Fibrations & Indexed Categories

Part 2

Reminder: We have 2 ways to describe the idea:

family of categories indexed by a category,

Easiest to think of families of sets indexed by a set

$\{X_i\}_{i \in I}$

Each X_i a set

I the indexing set

$\Downarrow$ leads to

Indexed categories

$$\begin{array}{c} \coprod_{i \in I} X_i \\ \Big\downarrow \text{canonical map} \\ I \end{array}$$

$\Downarrow$ leads to

fibred categories

or

fibrations

The tricky bit is what do arrows in the indexing category or base category do to the family.

Definition 1: Let $\mathbb{C}$ be a category. A <u>$\mathbb{C}$-indexed category</u> is a pseudofunctor $\Psi : \mathbb{C}^{op} \to CAT$

So, for every object C in $\mathbb{C}$, we have a category $\Psi(C)$. For every arrow $f : C \to D$ in $\mathbb{C}$, a functor

$$f^* : \Psi(D) \to \Psi(C)$$

remember contravariance

such that

$$id_{\Psi(D)} \cong (id_D)^*$$

& for composible pairs

$$u^* v^* \cong (v \circ u)^*$$

These isos must be specified and satisfy coherence conditions.

A coherence theorem was proved by Maclane & Paré.

Fibred categories are less intuitively clear but have several desirable properties.

Defn: $p : \mathbb{E} \to \mathbb{B}$

$f : X \to Y$ in $\mathbb{E}$ is <u>cartesian</u> over $u : I \to J$ if

1) $p(f) = u$

2) $\forall g : Z \to Y$ in $\mathbb{E}$ for which one has $p(g) = u \circ w$ where $w : pZ \to I$ in $\mathbb{B}$, $\exists$ a unique $h : Z \to X$ in $\mathbb{E}$ s.t. $p(h) = w$ & $f \circ h = g$

$$
\begin{tikzcd}
\mathbb{E} \arrow[ddd] & Z \arrow[d, "\exists! h"'] \arrow[dr, bend left, "g"] & \\
& X \arrow[r, "f"'] & Y \\
& pZ \arrow[d, "w"'] \arrow[dr, bend left, "p(g) = u \circ w"] & \\
\mathbb{B} & I \arrow[r, "u"'] & J
\end{tikzcd}
$$

Def'n: $p : \mathbb{E} \to \mathbb{B}$ is a fibration

if $\forall Y \in ob(\mathbb{E})$ and $u : I \to pY$

in $\mathbb{B}$, there is a cartesian morphism

$f : X \to Y$ in $\mathbb{E}$ s.t. $p(f) = u$.

p is called a fibration or fibred category

Ex: $cod : Set^{\to} \to Set$

the codomain functor is a fibration.

The cartesian morphisms are precisely the pullback squares.

More generally, while the functor

$$cod : \mathbb{C}^{\to} \to \mathbb{C}$$

always exists, it is a fibration if and only if $\mathbb{C}$ has all pullbacks.

Given a fibration, one can use

the uniqueness property to construct functors

$$\frac{u : I \to J \quad \text{in } \mathbb{B}}{u^* : \mathbb{E}_J \to \mathbb{E}_I}$$

where $p^{-1}(J) = \mathbb{E}_J$ is the fibre over J

This process ~~[illegible]~~ determines an indexed category. (Details last time)

Converse is tricky. Given an indexed category, can we construct a fibration?

Consider $\Psi : \mathbb{B}^{op} \to Cat$ be an indexed category (or pseudo functor)

Will define a category $\int_{\mathbb{B}} \Psi$

objects (I, X) with $I \in ob(\mathbb{B})$ and
$X \in ob(\Psi(I))$

arrows $(I, X) \xrightarrow{(u,f)} (J, Y)$

with ~~[illegible]~~ $u: I \to J$

$f: X \to \Psi(u)(Y)$ in $\Psi(I)$

There is a functor

$$\int_{\mathbb{B}} \Psi \to \mathbb{B}$$

$$(I, X) \mapsto I$$

$$(u, f) \mapsto u$$

Thm: This is a fibration, called the Grothendieck construction

It induces an equivalence between indexed categories & fibrations

There are many extensions of this idea.
We'll look at:

The Monoidal Grothendieck Construction
by Moeller & Vasilakopoulou

See also

Shulman: Framed bicategories & monoidal fibrations.

We need to understand monoidal 2-categories

Def'n: A <u>monoidal 2-category</u> $\mathbb{C}$ is a 2-category with a pseudofunctor

$$\otimes : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$$

and a 0-cell I with associativity & unitary isos satisfying coherence.

