Quantum Sets II

Thm: (Gelfand duality): The category of commutative, unital C^*-algebras is equivalent to the opposite of the category of compact hausdorff spaces.

$$\underbrace{\mathrm{Spec}}_{\text{spectrum of a } C^*\text{-alg}} : C^*\text{-Alg}_{c,u} \rightleftarrows \mathrm{CptHaus} : \underbrace{C(-)}_{\text{continuous maps to } \mathbb{C}}$$

Restricting this to the <u>finite</u> compact hausdorff spaces (which are just sets), we get:

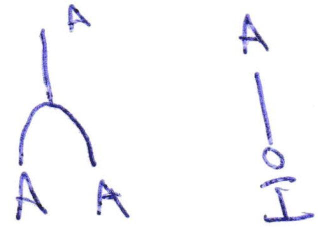
$$C^*\text{-Alg}_{fd,c,u} \cong \mathrm{FinSet}$$

This gives a more abstract characterization of the category of finite sets. But we would like one that doesn't require norm or linear structure

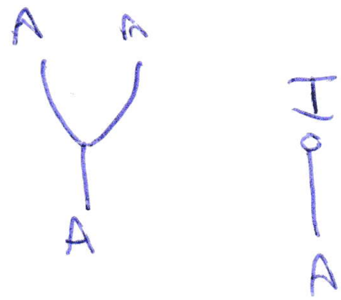
We'll work in a dagger symmetric monoidal category.

Defn: A <u>dagger frobenius algebra</u> is an algebra

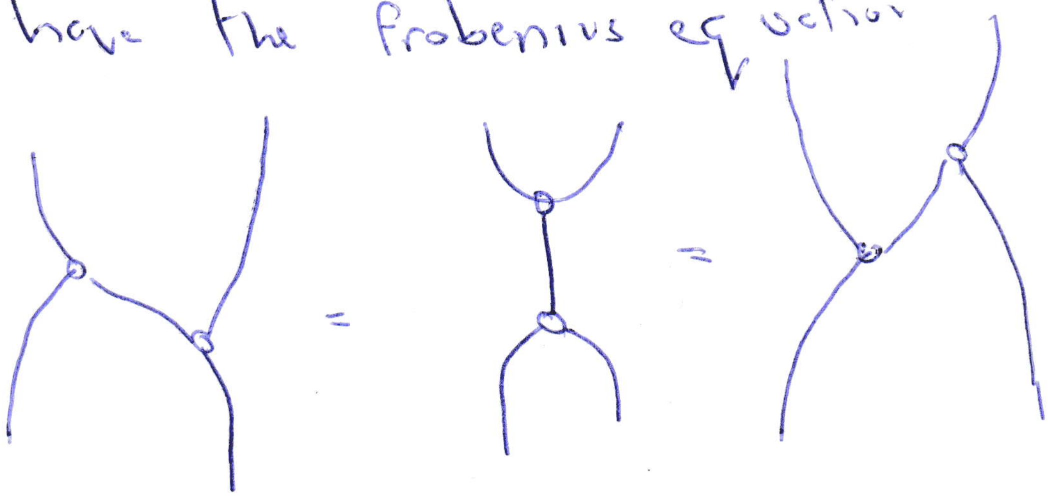
↑ read upward



such that with its adjoint coalgebra structure



We have the frobenius equation

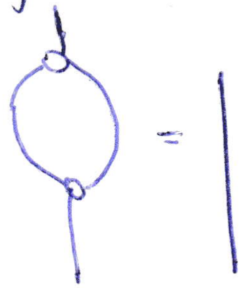


Exercise: Translate these into traditional diagrams.

See Musto, Reutter, Verdon.

