Life in Johnstone's Topological Topos

Chris Grossack
(they/them)

CT Octoberfest 2024

- A Topos is an "alternative universe of sets" in which we can do math.

- A Topos is an “alternative universe of sets” in which we can do math.
- Informally, we know how to “compile” mathematical statements down to statements about pure sets

- A Topos is an “alternative universe of sets” in which we can do math.
- Informally, we know how to “compile” mathematical statements down to statements about pure sets
- A topos is a different “implementation” of the same “interface”

- A Topos is an "alternative universe of sets" in which we can do math.
- Informally, we know how to "compile" mathematical statements down to statements about pure sets
- A topos is a different "implementation" of the same "interface"
- Except for choice and excluded middle

- A Topos is an “alternative universe of sets” in which we can do math.
- Informally, we know how to “compile” mathematical statements down to statements about pure sets
- A topos is a different “implementation” of the same “interface”
- Except for choice and excluded middle
- It often comes with a mechanical procedure for translating statements interpreted in the topos into statements about “the real world”

- A Topos is an "alternative universe of sets" in which we can do math.
- Informally, we know how to "compile" mathematical statements down to statements about pure sets
- A topos is a different "implementation" of the same "interface"
- Except for choice and excluded middle
- It often comes with a mechanical procedure for translating statements interpreted in the topos into statements about "the real world"
- Why care about this. . . ?

Punchline: (One of Many)

We can interpet *Old Theorems* in these *New Worlds* in order to prove new theorems without much work!

Punchline: (One of Many)

We can interpet *Old Theorems* in these *New Worlds* in order to prove new theorems without much work!

Punchline: (Slightly Harder)

Given a problem, we can also try to cook up a topos in which that problem is particularly easy to study! Then we work Synthetically until the very last step

Punchline: (One of Many)

We can interpet *Old Theorems* in these *New Worlds* in order to prove new theorems without much work!

Punchline: (Slightly Harder)

Given a problem, we can also try to cook up a topos in which that problem is particularly easy to study! Then we work Synthetically until the very last step

eg.

- Synthetic Differential Geometry (Kock-Lawvere)

Punchline: (One of Many)

We can interpet *Old Theorems* in these *New Worlds* in order to prove new theorems without much work!

Punchline: (Slightly Harder)

Given a problem, we can also try to cook up a topos in which that problem is particularly easy to study! Then we work Synthetically until the very last step

eg.

- Synthetic Differential Geometry (Kock-Lawvere)
- Synthetic Probability Theory (Simpson)

Punchline: (One of Many)

We can interpet *Old Theorems* in these *New Worlds* in order to prove new theorems without much work!

Punchline: (Slightly Harder)

Given a problem, we can also try to cook up a topos in which that problem is particularly easy to study! Then we work Synthetically until the very last step

eg.

- Synthetic Differential Geometry (Kock-Lawvere)
- Synthetic Probability Theory (Simpson)
- Synthetic Homotopy Theory* (HoTT)

Punchline: (One of Many)

We can interpet *Old Theorems* in these *New Worlds* in order to prove new theorems without much work!

Punchline: (Slightly Harder)

Given a problem, we can also try to cook up a topos in which that problem is particularly easy to study! Then we work Synthetically until the very last step

eg.

- Synthetic Differential Geometry (Kock-Lawvere)
- Synthetic Probability Theory (Simpson)
- Synthetic Homotopy Theory* (HoTT)
- Many others

- Today we'll be interested in Johnstone's Topological Topos $\mathcal{T}$

- Today we'll be interested in Johnstone's Topological Topos $\mathcal{T}$
- A world for Synthetic (Sequential) Topology

- Today we'll be interested in Johnstone's Topological Topos $\mathcal{T}$
- A world for Synthetic (Sequential) Topology
- (Compare with the Condensed Topos, a setting for synthetic *compact hausdorff* topology)

- Today we'll be interested in Johnstone's Topological Topos $\mathcal{T}$
- A world for Synthetic (Sequential) Topology
- (Compare with the Condensed Topos, a setting for synthetic *compact hausdorff* topology)
- The topological topos satisfies the bonus axiom that every function $\mathbb{R} \to \mathbb{R}$ is continuous.

- Today we'll be interested in Johnstone's Topological Topos $\mathcal{T}$
- A world for Synthetic (Sequential) Topology
- (Compare with the Condensed Topos, a setting for synthetic *compact hausdorff* topology)
- The topological topos satisfies the bonus axiom that every function $\mathbb{R} \to \mathbb{R}$ is continuous.
- Sometimes called "Brouwer's Axiom" in the literature.

