The Fundamental Groupoid in Discrete Homotopy Theory

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Based on joint work with Chris Kapulkin

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References

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What is discrete homotopy theory?

Dramatis Personae:

A graph is a set equipped with a symmetric, reflexive relation $\sim$.

If $x \sim x'$ and $x \neq x'$, we draw an edge between the vertices corresponding to x and x'. We suppress the unique loop present at each vertex.

A graph map $f \colon X \to Y$ is a function that preserves the relation.

Thus, edges in X can be mapped to edges in Y or be contracted to single vertex in Y.

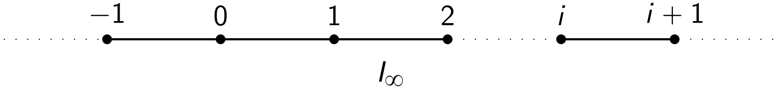
We write Graph for the the category of graphs and graph maps.

What is discrete homotopy theory?

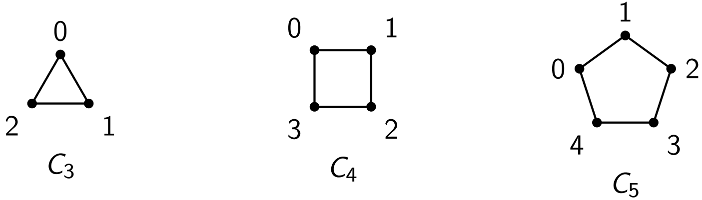
Examples:

- The n-interval I_n, for $n \in \mathbb{N}$

- The infinite interval I_∞



- The n-cycle C_n, for $n \geqslant 3$



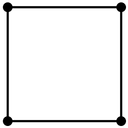
What is discrete homotopy theory?

The box product $X \square Y$ of two graphs X and Y is defined as follows:

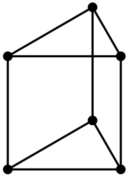
$$(X \square Y)_V = X_V \times Y_V$$

$$(X \square Y)_E = \left\{ (x, y) \sim (x', y') \;\middle|\; \begin{array}{r} \text{either } x \sim x' \text{ and } y = y' \\ \text{or } x = x' \text{ and } y \sim y' \end{array} \right\}$$

Examples:



$I_1 \square I_1$



$C_3 \square I_1$

What is discrete homotopy theory?

Let X and Y be two graphs, and let $f, g \colon X \to Y$ be two graph maps

An *A-homotopy* $H \colon f \Rightarrow g$ of length $n \in \mathbb{N}$ is a map

$$H \colon X \mathbin{\square} I_n \to Y$$

such that

$$H(-, 0) = f \text{ and } H(-, n) = g.$$

A graph map $f \colon X \to Y$ is an *A-homotopy equivalence* if there exists some graph map $g \colon Y \to X$ and homotopies $g \circ f \Rightarrow \mathrm{id}_X$ and $f \circ g \Rightarrow \mathrm{id}_Y$.

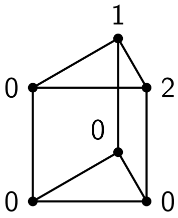
Examples:

- $I_n \xrightarrow{!} I_0$ is an A-homotopy equivalence for all $n \in \mathbb{N}$.
- $I_\infty \xrightarrow{!} I_0$ is NOT an A-homotopy equivalence.

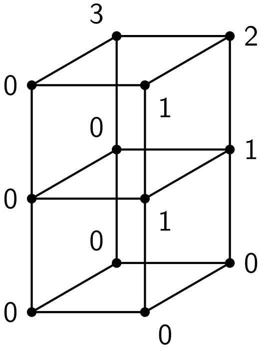
What is discrete homotopy theory?

Examples:

- $C_n \xrightarrow{!} I_0$ is an A-homotopy equivalence for $n = 3, 4$.



$C_3 \,\square\, I_1 \to C_3$



$C_4 \,\square\, I_2 \to C_4$

What is discrete homotopy theory?