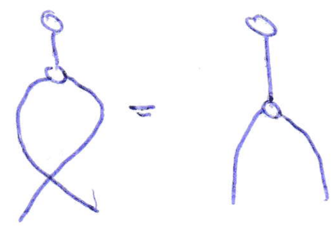
Different types of frobenius algebra

Special



$$\begin{tikzcd} A \arrow[r, "\Delta"] \arrow[dr, equal] & A\otimes A \arrow[d, "m"] \\ & A \end{tikzcd}$$

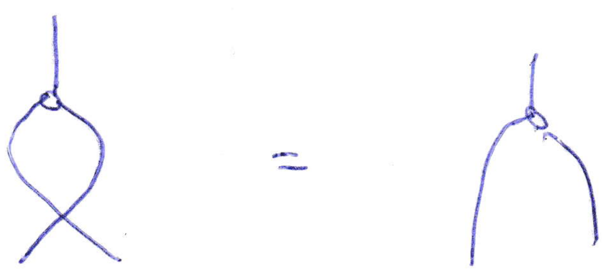
symmetric



$$\begin{tikzcd} A\otimes A \arrow[rr, "c"] \arrow[dr, "m"'] & & A\otimes A \arrow[d, "m"] \\ & A \arrow[dr, "\varepsilon"'] & A \arrow[d, "\varepsilon"] \\ & & \mathbb{C} \end{tikzcd}$$

SSFA

Commutative



$$\begin{tikzcd} A\otimes A \arrow[r, "c"] \arrow[dr, "m"'] & A\otimes A \arrow[d, "m"] \\ & A \end{tikzcd}$$

SCFA

Morphisms

A *-homomorphism between SSFAs or SCFAs is a linear map $f: A \to B$ which commutes with the algebra structure and the dagger.

A *-cohomomorphism is one which commutes with the coalgebra structure and dagger

Thm (Vicary)

1) Every finite dimensional C^*-algebra has a coalgebra structure making it a special symmetric dagger Frobenius algebra structure
2) The commutative finite-dimensional C^*-algebras correspond ~~to~~ special commutative dagger Frobenius algebras.
3) The notion of a *-homomorphism between †-SSFAs corresponds to *-homomorphism between finite-dimensional C^*-algebras.

4) The category of commutative special dagger Frobenius algebras and $*$-cohomomorphisms is equivalent to the category of finite sets and functions.

Note: If we used $*$-homomorphisms, we would have a contravariant equivalence

Note: When doing non-commutative topology, we'll still use symmetric FAs so we do need a bit of commutativity.

(ordinary) sets	SCFAs
(ordinary) functions	$*$-cohomomorphisms
quantum sets	SSFAs
quantum functions	$*$-cohomomorphisms

Def'n: A <u>copyable element of</u> a SCFA is a $*$-cohomomorphism $\psi : \mathbb{C} \to A$,

i.e. a vector $v \in A$ s.t.

$$\Delta(v) = v \otimes v \qquad \varepsilon(v) = 1_{\mathbb{C}}$$

In Hopf algebra theory, this is typically called a <u>group-like element</u>

Thm (Coecke, Pavlovic, Vicary)
The copyable elements of a SCFA form an ON basis of A

This theorem is significant because it allows for the definition of ON basis abstractly ~~without talking about elements~~, Is there a notion of quantum element?

In Set, an element is $\{*\} \longrightarrow X$

In QSet, we quantize

$$\underset{\text{Hilbert space}}{H} \rightarrow \underset{\text{quantum set}}{A} \otimes H$$

Def'n: If A is a quantum set, a <u>quantum element of A</u> is a function $Q : H \rightarrow A \otimes H$ such that:

$$\begin{tikzcd}
H \arrow[r, "Q"] \arrow[d, "Q"'] & A \otimes H \arrow[d, "id_A \otimes Q"] \\
A \otimes H \arrow[r, "\Delta \otimes id"'] & A \otimes A \otimes H
\end{tikzcd}
\qquad
\begin{tikzcd}
H \arrow[r, "Q"] \arrow[dd, equal] & A \otimes H \arrow[d, "\varepsilon \otimes id"] \\
 & k \otimes H \arrow[d, "\cong"] \\
 & H
\end{tikzcd}$$

Def'n: If A is a quantum set, define a category $QEl(A)$

So note A is <u>fixed</u>.

objects: quantum elements of A

arrows: $(H,Q) \to (H',Q')$ are <u>intertwiners</u>

i.e.