The Plan™

The Plan™

- First, we'll say how $\mathcal{T}$ is defined

The Plan™

- First, we'll say how $\mathcal{T}$ is defined
- Then we'll say how objects of $\mathcal{T}$ relate to topological spaces "in the real world"

The Plan™

- First, we'll say how $\mathcal{T}$ is defined
- Then we'll say how objects of $\mathcal{T}$ relate to topological spaces "in the real world"
- We'll talk about *truth* in $\mathcal{T}$, and various subobject classifiers

The Plan™

- First, we'll say how $\mathcal{T}$ is defined
- Then we'll say how objects of $\mathcal{T}$ relate to topological spaces "in the real world"
- We'll talk about *truth* in $\mathcal{T}$, and various subobject classifiers
- We'll give some quality-of-life axioms that $\mathcal{T}$ satisfies, and explain why certain other quality-of-life axioms are false

The Plan™

- First, we'll say how $\mathcal{T}$ is defined
- Then we'll say how objects of $\mathcal{T}$ relate to topological spaces "in the real world"
- We'll talk about *truth* in $\mathcal{T}$, and various subobject classifiers
- We'll give some quality-of-life axioms that $\mathcal{T}$ satisfies, and explain why certain other quality-of-life axioms are false
- We'll give two proofs that Brouwer's Axiom holds in $\mathcal{T}$.

Let's get to it!

Defn

Write $\{1, \mathbb{N}_\infty\}$ for the full subcategory of Top consisting of the objects 1 and $\mathbb{N}_\infty = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots, 0\}$ (the one point compactification of $\mathbb{N}$)

Defn

Write $\{1, \mathbb{N}_\infty\}$ for the full subcategory of Top consisting of the objects 1 and $\mathbb{N}_\infty = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots, 0\}$ (the one point compactification of $\mathbb{N}$)

The Topological Topos $\mathcal{T}$ is presented as sheaves on $\{1, \mathbb{N}_\infty\}$ with the *canonical (grothendieck) topology*

Defn

The Topological Topos $\mathcal{T}$ is presented as sheaves on $\{1, \mathbb{N}_\infty\}$ with the *canonical (grothendieck) topology*

Defn

The Topological Topos $\mathcal{T}$ is presented as sheaves on $\{1, \mathbb{N}_\infty\}$ with the *canonical (grothendieck) topology*

If $F \in \mathcal{T}$ is a sheaf, it's in particular a presheaf, so we get sets $F(1)$ and $F(\mathbb{N}_\infty)$

Defn

The Topological Topos $\mathcal{T}$ is presented as sheaves on $\{1, \mathbb{N}_\infty\}$ with the *canonical (grothendieck) topology*

If $F \in \mathcal{T}$ is a sheaf, it's in particular a presheaf, so we get sets $F(1)$ and $F(\mathbb{N}_\infty)$

- $F(1)$ should be thought of as the set of *points* of F

Defn

The Topological Topos $\mathcal{T}$ is presented as sheaves on $\{1, \mathbb{N}_\infty\}$ with the *canonical (grothendieck) topology*

If $F \in \mathcal{T}$ is a sheaf, it's in particular a presheaf, so we get sets $F(1)$ and $F(\mathbb{N}_\infty)$

- $F(1)$ should be thought of as the set of *points* of F
- $F(\mathbb{N}_\infty)$ should be thought of as the set of *convergent sequences* in $F\ldots$

Defn

The Topological Topos $\mathcal{T}$ is presented as sheaves on $\{1, \mathbb{N}_\infty\}$ with the *canonical (grothendieck) topology*

If $F \in \mathcal{T}$ is a sheaf, it's in particular a presheaf, so we get sets $F(1)$ and $F(\mathbb{N}_\infty)$

- $F(1)$ should be thought of as the set of *points* of F
- $F(\mathbb{N}_\infty)$ should be thought of as the set of *convergent sequences* in $F\ldots$
- Better to say *witnesses to convergence*. In particular, there might be multiple elements $p, q \in F(\mathbb{N}_\infty)$ showing that the same sequence converges!

Defn

The Topological Topos $\mathcal{T}$ is presented as sheaves on $\{1, \mathbb{N}_\infty\}$ with the *canonical (grothendieck) topology*

If $F \in \mathcal{T}$ is a sheaf, it's in particular a presheaf, so we get sets $F(1)$ and $F(\mathbb{N}_\infty)$

- $F(1)$ should be thought of as the set of *points* of F
- $F(\mathbb{N}_\infty)$ should be thought of as the set of *convergent sequences* in $F \ldots$
- Better to say *witnesses to convergence*. In particular, there might be multiple elements $p, q \in F(\mathbb{N}_\infty)$ showing that the same sequence converges!
- The canonical topology says, roughly, that if every subsequence has a further subsequence converging to a limit, then the whole sequence converges to the same limit.