Let X be a graph and $x, x' \in X_V$ be two vertices.

A path $\gamma \colon x \rightsquigarrow x'$ is a map

$$\gamma \colon I_\infty \to X$$

for which there exist integers $N_-, N_+ \in \mathbb{Z}$ such that

$$\gamma(i) = x \text{ for all } i \leqslant N_-$$

and

$$\gamma(i) = x' \text{ for all } i \geqslant N_+.$$

The fundamental group $A_1(X, x_0)$ of a pointed graph (X, x_0) is defined as follows:

$$A_1(X, x_0) = \{\text{paths in } X \text{ from } x_0 \text{ to itself}\} \,/ \sim_*$$

Applications

Some areas in which discrete homotopy theory has found applications:

1. Matroid theory
   S. B. Maurer, *Matroid basis graphs*, J. Combinatorial Theory Ser. B, 1973
2. Hyperplane arrangements
   H. Barcelo, C. Severs, and J. A. White, *k-Parabolic subspace arrangements*, Trans. Am. Math. Soc., 2011
3. Topological data analysis
   F. Mémoli and L. Zhou, *Persistent homotopy groups of metric spaces*, 2019.

The fundamental groupoid

The fundamental groupoid $\Pi_1 X$ of a graph X is defined as follows:

$$\Pi_1 X = \begin{cases} \text{objects:} & \text{vertices } x, x', \ldots \text{ of } X \\ \text{morphisms:} & \text{path-homotopy classes } [\gamma] : x \to x' \text{ of} \\ & \text{paths } \gamma\colon x \rightsquigarrow x' \text{ in } X \end{cases}$$

We have $A_1(X, x_0) \cong \Pi_1 X(x_0, x_0)$.

Theorem (Kapulkin-M.)

For any graphs X and Y, we have:

$$\Pi_1(X \,\square\, Y) \cong \Pi_1 X \times \Pi_1 Y$$

Overview of contributions

1. A finite formulation of the homotopy groups
2. Covering graphs
3. Van Kampen theorem for graphs
4. Graphs with a prescribed fundamental group(oid)

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Covering graphs

The (star) neighbourhood N_x of a vertex x in a graph X consists of x and its neighbours, with an edge drawn between x and each of its neighbours.

A map $p\colon Y \to X$ is a local isomorphism if it restricts to an isomorphism between N_y and N_{fy} for each $y \in Y$.

A map $p\colon Y \to X$ is a covering if, in addition to being a local isomorphism, it has the RLP against $\sqsubset\colon I_3 \to I_1 \square I_1$.

$$\begin{array}{ccc} I_3 & \longrightarrow & Y \\ {\scriptstyle \sqsubset}\big\downarrow & \nearrow & \big\downarrow{\scriptstyle p} \\ I_1 \square I_1 & \longrightarrow & X \end{array}$$

Example:

- $I_\infty \to C_n; i \mapsto i \mod n$ is a local isomorphism for all $n \geqslant 3$, but a covering only for $n \geqslant 5$.

Covering graphs

We write $\mathsf{Cov}(X)$ for the full subcategory of $\mathsf{Graph} \downarrow X$ on coverings.

Theorem (Kapulkin-M.)

1. *For any graph X, we have:* $\mathsf{Cov}(X) \simeq \mathsf{Set}^{\Pi_1 X}$.
2. *For any pointed, connected graph (X, x_0), we have:*

$$\left\{ \begin{array}{c} \text{connected coverings} \\ \text{of } (X, x_0) \end{array} \right\} \simeq \left\{ \begin{array}{c} \text{subgroups of } A_1(X, x_0) \\ \text{ordered by } \subseteq \end{array} \right\}^{\mathrm{op}}$$

Covering graphs

A pointed covering $(Y, y_0) \to (X, x_0)$ is called a universal cover if it is initial in the category of pointed coverings over (X, x_0).

Theorem (Kapulkin-M.)

