Grothendieck 2-topoi are elementary 2-topoi

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Subobject classifiers

Definition.

Let $\mathcal{C}$ be a 1-category with finite limits. A **subobject classifier** in $\mathcal{C}$ is a **monomorphism $\tau\colon 1 \hookrightarrow \Omega$** in $\mathcal{C}$ that is **universal** in the following sense:

$$\begin{array}{ccc} A & \longrightarrow & 1 \\ {\scriptstyle \forall i}\big\downarrow & \lrcorner & \big\downarrow{\scriptstyle \tau} \\ X & \underset{\exists ! \chi_i}{\dashrightarrow} & \Omega \end{array}$$

For every $X \in \mathcal{C}$ the function

$$\mathcal{G}_{\tau,X}\colon \mathcal{C}(X,\Omega) \to \mathrm{Sub}(X)$$

given by **pulling back τ is a bijection**.

Elementary toposes

An **elementary topos** is a category $\mathcal{C}$ with finite limits that has a subobject classifier and is cartesian closed.

An elementary topos has an internal logic!

Example.

The archetypal elementary topos is $\mathcal{S}et$.

$T\colon 1 \hookrightarrow \{T,F\}$ classifies subsets via the characteristic functions.

$$\begin{array}{ccc} A & \longrightarrow & 1 \\ {\scriptstyle \forall i_A}\big\downarrow & \lrcorner & \big\downarrow{\scriptstyle T} \\ X & \underset{\exists ! \chi_A}{\dashrightarrow} & \{T,F\} \end{array}$$

$$\chi_A\colon \; X \longrightarrow \{T,F\}$$

$$x \longmapsto \begin{cases} T \text{ if } x \in A \\ F \text{ if } x \notin A \end{cases}$$

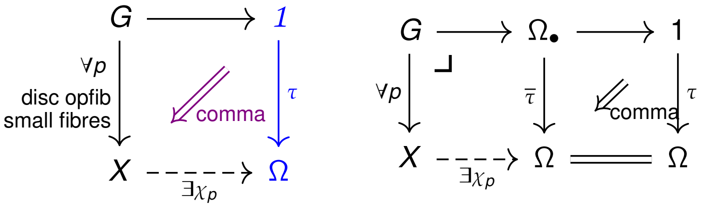
2-classifiers

Definition (Weber, slightly changed).

Let $\mathcal{K}$ be a 2-category with finite 2-limits. A **2-classifier** is a morphism $\tau\colon 1 \to \Omega$ in $\mathcal{K}$ such that for every $X \in \mathcal{K}$ the functor

$$\mathcal{G}_{\tau,X}\colon \mathcal{K}(X,\Omega) \to \mathcal{DOpFib}_{\mathcal{K}}(X)$$

given by **taking comma objects from $\tau\colon 1 \to \Omega$** is an **equivalence of categories**.



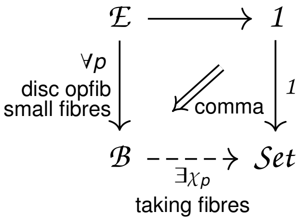
Discrete opfibrations

We upgrade the subobjects to **discrete opfibrations**, that (essentially) **have as fibres arbitrary sets** (thus of one dimension higher).

Example.

The archetypal elementary 2-topos is $\mathcal{CAT}$.

$1\colon 1 \to \mathcal{S}et$ classifies all discrete opfibrations with small fibres, via the **category of elements (Grothendieck construction)**.



equivalence of categories

$$[\mathcal{B}, \mathcal{S}et] \simeq \mathcal{DO}p\mathcal{F}ib(\mathcal{B})$$

Dense generators

Definition.

A 2-functor $J\colon \mathcal{Y} \to \mathcal{K}$ is **dense** if the restricted Yoneda embedding

$$\begin{aligned} \widetilde{J}\colon \ \mathcal{K} &\longrightarrow [\mathcal{Y}^{\mathrm{op}}, \mathcal{CAT}] \\ F &\mapsto \mathcal{K}(J(-), F) \end{aligned}$$

is fully faithful.

If J is fully faithful, this is equivalent to ask that **each object $F \in \mathcal{K}$ is a weighted 2-colimit of objects of $\mathcal{Y}$ which is preserved by $\widetilde{J}$**.

Example.

Representables are a dense generator for 2-presheaves. Every 2-presheaf is a weighted 2-colimit of representables.