Defn

The Topological Topos $\mathcal{T}$ is presented as sheaves on $\{1, \mathbb{N}_\infty\}$ with the *canonical (grothendieck) topology*

If $F \in \mathcal{T}$ is a sheaf, it's in particular a presheaf, so we get sets $F(1)$ and $F(\mathbb{N}_\infty)$

- $F(1)$ should be thought of as the set of *points* of F
- $F(\mathbb{N}_\infty)$ should be thought of as the set of *convergent sequences* in F...
- Better to say *witnesses to convergence*. In particular, there might be multiple elements $p, q \in F(\mathbb{N}_\infty)$ showing that the same sequence converges!
- The canonical topology says, roughly, that if every subsequence has a further subsequence converging to a limit, then the whole sequence converges to the same limit.
- So F is a "generalized sequential space"!

- Note that in Set a topological space is a set *equipped with* the bonus data of a topology.

- Note that in Set a topological space is a set *equipped with* the bonus data of a topology.
- When doing constructions we have to come up with the “right topology” to put on the new gadget

- Note that in Set a topological space is a set *equipped with* the bonus data of a topology.
- When doing constructions we have to come up with the "right topology" to put on the new gadget
- Moreover, when building maps we have to check *by hand* that they respect the topology (are continuous).

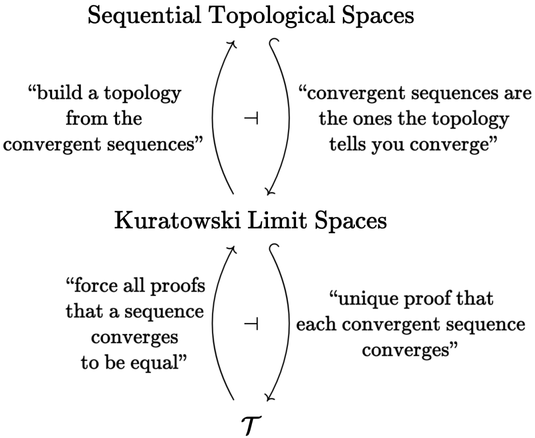
- Note that in Set a topological space is a set *equipped with* the bonus data of a topology.
- When doing constructions we have to come up with the "right topology" to put on the new gadget
- Moreover, when building maps we have to check *by hand* that they respect the topology (are continuous).
- In $\mathcal{T}$, though, every set *is* a space! *Intrinsically*!

- Note that in Set a topological space is a set *equipped with* the bonus data of a topology.
- When doing constructions we have to come up with the "right topology" to put on the new gadget
- Moreover, when building maps we have to check *by hand* that they respect the topology (are continuous).
- In $\mathcal{T}$, though, every set *is* a space! *Intrinsically*!
- No need to check continuity or to build new topologies from old. The topological structure is *automatic* and comes along for the ride! Our sets *themselves* are (possibly generalized) spaces!

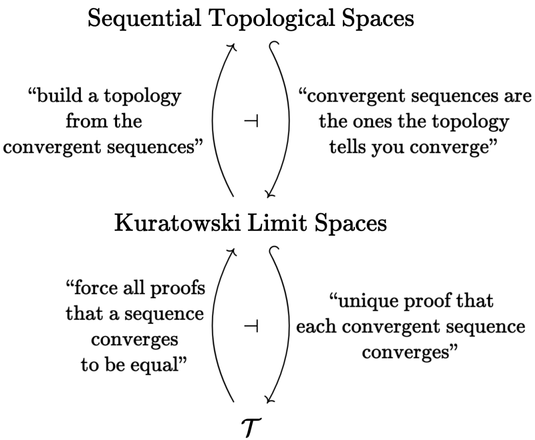
- With this context, you might ask how objects of $\mathcal{T}$ relate to "honest" sequential spaces.

- With this context, you might ask how objects of $\mathcal{T}$ relate to "honest" sequential spaces.
- They're *extremely* closely related!

- With this context, you might ask how objects of $\mathcal{T}$ relate to "honest" sequential spaces.
- They're *extremely* closely related!

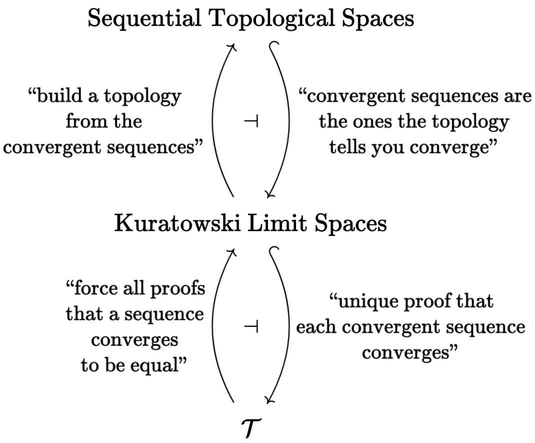


- With this context, you might ask how objects of $\mathcal{T}$ relate to "honest" sequential spaces.
- They're *extremely* closely related!



