from one in the image of

$$U \downarrow \cdot \text{Dlat}(D, \mathcal{Q}) \longrightarrow \text{Med}(U \downarrow D, U \downarrow \mathcal{Q})$$

by post-composing with an element of $\text{Med}(\mathcal{Q}, \mathcal{Q})$.

(ii) corresponds to the bijection on Hom-sets associated with the adjunction $F_{\downarrow} \dashv U \downarrow$ (up to type-checking)

$$\text{Dlat}(F_{\downarrow} M, \mathcal{Q}) \cong U_{\neg} \text{Med}(M, U \downarrow \mathcal{Q}).$$

(iv) now follows formally. Consider the dual of the counit

$$\text{Dlat}(D, \mathcal{Q}) \xrightarrow{(\varepsilon_D)^*} \text{Dlat}(F_{\downarrow} U \downarrow D, \mathcal{Q})$$

we use the identification in (ii) to rewrite the codomain as in

$$\text{Dlat}(D, \mathcal{Q}) \longrightarrow U_{\neg} \text{Med}(U \downarrow D, U \downarrow \mathcal{Q})$$

Claim: Under this identification $(\varepsilon_D)^* = U_{\downarrow}$

proof: let $\phi \in \text{Dlat}(D, \mathcal{Q})$, then the overall composite is

$$\begin{array}{c}
 \text{Dlat}(D, \mathcal{Q}) \xrightarrow{(\varepsilon_D)^*} \text{Dlat}(\underset{\cap}{F_{\downarrow} U_{\downarrow} D}, \mathcal{Q}) \xrightarrow{U_{\downarrow}} \text{Med}(\underset{\cap}{U_{\downarrow} F_{\downarrow} U_{\downarrow} D}, U_{\downarrow} \mathcal{Q}) \\
 \searrow \sim \quad \quad \quad \downarrow (\eta_{U_{\downarrow} D})^* \\
 \text{Med}(U_{\downarrow} D, U_{\downarrow} \mathcal{Q}) \\
 \quad \quad \quad \uparrow \psi \\
 U_{\downarrow} \phi = U_{\downarrow} \phi \circ U_{\downarrow} \varepsilon_D \circ \eta_{U_{\downarrow} D} \\
 \quad \quad \quad \uparrow \\
 \text{triangle identity } \square
 \end{array}$$

Using (i) we can rewrite the codomain once more as

$$\text{Dlat}(D, \mathcal{Q}) \longrightarrow U_{\downarrow} F_{\downarrow} \text{Dlat}(D, \mathcal{Q}).$$

Since every element in the image is of the form $U_{\downarrow} \phi$ for some $\phi \in \text{Dlat}(D, \mathcal{Q})$ we conclude that this map the $\text{Dlat}(D, \mathcal{Q})$ -Component of the unit η' .

of course, this proof of (iv) depends only on the nature of the identifications in (i) & (ii) as in this example.

$F^{\text{Poc}} + U^{\text{Poc}}$ is dual to $F^{\text{Med}} + U^{\text{Med}}$

$$(i) \quad \text{Poc}(U^{\text{Poc}} \mathbb{Z}^n, \mathbb{Z}) \cong F^{\text{Med}} \text{Bool}(\mathbb{Z}^n, \mathbb{Z})$$

This is true but not easy to prove. The idea is to show that every morphism in $\text{Poc}(U^{\text{Poc}} \mathbb{Z}^n, \mathbb{Z})$ can be obtained by the following iterative process: beginning with the

set of coordinate projections, take all possible triples

(f, g, h) and form the composite $m \circ \langle f, g, h \rangle$

$\uparrow$
the median operation on $\mathbb{Z}$

then repeat with this new set of maps until new maps are no longer produced. (This is the hard part)

Then identify the free median algebra on $\text{Bool}(\mathbb{Z}^n, \mathbb{Z})$

with the smallest median subalgebra of the free Boolean algebra on n -elements containing the generators (We do this using the Stone representation of median algebras, i.e., the fact that the unit of $F^{\text{Med}} + U^{\text{Med}}$, which tuples all maps in $\text{Med}(M, \mathbb{Z})$, is monic). Then since the generating set in the free Boolean algebra can be

identified with the coordinate projections in

$$\text{Set}(\mathcal{Q}^n, \mathcal{Q}),$$

We have a commutative diagram

$$\begin{array}{ccccc} \text{Bool}(\mathcal{Q}^n, \mathcal{Q}) & \xrightarrow{U^{\text{Poc}}} & \text{Poc}(U^{\text{Poc}} \mathcal{Q}^n, U^{\text{Poc}} \mathcal{Q}) & \xrightarrow{U^{\text{Poc}}} & \text{Set}(\mathcal{Q}^n, \mathcal{Q}) \\ \parallel & & \uparrow & & \parallel \\ \{\pi_1, \dots, \pi_n\} & \longleftrightarrow & F^{\text{Med}} \{\pi_1, \dots, \pi_n\} & \longleftrightarrow & F_{\text{Bool}} \{\pi_1, \dots, \pi_n\} \end{array}$$

& since all functions in the outer rectangle are injective, so is the one in the middle.

(ii) is true by adjunction

$$\text{Bool}(F^{\text{Poc}} P, \mathcal{Q}) \cong U^{\text{Med}} \text{Poc}(P, U^{\text{Poc}} \mathcal{Q})$$

Claim: Dual adjunctions compose (this is formal)

Corollary $F_{\text{Bool}} \dashv U_{\text{Bool}}$ is self-dual.