If $\mathbb{C}$, $\mathbb{D}$ are monoidal 2-cats, a <u>weak monoidal pseudofunctor</u> is a pseudofunctor $\mathbb{F}: \mathbb{C} \to \mathbb{D}$ with pseudonatural transformations

$$
\begin{tikzcd}
\mathbb{C} \times \mathbb{C} \arrow[r, "\mathbb{F} \times \mathbb{F}"] \arrow[d, "\otimes"'] \arrow[dr, phantom, "\overset{M}{\swarrow}"] & \mathbb{D} \times \mathbb{D} \arrow[d, "\otimes"] \\
\mathbb{C} \arrow[r, "\mathbb{F}"'] & \mathbb{D}
\end{tikzcd}
\qquad I \xrightarrow{M} \mathbb{F} I
$$

satisfying horrifying coherence conditions. In particular, to explain properly requires morphisms between pseudonatural transformations

Defn: Let $\mathbb{C}$ be a monoidal 2-category. A <u>pseudomonoid</u> (or <u>monoidale</u>)

is an object A together with 1-cells

$$m: A\otimes A \to A \qquad j: I \to A$$

and invertible 2-cells

$$\begin{tikzcd} & A\otimes A \arrow[dr] & \\ A\otimes A\otimes A \arrow[ur] \arrow[dr] & \cong & A \\ & A\otimes A \arrow[ur] & \end{tikzcd} \qquad \begin{tikzcd} A\otimes I \arrow[r] \arrow[dr] \arrow[dr, phantom, bend left=10, "\cong"] & A\otimes A \arrow[d] \arrow[d, phantom, shift left=4, "\cong"] & I\otimes A \arrow[l] \arrow[dl] \\ & A & \end{tikzcd}$$

satisfying coherence

Key Ex: CAT, with $\times$ as tensor product.

The pseudomonoids are the monoidal categories.

Let Fib be the 2-category whose objects are fibrations, 1-cells are structure-preserving maps & 2-cells are appropriate natural transformations

Lemma: If $P: \mathbb{C} \longrightarrow \mathbb{D}$ and $Q: \mathbb{C}' \longrightarrow \mathbb{D}'$ are fibrations

then $P \times Q : \mathbb{C} \times \mathbb{C}' \longrightarrow \mathbb{D} \times \mathbb{D}'$
is a fibration as well.

In particular, Fib is a monoidal 2-category. It inherits its monoidal 2-structure from $CAT^{\to}$

Def'n: A pseudomonoid in Fib is called a <u>monoidal fibration</u>.

Thm: A monoidal fibration $P : \mathbb{C} \to \mathbb{D}$ is a fibration for which both $\mathbb{C}$ and $\mathbb{D}$ are monoidal, P is a strong monoidal functor and $\otimes$ preserves cartesian liftings.

In other words, if $f : D_1 \to D_1'$, $g : D_2 \to D_2'$ in $\mathbb{D}$, and $\hat{f} : C_1 \to C_1'$, ~~$g: D_2 \to D_2'$~~ $\hat{g} : C_2 \to \hat{C}_2$ are cartesian liftings of f and g in $\mathbb{C}$, then $\hat{f} \otimes \hat{g}$ is a cartesian lifting of $f \otimes g$.

Monoidal Indexed Categories

The 2-category of indexed category is denoted ICAT. It's monoidal

$$\mathbb{F} : \mathbb{C}^{op} \to CAT$$

$$\mathbb{G} : \mathbb{D}^{op} \to CAT$$

$$\mathbb{F} \otimes \mathbb{G} : (\mathbb{C} \times \mathbb{D})^{op} \xrightarrow{\simeq} \mathbb{C}^{op} \times \mathbb{D}^{op} \xrightarrow{\mathbb{F} \times \mathbb{G}} CAT \times CAT \xrightarrow{\times} CAT$$

Def'n: A pseudomonoid in ICAT is a monoidal indexed category

Lemma: A monoidal indexed category is a monoidal pseudofunctor

$$\mathbb{C}^{op} \longrightarrow CAT$$

with $\mathbb{C}$ a monoidal category

So we have the following 2-categories

$$
\begin{tikzcd}
\mathrm{MonFib} \arrow[r, hook] & \mathrm{Fib} \arrow[d, phantom, "\simeq"] \\
\mathrm{MonICAT} \arrow[r, hook] & \mathrm{ICAT}
\end{tikzcd}
$$

Grothendieck construction establishes the equivalence

Thm: The equivalence induced by the Grothendieck construction is monoidal, and so induces an equivalence between

$$\mathrm{MonFib} \simeq \mathrm{Mon\,ICAT}$$

The equivalence also holds in the braided & symmetric cases as well.