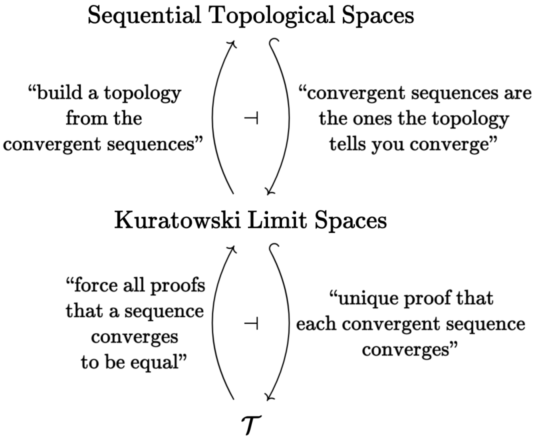
- The downwards arrows are embeddings

- With this context, you might ask how objects of $\mathcal{T}$ relate to "honest" sequential spaces.
- They're *extremely* closely related!



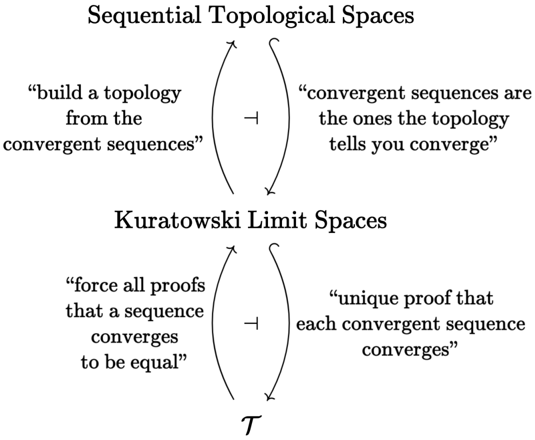
- The downwards arrows are embeddings
- They preserve "nice" colimits

- With this context, you might ask how objects of $\mathcal{T}$ relate to "honest" sequential spaces.
- They're *extremely* closely related!

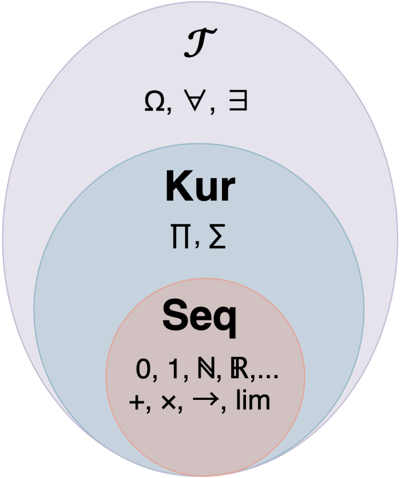


- The downwards arrows are embeddings
- They preserve "nice" colimits
- The upwards arrows preserve finite products

- With this context, you might ask how objects of $\mathcal{T}$ relate to "honest" sequential spaces.
- They're *extremely* closely related!



- The downwards arrows are embeddings
- They preserve "nice" colimits
- The upwards arrows preserve finite products
- More structure as we move down


$\mathcal{T}$
Ω, ∀, ∃
Kur
Π, Σ
Seq
0, 1, ℕ, ℝ,...
+, ×, →, lim


I'VE GOT A PROPOSITION
FOR YOU.

Clopen Subobjects = Decidable Propositions

The object $2 = \{\top, \bot\}$ with the discrete topology is the Clopen Subobject Classifier.

Clopen Subobjects = Decidable Propositions

The object $2 = \{\top, \bot\}$ with the discrete topology is the Clopen Subobject Classifier.

Sometimes written $\Delta 2$ and called the *Complemented Subobject Classifier* or *Decidable Subobject Classifier*

Clopen Subobjects = Decidable Propositions

The object $2 = \{\top, \bot\}$ with the discrete topology is the Clopen Subobject Classifier.

Sometimes written $\Delta 2$ and called the *Complemented Subobject Classifier* or *Decidable Subobject Classifier*

- Maps $f : X \to 2$ are in natural bijection with clopen subsets of X.

Clopen Subobjects = Decidable Propositions

The object $2 = \{\top, \bot\}$ with the discrete topology is the Clopen Subobject Classifier.

Sometimes written $\Delta 2$ and called the *Complemented Subobject Classifier* or *Decidable Subobject Classifier*

- Maps $f : X \to 2$ are in natural bijection with clopen subsets of X.
- Look at $f^{-1}(\top)$, which is clopen since $\{\top\}$ is

Clopen Subobjects = Decidable Propositions

The object $2 = \{\top, \bot\}$ with the discrete topology is the Clopen Subobject Classifier.

Sometimes written $\Delta 2$ and called the *Complemented Subobject Classifier* or *Decidable Subobject Classifier*

- Maps $f : X \to 2$ are in natural bijection with clopen subsets of X.
- Look at $f^{-1}(\top)$, which is clopen since $\{\top\}$ is
- Note 2 is *not* a complete lattice! We have finite $\wedge$ and $\vee$ but not even countable $\bigwedge$ and $\bigvee$!

