What's wrong with higher-order derivatives?

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Introduction

We will discuss two questions related to higher-order derivatives:

(1) How to deal with the derivative of polynomials in positive characteristic? (and Hasse-Schmidt differential categories)

(2) Can we define an extension of tangent categories with functors T_n which differentiate n times?

We'll see four clues towards the existence of a notion of higher-order tangent category.

1. Derivatives of polynomials in positive characteristic

Differentiating polynomials

Let R be a commutative ring. We can define the derivative of a polynomial $f \in R[x]$. We start with:

$$1' := 0$$

$$(x^n)' := x^{n-1} \quad (n \geq 1)$$

Then, we extend to an R-linear map:

$$\begin{aligned} R[x] &\longrightarrow R[x] \\ f &\longmapsto f' \end{aligned}$$

We get the usual properties ($r \in R$):

- $(r)' = 0$,
- $(rx)' = r$,
- $(fg)' = f'g + fg'$,
- $(f \circ g)' = (f' \circ g)g'$.

Derivative in positive characteristic. What's wrong?

Let R be a commutative ring of positive characteristic (i.e. there exists $n \in \mathbb{N} \setminus \{0\}$ such that $n.1_R = 0$).

It is false that $f \in R[x]$ is a constant $\Leftrightarrow f' = 0$! (consider $f(x) = x^n$ with $n = \mathrm{char} R$).

We will not solve this issue, but we can still improve a bit the derivative in this context.

Theorem (Taylor expansion for polynomials over a $\mathbb{Q}$-algebra)

Let R be a commutative ring with a ring homomorphism $\mathbb{Q} \to R$, let $f \in R[x]$ and let $a \in R$. Then:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}}{n!}(a)(x-a)^n$$

A nonzero ring of positive characteristic certainly does not have a ring homomorphism $\mathbb{Q} \to R$, so we can't make sense of this theorem in positive characteristic.

This theorem seems to tell us that $\frac{f^{(n)}}{n!}$ is more important than $f^{(n)}$ alone… How can we write this in positive characteristic?

We can't divide by $n!$ in positive characteristic. But we can still **not multiply** by $n!$.

We have $(x^{n+p})^{(n)} = (p+n)\dots(p+1)x^{(p)} = n!\binom{n+p}{n}x^{(p)}$.

If we have a ring homomorphism $\mathbb{Q} \to R$, we can write:

$$\frac{(x^{n+p})^{(n)}}{n!} = \frac{(p+n)\dots(p+1)}{n!}x^{(p)} = \binom{n+p}{n}x^{(p)}$$

If we don't have a ring homomorphism $\mathbb{Q} \to R$, we can still define:

$$D^n(x^{n+p}) = \binom{n+p}{n}x^p \qquad D^n(x^k) = 0 \quad (k < n)$$

In some cases, it will resurrect some higher-derivatives which were equal to 0!

For instance if $\mathrm{char}(R) = 2$, then $(x^3)^{(2)} = 3.2.x = 0$ but $D^2(x^3) = \binom{3}{2}x = 3x$.

We extend D^n to an R-linear map $D^{(n)} : R[x] \to R[x]$.

$D^n(f)$ is called the Hasse-Schmidt derivative of order n of f.

Theorem (Taylor expansion for polynomials over a com. ring)

Let R be a commutative ring, let $f \in R[x]$ and let $a \in R$. Then:

$$f(x) = \sum_{n=0}^{\infty} D^n(f)(a)(x-a)^n$$

Corollary

Let $f \in R[x]$ and let $a \in R$. Then:

$$f \text{ is a constant} \Leftrightarrow D^n(f)(a) = 0 \text{ for every } n \geq 1$$

Multivariate Hasse-Schmidt derivative

Let R be a commutative ring. We define

$$D^{(r_1,\ldots,r_p)} : R[x_1,\ldots,x_p] \to R[x_1,\ldots,x_p]$$

as the unique R-linear map such that

$$D^{(n_1,\ldots,n_p)}(x_1^{n_1+r_1}\ldots x_p^{n_p+r_p}) = \binom{n_1+r_1}{n_1}\ldots\binom{n_p+r_p}{n_p}x_1^{n_1}\ldots x_p^{r_p}$$

$$D^{(n_1,\ldots,n_p)}(x_1^{k_1}\ldots x_p^{k_p}) = 0 \quad (k_1 < n_1 \text{ or } \ldots \text{ or } k_p < n_p)$$

Differential categories

An algebraic differential category is an additive (i.e. enriched over commutative monoids) symmetric monoidal category $(\mathcal{C}, \otimes, I)$ together (in particular) with a monad $S : \mathcal{C} \to \mathcal{C}$ and a natural transformation

$$d_A : SA \to SA \otimes A$$

which has properties similar to the R-linear map given by:

$$x_1 \otimes_s \cdots \otimes_s x_{p+1} \mapsto \sum_{1 \le i \le p} (x_1 \otimes_s \cdots \otimes_s \hat{x}_i \otimes_s \cdots \otimes_s x_p) \otimes x_i$$

when A is an R-module and SA is the symmetric algebra over A.