Reduction to dense generators

We reduce the study of 2-classifiers to dense generators.

Let $J\colon \mathcal{Y} \to \mathcal{K}$ be a fully faithful dense 2-functor. Consider a discrete opfibration $\tau\colon \Omega_{\bullet} \to \Omega$ in $\mathcal{K}$.

Proposition (M.).

If $\forall Y \in \mathcal{Y}$ the functor $\mathcal{G}_{\tau,Y}\colon \mathcal{K}(Y,\Omega) \to \mathcal{DOpFib}(Y)$ is **faithful**, **then** $\forall F \in \mathcal{K}$ the functor $\mathcal{G}_{\tau,F}\colon \mathcal{K}(F,\Omega) \to \mathcal{DOpFib}(F)$ is **faithful**.

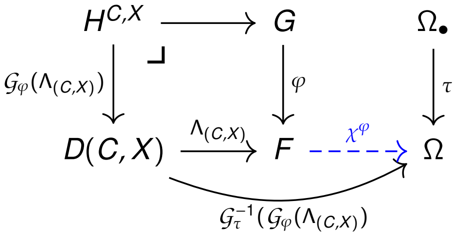
Theorem (M.).

If $\forall Y \in \mathcal{Y}$ the functor $\mathcal{G}_{\tau,Y}\colon \mathcal{K}(Y,\Omega) \to \mathcal{DOpFib}(Y)$ is **fully faithful**, **then** $\forall F \in \mathcal{K}$ the functor $\mathcal{G}_{\tau,F}\colon \mathcal{K}(F,\Omega) \to \mathcal{DOpFib}(F)$ is **full** (and faithful).

Reduction to dense generators

Theorem (M.).

Assume that $\forall Y \in \mathcal{Y}$ the functor $\mathcal{G}_{\tau,Y}\colon \mathcal{K}(Y,\Omega) \to \mathcal{DOpFib}(Y)$ is an **equivalence of categories**. If an operation of normalization is possible, **then** $\forall F \in \mathcal{K}$ the functor $\mathcal{G}_{\tau,F}\colon \mathcal{K}(F,\Omega) \to \mathcal{DOpFib}(F)$ is **essentially surjective on objects** (and fully faithful).



Reduction applied to $\mathcal{CAT}$

Example (Reduction of the category of elements).

The singleton category **$\mathit{1}$ is a dense generator in $\mathcal{CAT}$**. So we can just look at the discrete opfibrations over $\mathit{1}$.

$$\mathcal{G}_{\tau,1}\colon \mathcal{CAT}(1, \mathcal{S}et) \to \mathcal{S}et$$

sends a functor $1 \to \mathcal{S}et$ to the set it picks, so it is an equivalence of categories. By the theorems of reduction, the construction of the **category of elements is fully faithful and classifies all discrete opfibrations with small fibres.**

Moreover, following the proof, we obtain a classifying morphism for a φ by collecting all its fibres, since the pullback of φ along $B\colon 1 \to \mathcal{B}$ gives precisely the fibre over B.

A 2-classifier in prestacks

Let $\mathcal{C}$ be a category and consider $\mathcal{K} = [\mathcal{C}^{\mathrm{op}}, \mathcal{CAT}]$. **Representables form a dense generator**, so we can just look at

$$\mathcal{G}_{\tau,\mathrm{y}(C)} \colon [\mathcal{C}^{\mathrm{op}}, \mathcal{CAT}]\,(\mathrm{y}(C), \Omega) \to \mathcal{DOpFib}\,(\mathrm{y}(C))$$

and ask this to be an equivalence of categories for every $C \in \mathcal{C}$.

$C \overset{\Omega}{\mapsto} \mathcal{DOpFib}\,(\mathrm{y}(C))$ would only give a pseudofunctor $\mathcal{C}^{\mathrm{op}} \to \mathcal{CAT}$.
Thanks to the **indexed Grothendieck construction (joint work with Elena Caviglia)**, we can take the strict replacement

$$C \overset{\widetilde{\Omega}}{\mapsto} \left[\left(\mathcal{C}/_C\right)^{\mathrm{op}}, \mathcal{Set}\right]$$

This is in line with Hofmann-Streicher.

A 2-classifier in prestacks

Theorem (M.).