Clopen Subobjects = Decidable Propositions

The object $2 = \{\top, \bot\}$ with the discrete topology is the Clopen Subobject Classifier.

Sometimes written $\Delta 2$ and called the *Complemented Subobject Classifier* or *Decidable Subobject Classifier*

- Maps $f : X \to 2$ are in natural bijection with clopen subsets of X.
- Look at $f^{-1}(\top)$, which is clopen since $\{\top\}$ is
- Note 2 is *not* a complete lattice! We have finite $\wedge$ and $\vee$ but not even countable $\bigwedge$ and $\bigvee$!
- After all, $\bigvee : 2^{\mathbb{N}} \to 2$ is not continuous!

Clopen Subobjects = Decidable Propositions

The object $2 = \{\top, \bot\}$ with the discrete topology is the Clopen Subobject Classifier.

Sometimes written $\Delta 2$ and called the *Complemented Subobject Classifier* or *Decidable Subobject Classifier*

- Maps $f : X \to 2$ are in natural bijection with clopen subsets of X.
- Look at $f^{-1}(\top)$, which is clopen since $\{\top\}$ is
- Note 2 is *not* a complete lattice! We have finite $\wedge$ and $\vee$ but not even countable $\bigwedge$ and $\bigvee$!
- After all, $\bigvee : 2^{\mathbb{N}} \to 2$ is not continuous!
- $1^n 0^\omega \to 1^\omega$ but $\bigwedge 1^n 0^\omega \not\to \bigwedge 1^\omega$

Open Subobjects = Semidecidable Propositions

The object $\Sigma = \{\top, \bot\}$ with the topology where $\{\top\}$ is open and $\{\bot\}$ isn't is called Sierpinski Space.

Open Subobjects = Semidecidable Propositions

The object $\Sigma = \{\top, \bot\}$ with the topology where $\{\top\}$ is open and $\{\bot\}$ isn't is called Sierpinski Space.

- Maps $f : X \to \Sigma$ are in natural bijection with open subsets of X

Open Subobjects = Semidecidable Propositions

The object $\Sigma = \{\top, \bot\}$ with the topology where $\{\top\}$ is open and $\{\bot\}$ isn't is called Sierpinski Space.

- Maps $f : X \to \Sigma$ are in natural bijection with open subsets of X
- Again, look at $f^{-1}(\top)$. This is open since $\{1\}$ is, but need not be closed (since $\{\bot\}$ isn't open).

Open Subobjects = Semidecidable Propositions

The object $\Sigma = \{\top, \bot\}$ with the topology where $\{\top\}$ is open and $\{\bot\}$ isn't is called Sierpinski Space.

- Maps $f : X \to \Sigma$ are in natural bijection with open subsets of X
- Again, look at $f^{-1}(\top)$. This is open since $\{1\}$ is, but need not be closed (since $\{\bot\}$ isn't open).
- This forms a $\bigvee$-lattice, since arbitrary joins of open subsets are open (yoneda).

Open Subobjects = Semidecidable Propositions

The object $\Sigma = \{\top, \bot\}$ with the topology where $\{\top\}$ is open and $\{\bot\}$ isn't is called Sierpinski Space.

- Maps $f : X \to \Sigma$ are in natural bijection with open subsets of X
- Again, look at $f^{-1}(\top)$. This is open since $\{1\}$ is, but need not be closed (since $\{\bot\}$ isn't open).
- This forms a $\bigvee$-lattice, since arbitrary joins of open subsets are open (yoneda).
- This is also easy to check by hand, if you prefer.

Arbitrary Subobjects = Arbitrary Propositions

The object Ω has two points $\top$ and $\bot$, and for any sequence ω_n

Arbitrary Subobjects = Arbitrary Propositions

The object Ω has two points $\top$ and $\bot$, and for any sequence ω_n

- There's exactly one proof $\omega_n \to \bot$ in $\Omega(\mathbb{N}_\infty)$

Arbitrary Subobjects = Arbitrary Propositions

The object Ω has two points $\top$ and $\bot$, and for any sequence ω_n

- There's exactly one proof $\omega_n \to \bot$ in $\Omega(\mathbb{N}_\infty)$
- There's *lots* of proofs $\omega_n \to \top$

Arbitrary Subobjects = Arbitrary Propositions

The object Ω has two points $\top$ and $\bot$, and for any sequence ω_n

- There's exactly one proof $\omega_n \to \bot$ in $\Omega(\mathbb{N}_\infty)$
- There's *lots* of proofs $\omega_n \to \top$

- Maps $f : X \to \Omega$ are in natural bijection with subobjects of X as computed in $\mathcal{T}$

Arbitrary Subobjects = Arbitrary Propositions

The object Ω has two points $\top$ and $\bot$, and for any sequence ω_n

- There's exactly one proof $\omega_n \to \bot$ in $\Omega(\mathbb{N}_\infty)$
- There's *lots* of proofs $\omega_n \to \top$

- Maps $f : X \to \Omega$ are in natural bijection with subobjects of X as computed in $\mathcal{T}$
- This is a complete lattice, of course.