1. *Every pointed graph* (X, x_0) *admits a universal cover* $(\tilde{X}_{x_0}, [c_{x_0}])$.
2. *A pointed covering* $(Y, y_0) \to (X, x_0)$ *is universal if and only if the graph* Y *is simply connected.*
3. $A_1(X, x_0) \cong Aut_{\mathsf{Cov}(X)}(\tilde{X}_{x_0})$.

Examples:

- The cycle graph C_n for $n = 3, 4$ has C_n itself as its universal cover.
- The cycle graph C_n for $n \geqslant 5$ has I_∞ as its universal cover.

Corollary

$$A_1(C_n, *) \cong \begin{cases} 0 & \text{if } n = 3, 4 \\ \mathbb{Z} & \text{if } n \geqslant 5 \end{cases}$$

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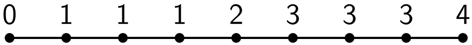
Van Kampen theorem for graphs

A map $s\colon I_m \to I_n$ is shrinking if it is order-preserving and surjective.

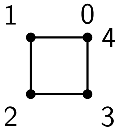


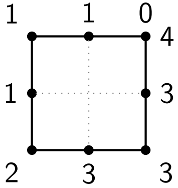
a shrinking map $I_8 \to I_4$

Let X be a graph and $h\colon I_1 \,\square\, I_1 \to X$ be any map.
A net of h is a pair (H, s) consisting of a map $H\colon I_m \,\square\, I_n \to X$ for some $m, n \in \mathbb{N}$, together with a shrinking map $s\colon I_{2m+2n} \to I_4$, such that:

$$H \circ \partial_{m,n} = h \circ \partial_{1,1} \circ s$$

Example:





Van Kampen theorem for graphs

Theorem (Kapulkin-M.)

Consider a pushout square in Graph *as follows:*

$$\begin{array}{ccc} X_0 & \longrightarrow & X_1 \\ \downarrow & & \downarrow \\ X_2 & \longrightarrow & X \end{array}$$

If every map $h\colon I_1 \square I_1 \to X$ whose image in X is a 3- or 4-cycle admits a net (H, s) such that each cell $H_{i,j}$ of H factors through X_1 or X_2, then the pushout square is preserved by the functor $\Pi_1\colon \mathsf{Graph} \to \mathsf{Gpd}$. That is, we have the following pushout square in Gpd*:*

$$\begin{array}{ccc} \Pi_1 X_0 & \longrightarrow & \Pi_1 X_1 \\ \downarrow & & \downarrow \\ \Pi_1 X_2 & \longrightarrow & \Pi_1 X \end{array}$$

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Graphs with a prescribed fundamental group(oid)

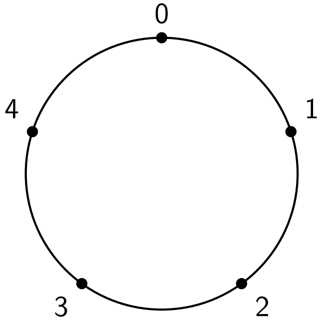
For $m \geqslant 3$ and $n \in \mathbb{N}$, let the (m,n)-disk $D_{m,n}$ be the following quotient:

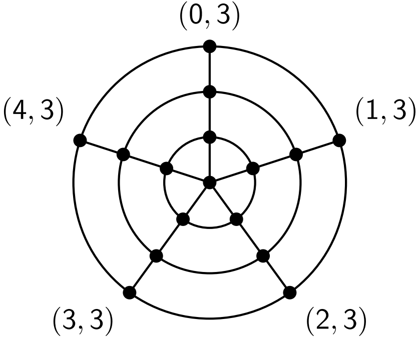
$$D_{m,n} = C_m \,\square\, I_n / \sim \qquad\qquad (i,0) \sim (j,0) \text{ for all } i,j \in C_m$$

and let the boundary map $\partial_{m,n} \colon C_m \to D_{m,n}$ be given by

$$\partial_{m,n}(i) = (i,n).$$

Example:





Graphs with a prescribed fundamental group(oid)

Let F_S be the free group generated by a set S.