Consider $\tau\colon 1 \to \Omega$ such that τ_C picks $\mathrm{id}_{y(C)} \in \mathcal{DOpFib}(y(C))$. Then

$$\mathcal{G}_{\tau,y(C)}\colon \mathrm{Ps}\,[\mathcal{C}^{\mathrm{op}},\mathcal{CAT}]\,(y(C),\Omega) \to \mathcal{DOpFib}(y(C)) = \Omega(C)$$

is **isomorphic to Yoneda's lemma map and is thus an equivalence**.
The corresponding $\widetilde{\tau}\colon 1 \to \widetilde{\Omega}$ such that $\widetilde{\tau}_C$ picks $\Delta 1$ is a **2-classifier in $[\mathcal{C}^{\mathrm{op}},\mathcal{CAT}]$**.

Idea of the normalization process:

$$\begin{array}{ccc} H^{C,X} & \longrightarrow & G \\ {\scriptstyle \psi^{C,X}}\downarrow & \lrcorner & \downarrow{\scriptstyle \varphi} \\ y(C) & \underset{X}{\longrightarrow} & F \end{array}$$

change the fibre $\left(\psi^{C,X}_D\right)_{D \xrightarrow{f} C}$
into the fibre $(\varphi_D)_{F(f)(X)}$
(fibres of φ are global).

Definition (Idea).

A **stack** is a **bicategorical sheaf**. Matching families are only required to satisfy the **descent compatibility up to isomorphism**.

A **descent datum** for $F\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{CAT}$ w.r.t. a sieve S on $C \in \mathcal{C}$ is

$$(D \xrightarrow{f} C) \in S \quad \longmapsto \quad x_f \in F(D)$$

and for every $D'' \xrightarrow{h} D' \xrightarrow{g} D \xrightarrow{f} C$ with $f \in S$ an isomorphism

$$\boldsymbol{\varphi^{f,g}\colon g^* x_f \xrightarrow{\simeq} x_{f\circ g}}$$

$$\begin{array}{ccc} h^*(g^* x_f) & \xrightarrow{h^*\varphi^{f,g}} & h^*(x_{f\circ g}) \\ \wr\Vert & & \big\downarrow \varphi^{f\circ g,h} \\ (g\circ h)^*(x_f) & \xrightarrow{\varphi^{f,g\circ h}} & x_{f\circ g\circ h}. \end{array}$$

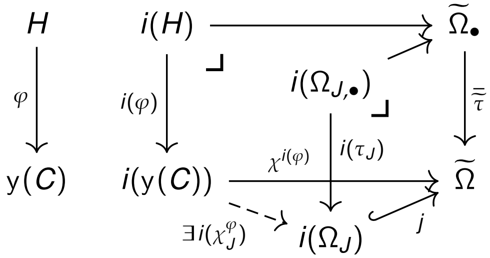
A 2-classifier in stacks

We want to restrict the 2-classifier

$$\begin{aligned} \widetilde{\Omega}: \ \mathcal{C}^{\mathrm{op}} &\longrightarrow \mathcal{CAT} \\ C &\mapsto \left[\left(\mathcal{C}/_{C}\right)^{\mathrm{op}}, \mathcal{S}et\right] \end{aligned}$$

in $[\mathcal{C}^{\mathrm{op}}, \mathcal{CAT}]$ to a **2-classifier Ω_J in functorial stacks $\mathcal{S}t(\mathcal{C}, J)$**.

Ω_J needs to be **tight enough to be a stack** but at the same time **loose enough to host the classification**.



$$\mathcal{S}t(\mathcal{C}, J) \overset{i}{\hookrightarrow} [\mathcal{C}^{\mathrm{op}}, \mathcal{CAT}]$$

A 2-classifier in stacks

Theorem (M.).

$$\begin{aligned} \Omega_J : \ \mathcal{C}^{\mathrm{op}} &\longrightarrow \mathcal{CAT} \\ C &\mapsto \mathcal{Sh}\left(\mathcal{C}/_{C}\right) \end{aligned}$$

is a **functorial stack**.

For discrete opfibrations $\varphi\colon \mathcal{H} \to \mathrm{y}(C)$ in $\mathcal{St}(\mathcal{C}, J)$ over representables, the **characteristic morphism $\chi^{i(\varphi)}$ factorizes through $i(\Omega_J)$**.

And the normalization process is done in prestacks.

Then **$\tau_J\colon \Omega_{J,\bullet} \to \Omega_J$ is a 2-classifier in functorial stacks $\mathcal{St}(\mathcal{C}, J)$**, classifying all discrete opfibrations with small fibres.

References

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