Arbitrary Subobjects = Arbitrary Propositions

The object Ω has two points $\top$ and $\bot$, and for any sequence ω_n

- There's exactly one proof $\omega_n \to \bot$ in $\Omega(\mathbb{N}_\infty)$
- There's *lots* of proofs $\omega_n \to \top$

- Maps $f : X \to \Omega$ are in natural bijection with subobjects of X as computed in $\mathcal{T}$
- This is a complete lattice, of course.
- Its power comes from all the data stored in the proofs of convergence!

Arbitrary Subobjects = Arbitrary Propositions

The object Ω has two points $\top$ and $\bot$, and for any sequence ω_n

- There's exactly one proof $\omega_n \to \bot$ in $\Omega(\mathbb{N}_\infty)$
- There's *lots* of proofs $\omega_n \to \top$

- Maps $f : X \to \Omega$ are in natural bijection with subobjects of X as computed in $\mathcal{T}$
- This is a complete lattice, of course.
- Its power comes from all the data stored in the proofs of convergence!
- This is a common pattern. . . which we won't explore in this talk.

$\mathcal{T}$ doesn't satisfy LEM or AC, but it gets pretty far!

$\mathcal{T}$ doesn't satisfy LEM or AC, but it gets pretty far!

The Following Hold in $\mathcal{T}$: (Folklore/G.)

$\mathcal{T}$ doesn't satisfy LEM or AC, but it gets pretty far!

The Following Hold in $\mathcal{T}$: (Folklore/G.)

- LLPO – $\forall x : \mathbb{R}.(x \geq 0) \vee (x \leq 0)$

$\mathcal{T}$ doesn't satisfy LEM or AC, but it gets pretty far!

The Following Hold in $\mathcal{T}$: (Folklore/G.)

- LLPO – $\forall x : \mathbb{R}.(x \geq 0) \vee (x \leq 0)$
- MP – $\forall x : \mathbb{R}.\neg(x = 0) \to (x \# 0)$

$\mathcal{T}$ doesn't satisfy LEM or AC, but it gets pretty far!

The Following Hold in $\mathcal{T}$: (Folklore/G.)

- LLPO – $\forall x : \mathbb{R}.(x \geq 0) \vee (x \leq 0)$
- MP – $\forall x : \mathbb{R}.\neg(x = 0) \to (x \# 0)$
- These are weak versions of excluded middle

$\mathcal{T}$ doesn't satisfy LEM or AC, but it gets pretty far!

The Following Hold in $\mathcal{T}$: (Folklore/G.)

- LLPO – $\forall x : \mathbb{R}.(x \geq 0) \vee (x \leq 0)$
- MP – $\forall x : \mathbb{R}.\neg(x = 0) \to (x \# 0)$
- These are weak versions of excluded middle

The Following Holds in $\mathcal{T}$: (Shulman-Simpson)

$\mathcal{T}$ doesn't satisfy LEM or AC, but it gets pretty far!

The Following Hold in $\mathcal{T}$: (Folklore/G.)

- LLPO – $\forall x : \mathbb{R}.(x \geq 0) \vee (x \leq 0)$
- MP – $\forall x : \mathbb{R}.\neg(x = 0) \to (x \# 0)$
- These are weak versions of excluded middle

The Following Holds in $\mathcal{T}$: (Shulman-Simpson)

- DC – "recursive definitions work"

$\mathcal{T}$ doesn't satisfy LEM or AC, but it gets pretty far!

The Following Hold in $\mathcal{T}$: (Folklore/G.)

- LLPO – $\forall x : \mathbb{R}.(x \geq 0) \vee (x \leq 0)$
- MP – $\forall x : \mathbb{R}.\neg(x = 0) \to (x \# 0)$
- These are weak versions of excluded middle

The Following Holds in $\mathcal{T}$: (Shulman-Simpson)

- DC – "recursive definitions work"
- This is a weak version of the axiom of choice

$\mathcal{T}$ doesn't satisfy LEM or AC, but it gets pretty far!

The Following Hold in $\mathcal{T}$: (Folklore/G.)

- LLPO – $\forall x : \mathbb{R}.(x \geq 0) \vee (x \leq 0)$
- MP – $\forall x : \mathbb{R}.\neg(x = 0) \to (x \# 0)$
- These are weak versions of excluded middle

The Following Holds in $\mathcal{T}$: (Shulman-Simpson)

- DC – "recursive definitions work"
- This is a weak version of the axiom of choice

The Following Holds in $\mathcal{T}$: (Folklore/G. again)

$\mathcal{T}$ doesn't satisfy LEM or AC, but it gets pretty far!