Given any word $r = s_1^{d_1} \cdots s_k^{d_k} \in F_S$, we can define its degree as follows:

$$\deg(r) = |d_1| + \cdots + |d_k|$$

We can then define a map ω_r corresponding to the word r as follows:

$$\omega_r \colon C_{5 \cdot \deg(r)} \longrightarrow \bigvee_{s \in S} C_5$$

First, wrap d_1 times around the C_5 corresponding to $s_1 \in S$ (clockwise if $d_1 > 0$ and counterclockwise if $d_1 < 0$), then d_2 times around the C_5 corresponding to $s_2 \in S$, and so on.

Graphs with a prescribed fundamental group(oid)

Let G be any group, along with a presentation $G = \langle S \mid R \rangle$.

Define $X_{\langle S|R \rangle}$ to be the graph given by the following pushout:

$$\begin{array}{ccc} \coprod_{r \in R} C_{5 \cdot \deg(r)} & \xrightarrow{(\omega_r)_{r \in R}} & \bigvee_{s \in S} C_5 \\ {\scriptstyle \coprod_{r \in R} \partial_{5 \cdot \deg(r),3}} \downarrow & \ulcorner & \downarrow \\ \coprod_{r \in R} D_{5 \cdot \deg(r),3} & \longrightarrow & X_{\langle S|R \rangle} \end{array}$$

Theorem (Kapulkin-M.)

Given any group G, with a presentation $G = \langle S \mid R \rangle$, we have:

$$A_1(X_{\langle S|R \rangle}, *) \cong G$$

Graphs with a prescribed fundamental group(oid)

Corollary (Approximation property)

Given any groupoid $\mathscr{G}$, graph Y, and functor $F\colon \mathscr{G} \to \Pi_1 Y$, there exists a graph X, graph map $f\colon X \to Y$, and a zigzag of equivalences of groupoids $\mathscr{G} \xleftarrow{\sim} \mathscr{G}' \xrightarrow{\sim} \Pi_1 X$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathscr{G}' & \xrightarrow{\sim} & \Pi_1 X \\ {\scriptstyle\sim}\downarrow & & \downarrow{\scriptstyle \Pi_1 f} \\ \mathscr{G} & \xrightarrow{F} & \Pi_1 Y \end{array}$$

Goals

A map $f\colon X \to Y$ is a weak 1-equivalence if it induces a bijection $\pi_0 X \to \pi_0 Y$ and an isomorphism $A_1(X, x) \to A_1(Y, fx)$ for all $x \in X$.

Let $W_{\leqslant 1}$ be the class of weak 1-equivalences in Graph.

Do we have $(\mathrm{Graph}, W_{\leqslant 1}) \simeq_H \mathrm{Gpd}$?

More generally, do we have $(\mathrm{Graph}, W_{\leqslant n}) \simeq_H n\text{-}\mathrm{Gpd}$?

Approach: use fibration categories

Theorem (Cisinski)

An exact functor $F\colon \mathscr{C} \to \mathscr{D}$ between fibration categories induces an equivalence $\operatorname{Ho}\mathscr{C} \to \operatorname{Ho}\mathscr{D}$ if and only if it satisfies the following approximation properties:

1. *F reflects weak equivalences*
2. *Given any morphism $f\colon B \to FY$ in $\mathscr{D}$, there exists a morphism $\hat{f}\colon X \to Y$ in $\mathscr{C}$ such that there is a commutative square of the form:*

$$\begin{array}{ccc} A & \xrightarrow{\sim} & FX \\ {\scriptstyle\sim}\downarrow & & \downarrow{\scriptstyle \hat{f}} \\ B & \xrightarrow{f} & FY \end{array}$$

where $A \xrightarrow{\sim} B$ and $A \xrightarrow{\sim} FX$ are weak equivalences in $\mathscr{D}$.