Quantum Sets

Gelfand Duality

All vector spaces are complex. A <u>normed vector space</u> is a vector space V with a norm function

$$\|-\|: V \to \mathbb{R}^+$$

A norm induces a metric on V, and V is a <u>Banach space</u> if it is complete in this metric.

The category Ban is the category whose objects are Banach spaces and whose arrows are continuous linear maps.

Thm: Ban is monoidal via the projective tensor product. In fact, it's symmetric monoidal closed.

Defn: A <u>Banach algebra</u> is a Banach space A with an associative multiplication

$$m: A \otimes A \to A$$

Warning: Some sources require a unit. Some don't. We'll say unital if we want a unit.

A C*-algebra B is a Banach algebra with an operation $()^*: B \to B$ s.t.

$$x^{**} = x$$

$$(x+y)^* = x^* + y^*$$

$$(xy)^* = y^* x^*$$

$$(\lambda x)^* = \overline{\lambda} x^* \qquad \lambda \in \mathbb{C}$$

$$\|x^* x\| = \|x\| \cdot \|x^*\|$$

Ex: 1) Let X be a compact Hausdorff space.

$C(X) = \{ f: X \to \mathbb{C} \mid \text{continuous} \}$

$\|f\| = \sup \{ \|f(x)\| \mid x \in X \}$ $\quad f^*(x) = \overline{f(x)}$

is a unital commutative C*-algebra.

2) Let X be locally compact Hausdorff

$C_0(X) = \{ f: X \to \mathbb{C} \mid \text{continuous \& "vanishes at } \infty\text{"} \}$

3) If H is a Hilbert space,

$\mathcal{B}(H) = \{ f: H \to H \mid f \text{ is bounded \& linear} \}$

$B(H)$ is a noncommutative unital C^*-Algebra

4) $K(H) = \{ f : H \to H \mid f$ is linear and compact$\}$.
is a non-unital C^*-algebra

Let Cpt be the category of compact hausdorff topological spaces.

Let C^*-$Alg_{c,u}$ be the category of commutative, unital C^*-algebras with unit, (and obvious notion of morphism)

$$C : Cpt^{op} \to C^*\text{-}Alg_{c,u}$$

There's a functor

$$sp : C^*\text{-}Alg_{c,u}^{op} \to Cpt$$

and these two functors form an equivalence

$sp(A) = \{ x : A \to \mathbb{C} \mid x$ is an algebra homomorphism$\}$. It can be given a topology that makes this work

Noncommutative Topology/Geometry.

Since commutative C^*-algebras can be thought of as functions on a space, then a noncommutative C^*-algebra can be thought of as functions on a noncommutative space. Can we redo geometry at this abstract level?

The theory of quantum sets is only finite for the moment.

Finite Gelfand duality

The category of finite-dimensional commutative C^*-algebras is equivalent to the opposite of the category of finite sets and functions.

We'd like a more abstract characterization which didn't rely on the existence of norms.

Compact Closed Categories (Max Kelly)

A symmetric monoidal ~~[illegible]~~ category is **compact closed** is it comes equipped with an operation on objects

$$C \mapsto C^*$$

and for all objects, maps

$$I \xrightarrow{\varepsilon} C \otimes C^* \qquad C^* \otimes C \xrightarrow{\eta} I$$

satisfying 2 equations. Here's one of them, ignoring associativities

$$\begin{tikzcd}
C \arrow[r, "\cong"] \arrow[ddrr, equal, "\mathrm{id}"'] & I \otimes C \arrow[r, "\varepsilon \otimes C"] & C \otimes C^* \otimes C \arrow[d, "C \otimes \eta"] \\
 & & C \otimes I \arrow[d, "\cong"] \\
 & & C
\end{tikzcd}$$

The other equation begins with C^*.

You can prove many things:

1) $(\,)^*$ can be extended to a functor

$$(\,)^* : \mathbb{C}^{op} \to \mathbb{C}$$

2) $(\,)^{**} \simeq \mathrm{id} : \mathbb{C} \to \mathbb{C}$

3) A compact closed category is closed with $A \Rightarrow B \triangleq A^* \otimes B$

4) $(A \otimes B)^* \cong A^* \otimes B^*$

5) Endomorphisms in a compact closed category have a <u>trace</u>

$$\frac{f : A \to A}{Tr(f) : I \to I}$$

satisfying $Tr(fg) = Tr(gf)$

This leads to the theory of <u>traced monoidal categories</u>

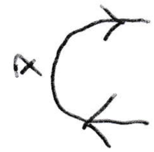
We can also talk about string diagrams in compact closed categories

$$\cap_X : I \to X \otimes X^*$$

$$\cup_X : X^* \otimes X \to I$$

We'll use the convention that on a X arrow, pointing left means X, but pointing right means X^*, and we'll draw our string diagrams left to right.