The Following Hold in $\mathcal{T}$: (Folklore/G.)

- LLPO – $\forall x : \mathbb{R}.(x \geq 0) \vee (x \leq 0)$
- MP – $\forall x : \mathbb{R}.\neg(x = 0) \rightarrow (x\#0)$
- These are weak versions of excluded middle

The Following Holds in $\mathcal{T}$: (Shulman-Simpson)

- DC – "recursive definitions work"
- This is a weak version of the axiom of choice

The Following Holds in $\mathcal{T}$: (Folklore/G. again)

- Every function $\mathbb{R} \rightarrow \mathbb{R}$ is continuous!

$\mathcal{T}$ doesn't satisfy LEM or AC, but it gets pretty far!

The Following Hold in $\mathcal{T}$: (Folklore/G.)

- LLPO – $\forall x : \mathbb{R}.(x \geq 0) \vee (x \leq 0)$
- MP – $\forall x : \mathbb{R}.\neg(x = 0) \to (x \# 0)$
- These are weak versions of excluded middle

The Following Holds in $\mathcal{T}$: (Shulman-Simpson)

- DC – "recursive definitions work"
- This is a weak version of the axiom of choice

The Following Holds in $\mathcal{T}$: (Folklore/G. again)

- Every function $\mathbb{R} \to \mathbb{R}$ is continuous!

This makes $\mathcal{T}$ like a strong version of Solovay's Model

- In fact there's another nonconstructive theorem that makes life in $\mathcal{T}$ particularly nice:

- In fact there's another nonconstructive theorem that makes life in $\mathcal{T}$ particularly nice:
- You may know that constructively it's preferable to work with *locales* rather than spaces.

- In fact there's another nonconstructive theorem that makes life in $\mathcal{T}$ particularly nice:
- You may know that constructively it's preferable to work with *locales* rather than spaces.
- But also in $\mathcal{T}$ everything is automatically a space...

- In fact there's another nonconstructive theorem that makes life in $\mathcal{T}$ particularly nice:
- You may know that constructively it's preferable to work with *locales* rather than spaces.
- But also in $\mathcal{T}$ everything is automatically a space...
- Can we say that the intrinsic topology on the points of a locale in $\mathcal{T}$ agrees with the topology coming from the locale?

Thm: Yes (G.)

Let X be a regular locale. Let $X^{\mathcal{T}}$ be the $\mathcal{T}$-object of points of X (as computed in $\mathcal{T}$).
Then the intrinsic topology on $X^{\mathcal{T}}$ agrees with the "extrinsic" topology coming from the locale structure! In particular. . .

Thm: Yes (G.)

Let X be a regular locale. Let $X^{\mathcal{T}}$ be the $\mathcal{T}$-object of points of X (as computed in $\mathcal{T}$).
Then the intrinsic topology on $X^{\mathcal{T}}$ agrees with the "extrinsic" topology coming from the locale structure! In particular...

- The object of dedekind reals in $\mathcal{T}$ is represented by the reals with its usual topology

Thm: Yes (G.)

Let X be a regular locale. Let $X^{\mathcal{T}}$ be the $\mathcal{T}$-object of points of X (as computed in $\mathcal{T}$).
Then the intrinsic topology on $X^{\mathcal{T}}$ agrees with the "extrinsic" topology coming from the locale structure! In particular...

- The object of dedekind reals in $\mathcal{T}$ is represented by the reals with its usual topology
- The object of infinite binary sequences in $\mathcal{T}$ is represented by the cantor space with its usual topology

Thm: Yes (G.)

Let X be a regular locale. Let $X^{\mathcal{T}}$ be the $\mathcal{T}$-object of points of X (as computed in $\mathcal{T}$).
Then the intrinsic topology on $X^{\mathcal{T}}$ agrees with the “extrinsic” topology coming from the locale structure! In particular...

- The object of dedekind reals in $\mathcal{T}$ is represented by the reals with its usual topology
- The object of infinite binary sequences in $\mathcal{T}$ is represented by the cantor space with its usual topology
- The object of infinite sequences of naturals in $\mathcal{T}$ is represented by the baire space with its usual topology

Thm: Yes (G.)

Let X be a regular locale. Let $X^{\mathcal{T}}$ be the $\mathcal{T}$-object of points of X (as computed in $\mathcal{T}$).
Then the intrinsic topology on $X^{\mathcal{T}}$ agrees with the "extrinsic" topology coming from the locale structure! In particular...

- The object of dedekind reals in $\mathcal{T}$ is represented by the reals with its usual topology
- The object of infinite binary sequences in $\mathcal{T}$ is represented by the cantor space with its usual topology
- The object of infinite sequences of naturals in $\mathcal{T}$ is represented by the baire space with its usual topology

This says, basically, that these objects behave the way you would expect!