If $A \overset{\psi}{\simeq} R^p$ as an R-module, then we have $SA \overset{\phi}{\simeq} R[x_1, \dots, x_p]$ and $\phi^{-1}; d_A; (\phi \otimes \psi)$ is given by

$$R[x_1, \dots, x_p] \to R[x_1, \dots, x_p] \otimes R^p$$

$$f \mapsto \sum_{1 \le i \le p} \frac{\partial f}{\partial x_i} \otimes e_i$$

Hasse-Schmidt differential categories

An (algebraic) Hasse-Schmidt differential category is an additive symmetric monoidal category together (in particular) with a monad $S : \mathcal{C} \to \mathcal{C}$, symmetric powers[1] and a family of natural transformations

$$D^n_A : SA \to SA \otimes S^n A$$

which behave like

$$R[x_1, \dots, x_p] \to R[x_1, \dots, x_p] \otimes R_n[x_1, \dots, x_p]$$
$$f \mapsto \sum_{I \in \mathbb{N}^n} D^I f \otimes x_I$$

where $R_n[x_1, \dots, x_p]$ is the set of homogeneous polynomials of degree n $\subseteq R[x_1, \dots, x_p]$.

Theorem

Every Hasse-Schmidt differential category is a differential category.

Theorem

Every $\mathbb{Q}_{\geq 0}$-linear [2] differential category is a Hasse-Schmidt differential category.

[1] we define $S^n A$ as the coequalizer of all the permutations $\sigma : A^{\otimes n} \to A^{\otimes n}$

[2] i.e. enriched over $\mathbb{Q}_{\geq 0}$-modules

2. Higher-order tangent categories?

Tangent categories

A tangent category is a category $\mathcal{C}$ with a functor $T : \mathcal{C} \to \mathcal{C}$ which behaves like the tangent functor on smooth manifolds. More precisely, we ask for:

- a natural transformation $p_A : TA \to A$,
- for each $A \in \mathcal{C}$ and $n \geq 0$, the pullback $T_n A$ of n copies of p_A exists and is preserved by every T^m, $m \geq 0$,
- natural transformations $m_A : T_2 A \to TA$, $e_A : A \to TA$, $l_A : TA \to T^2 A$, $c_A : T^2 A \to T^2 A$ which satisfy a bunch of identities. In particular every (TA, m_A, e_A) must be a commutative monoid, $c_A^2 = \mathrm{id}_{T^2 A}$ and $l_A ; c_A = l_A$.

The category of smooth manifolds is an example.

Clue 1 (ft. commutative rings)

A simpler example is given by the category of commutative rings. If R is a commutative ring, we define $TR = R[x]/(x^2) \simeq R[\epsilon]$ with $\epsilon^2 = 0$.

The idea is that an element $a + b\epsilon \in TR$ is an infinitesimal vector starting from a and pointing in the direction b. We have $T_n R = R[x_1, \dots, x_n]/(x_i x_j) \simeq R[\epsilon_1, \dots, \epsilon_n]$ with $\epsilon_i \epsilon_j = 0$ and $T^2 R \simeq R[x, y]/(x^2, y^2) \simeq R[\epsilon, \eta]$ with $\epsilon^2 = \eta^2 = 0$.

We have $p_R(a + b\epsilon) = a$, $m_R(a + b\epsilon_1 + c\epsilon_2) = a + b\epsilon_1$, $e_R(r) = r$, $l_R(a + b\epsilon) = a + b\epsilon\eta$, $c_R(a + b\epsilon + (c + d\epsilon)\eta) = a + c\epsilon + (b + d\epsilon)\eta$.

We would like to define ${}_nTR = R[x]/(x^n) \simeq R[\epsilon]$ with $\epsilon^n = 0$. We have projections ${}_np_R : {}_nTR \to R$. Of course we have the pullbacks of ${}_{n_1}p_R, \dots, {}_{n_r}p_R$. Do we have canonical flips $({}_nT)^2 R \to ({}_nT)^2 R$? etc...

The idea would be that an element $a_0 + a_1\epsilon + \cdots + a_n\epsilon^{n-1}$ is an infinitesimal curve starting from a_0 and going in the direction given by $x \mapsto a_0 + a_1 x + \cdots + a_n x^{n-1}$.

Clue 2 (ft. coEilenberg-Moore categories)

Theorem

The coEilenberg-Moore category of a coalgebraic differential category is a tangent category.

We would like to write:

The coEilenberg-Moore category of a coalgebraic Hasse-Schmidt differential category[3] is a . . . ?. . .

[3] i.e. the opposite of an algebraic Hasse-Schmidt differential category

Clue 3 (ft. higher-order tangent bundles)

We have functors ${}_nT$: **SMan** $\to$ **SMan** such that ${}_nTM$ gives the approximations of M by polynomials curves of degree n around every point $a \in M$, and ${}_nTf$, the n^{th} derivative of f sends an approximation of order n around a to a polynomial curve of order n around $f(a)$, in the best possible way.

SMan together with the functors ${}_nT$ should give a higher-order tangent category!

Clue 4 (ft. Poon Leung)

From "Classifying tangent structures using Weil algebras":

Theorem

If $\mathcal{C}$ is a category, then a tangent structure on $\mathcal{C}$ is equivalent (up to isomorphism) to a strong monoidal functor $F : \mathbb{N} - \mathbf{Weil}_1 \to \mathrm{End}(\mathcal{C})$ satisfying two additional conditions.

We would like to have a definition of higher-order tangent category of order n such that we have a similar theorem where we replace $\mathbb{N} - \mathbf{Weil}_1$ by some category $\mathbb{N} - \mathbf{Weil}_n$.