$I \to X \otimes X^*$ $\qquad\qquad$ $X^* \otimes X \to I$



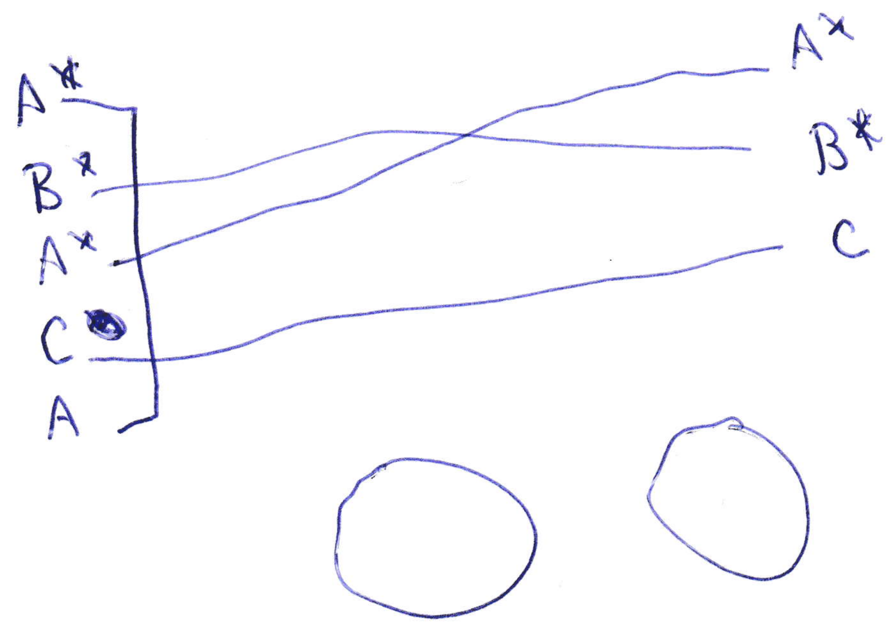
our first equation becomes:

Most equations in monoidal categories can be expressed in terms of

Loops can be created in these string diagrams

$I \to A \otimes A^*$ $\qquad\qquad$ $A \otimes A^* \to I$

So in the free compact closed category, a typical morphism will be a string diagram with some loops



Kelly & Laplaza

This sort of diagram

COHERENCE THEOREMS

We're going to follow:

A compositional approach to quantum functions
by Musto, Reutter, Verdon

They're working with Hilbert spaces,
not just vector spaces. So there's additional
structure

$$(\ )^\dagger : \mathrm{Hilb}^{op} \to \mathrm{Hilb}$$

which is an involution, and the <u>identity on objects</u>

$$\frac{f : H \to K}{f^\dagger : K \to H}$$

and

$$\frac{f \otimes g : H_1 \otimes H_2 \to K_1 \otimes K_2}{f^\dagger}$$

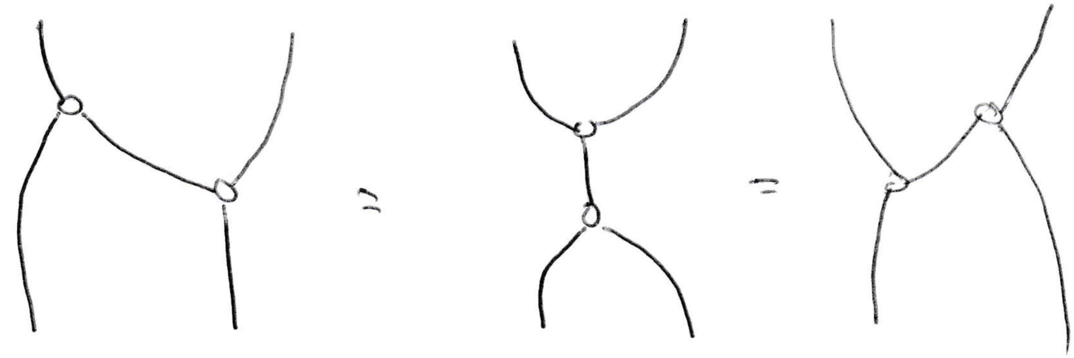
Following the paper, we'll draw our diagrams going upward

$m : H \otimes H \to H$ $\qquad$ $u : \mathbb{C} \to H$

For an associative, unital algebra these satisfy

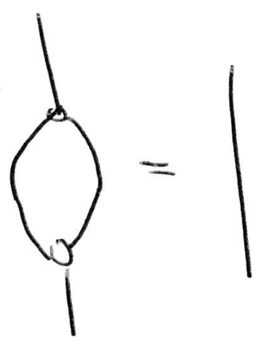
In a dagger category, one can automatically turn an algebra into a coalgebra. This is called the adjoint coalgebra structure

Def'n: A dagger Frobenius algebra is an algebra in a monoidal dagger category such that

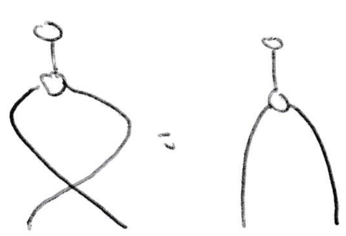


This is the <u>Frobenius equation</u>. In general, in a monoidal category (not necessarily dagger), a Frobenius algebra has an algebra & coalgebra satisfying this equation.