Corollary: Bar and Fan Theorems

Internal to $\mathcal{T}$, the dedekind reals, cantor space, and baire space all have enough points!

Corollary: Bar and Fan Theorems

Internal to $\mathcal{T}$, the dedekind reals, cantor space, and baire space all have enough points!

In particular, the fan theorem and the monotone bar theorems hold in $\mathcal{T}$

Corollary: Bar and Fan Theorems

Internal to $\mathcal{T}$, the dedekind reals, cantor space, and baire space all have enough points!

In particular, the fan theorem and the monotone bar theorems hold in $\mathcal{T}$

Corollary: Brouwer's Axiom for $\mathbb{R}$

In $\mathcal{T}$, every function $\mathbb{R} \to \mathbb{R}$ is (ϵ-δ)-continuous

Corollary: Bar and Fan Theorems

Internal to $\mathcal{T}$, the dedekind reals, cantor space, and baire space all have enough points!

In particular, the fan theorem and the monotone bar theorems hold in $\mathcal{T}$

Corollary: Brouwer's Axiom for $\mathbb{R}$

In $\mathcal{T}$, every function $\mathbb{R} \to \mathbb{R}$ is (ϵ-δ)-continuous

This is suddenly easy, since if $f : \mathbb{R} \to \mathbb{R}$ is a function in $\mathcal{T}$, it's intrinsically continuous. But the previous theorem tells us that intrinsic continuity agrees with localic continuity, which is the usual ϵ-δ notion.

- Here, you might wonder how we know that "f is intrinsically continuous", or indeed what that even means.

- Here, you might wonder how we know that "f is intrinsically continuous", or indeed what that even means.
- Recall that open subsets on a object of $\mathcal{T}$ are in natural bijection with maps into Σ.

- Here, you might wonder how we know that "f is intrinsically continuous", or indeed what that even means.
- Recall that open subsets on a object of $\mathcal{T}$ are in natural bijection with maps into Σ.
- Then if $f : X \to Y$ and $U : Y \to \Sigma$ is an open, pulling back along f always gives an open of X, called $U \circ f : X \to \Sigma$.

In fact, this idea lets us prove a souped-up version of Brouwer's Axiom:

In fact, this idea lets us prove a souped-up version of Brouwer's Axiom:

Cor: Brouwer's Axiom for Metric Spaces

If X and Y are any (external) metric spaces, they're regular locales, so the interpretation of X and Y in $\mathcal{T}$ (as locales) agrees with their metric topology, thus every function $f : X \to Y$ is ϵ-δ continuous!

In fact, this idea lets us prove a souped-up version of Brouwer's Axiom:

Cor: Brouwer's Axiom for Metric Spaces

If X and Y are any (external) metric spaces, they're regular locales, so the interpretation of X and Y in $\mathcal{T}$ (as locales) agrees with their metric topology, thus every function $f : X \to Y$ is ϵ-δ continuous!

But why stop there?

Defn (≈ Chapter 4 of Lešnik's PhD thesis)

Let (X, d) be an *internal* metric space. That is, $d : X \times X \to \mathbb{R}_{\geq 0}$ satisfying the usual axioms (as interpreted in the internal logic).

Defn (≈ Chapter 4 of Lešnik's PhD thesis)

Let (X, d) be an *internal* metric space. That is, $d : X \times X \to \mathbb{R}_{\geq 0}$ satisfying the usual axioms (as interpreted in the internal logic).

- It's always true that open balls $B(x, \epsilon)$ are intrinsically open in X

Defn (≈ Chapter 4 of Lešnik's PhD thesis)

Let (X, d) be an *internal* metric space. That is, $d : X \times X \to \mathbb{R}_{\geq 0}$ satisfying the usual axioms (as interpreted in the internal logic).

- It's always true that open balls $B(x, \epsilon)$ are intrinsically open in X
- Say that d Intrinsically Metrizes X whenever every intrinsic open is a union of open balls.

Defn (≈ Chapter 4 of Lešnik's PhD thesis)

Let (X, d) be an *internal* metric space. That is, $d : X \times X \to \mathbb{R}_{\geq 0}$ satisfying the usual axioms (as interpreted in the internal logic).

- It's always true that open balls $B(x, \epsilon)$ are intrinsically open in X
- Say that d Intrinsically Metrizes X whenever every intrinsic open is a union of open balls.

Theorem: (4.58 in Lešnik's thesis, independently G.)

If (X, d) and (Y, d') are internal metric spaces and d intrinsically metrizes X, then $\mathcal{T}$ proves that every function $X \to Y$ is (ϵ-δ) continuous.

Thank You! ^_^

- Grossack's Blog Post – "Life in Johnstone's Topological Topos (pts. 1,2,3)
- Johnstone's Paper – "On a Topological Topos"
- Lešnik's PhD Thesis – "Synthetic Topology and Constructive" Metric Spaces