$f : H \to H'$ s.t.

$$\begin{tikzcd} H \arrow[r, "Q"] \arrow[d, "f"'] & A \otimes H \arrow[d, "A \otimes f"] \\ H' \arrow[r, "Q'"] & A \otimes H' \end{tikzcd}$$

Thm (stated vaguely)

The quantum set A can be reconstructed from $QEl(A)$

This is an instance of <u>Tannaka–Krein duality</u>

Def'n: A <u>quantum function</u> $A \to B$ is a liner map

$$P : H \otimes A \to B \otimes H$$

A quick introduction to Tannaka-Krein duality

Suppose I have a category $\mathcal{C}$ which I know is of the form $\mathrm{Rep}_{fd}(G)$ but I don't know what G is? Can I reconstruct G from $\mathcal{C}$?

Thm: Every "sufficiently nice" compact closed category is equiv to $\mathrm{Rep}_{fd}(G)$ for an essentially unique G. The key is to have a "fibre functor" thought of as a forgetful functor

$$\mathcal{C} \xrightarrow{U} \mathrm{Vec}_{fd}$$

Then the group can be recovered as structure preserving endomorphisms of U

monoidal natural transformation

In the non-commutative case, one gets Hopf algebras instead of groups. This is due to KH Ulbrich.

Harder Theorem due to Doplicher-Roberts doesn't give the fibre functor.

Such that

$$\begin{tikzcd}
H \otimes A \arrow[rr, "P"] \arrow[d, "id \otimes \Delta"'] & & B \otimes H \arrow[d, "\Delta \otimes id"] \\
H \otimes A \otimes A \arrow[r, "P \otimes id"'] & B \otimes H \otimes A \arrow[r, "id \otimes B"'] & B \otimes B \otimes H
\end{tikzcd}$$

$$\begin{tikzcd}
H \otimes A \arrow[r, "P"] \arrow[dr, "id \otimes \varepsilon"'] & B \otimes H \arrow[r, "\varepsilon \otimes id"] & k \otimes H \arrow[d, "\cong"] \\
& H \otimes k \arrow[r, "\cong"'] & H
\end{tikzcd}$$

There's also an equation involving the dagger

Lemma: Consider $\mathbb{C}$ as a 1-dimensional SCFA. (This corresponds to the classical 1-point set.) A quantum element of a quantum set A is the same thing as a quantum function $\mathbb{C} \to A$

Note: Suppose X, Y are classical sets. Let A_X and A_Y be the corresponding

SCFA's. You might expect that a quantum function $A_X \to A_Y$ corresponded to an ordinary function $X \to Y$. But that's not what happens.

Instead what we get is

"an X-indexed family of projective measurements with outcomes in Y"

But if $H = \mathbb{C}$, then we do get an ordinary function.

Def'n:

$$\begin{tikzcd} A \arrow[r, bend left, "{(H,P)}"] \arrow[r, bend right, "{(H',P')}"'] & B \end{tikzcd}$$

An intertwiner of quantum functions is $f: H \to H'$ s.t.

$$\begin{tikzcd} H \otimes A \arrow[r, "f \otimes \mathrm{id}"] \arrow[dr, "P"'] & H' \otimes A \arrow[r, "P'"] & B \otimes H' \\ & B \otimes H \arrow[ur, "\mathrm{id} \otimes f"'] & \end{tikzcd}$$

We get a 2-category:

objects: quantum sets
1-cells - quantum functions
2-cells - intertwiners

Lemma: This is a dagger 2-category