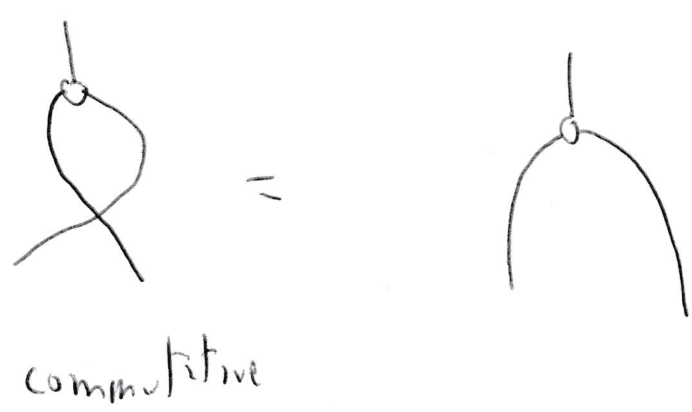
There are several types of Frobenius algebras



special



symmetric

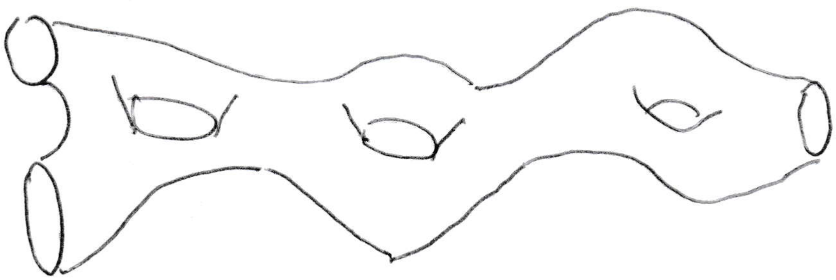


commutative

2D TQFT

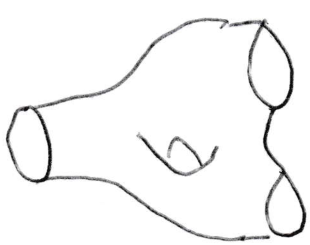
See Joachim Kock - Frobenius Algebras & 2D TQFT

Build a category (roughly speaking)

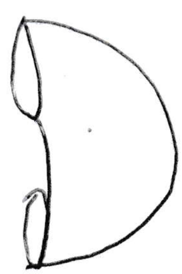


$2 \to 1$

objects are a finite disjoint union of circles
arrows are <u>cobordisms</u> as above.
Composition is to glue along boundaries



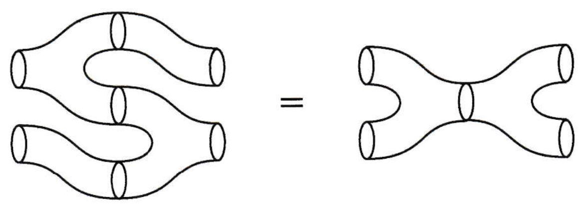
$1 \to 2$



$2 \to 0$

$1 \longrightarrow 0$

This is a category 2Cob



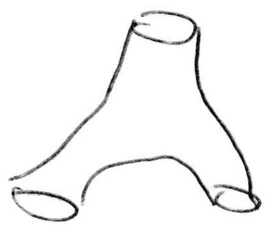
FRONTISPIECE *The topological expression of the main axiom of a Frobenius algebra.* This figure illustrates the content of the main theorem, and captures the whole spirit of the book.

This is a monoidal category with disjoint union as $\otimes$

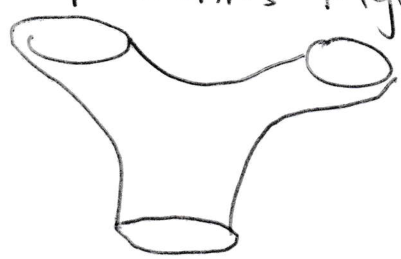
Def'n: A <u>2D TQFT</u> is a symmetric monoidal functor from

$$F: 2Cob \longrightarrow \underbrace{Vec_{fd}}_{\text{complex fd vector spaces.}}$$

Note: In 2Cob, the object 1 (1-circle) has the structure of a Frobenius Algebra (commutative)



Algebra



Coalgebra

Thm: The category 2Cob is the symmetric monoidal category freely generated by the above commutative Frobenius algebra

Thm: There is an equivalence of categories

$$\underset{\substack{\nearrow \\ \text{2d TQFTs} \\ \text{and appropriate} \\ \text{morphisms}}}{\text{2TQFTs}} \simeq \underset{\substack{\uparrow \\ \text{commutative FAs}}}{\text{cFA}}$$

Thm (Vicary)

1) Every finite-dimensional C^*-algebra has a canonical structure of a special commutative ~~symmetric~~ dagger Frobenius algebra.

2) The category of commutative special dagger Frobenius and an appropriate notion of morphism is equivalent to the category of finite sets and functions