A characterization of regular and exact completions of existential completions

Davide Trotta
j.w.w. M.E. Maietti

University of Pisa

Regular and Exact completions in category theory

The process of completing a category with finite limits to a regular category or exact category has been well-studied in categorical logic, raising to the so-called notion of $\mathsf{reg}/\mathsf{lex}$- and ex/lex- completions.

Theorem

A regular category $\mathcal{A}$ is the regular completion (necessarily) of the full subcategory $\mathcal{P}_{\mathcal{A}}$ of its regular projectives if and only if $\mathcal{P}_{\mathcal{A}}$ is closed under finite limits in $\mathcal{A}$, $\mathcal{A}$ has enough projectives, and every object of $\mathcal{A}$ can be embedded in a regular projective.

Theorem

An exact category $\mathcal{A}$ is the exact completion (necessarily) of the full subcategory $\mathcal{P}_{\mathcal{A}}$ of its regular projectives if and only if $\mathcal{P}_{\mathcal{A}}$ is closed under finite limits in $\mathcal{A}$ and $\mathcal{A}$ has enough projectives.

A. Carboni (1995), *Some free constructions in realizability and proof theory*, J. Pure Appl.

Introduction: doctrines

Definition (primary doctrine)

A **primary doctrine** is a functor $P\colon \mathcal{C}^{op} \longrightarrow \mathbf{InfSL}$ from the opposite of a category $\mathcal{C}$ with finite products to the category of inf-semilattices.

Definition (existential doctrine)

A primary doctrine $P\colon \mathcal{C}^{op} \longrightarrow \mathbf{InfSL}$ is **existential** if, for every object A and B in $\mathcal{C}$ for any product projection $\pi\colon A \to B$, the functor

$$P_\pi\colon P(B) \to P(A)$$

has a left adjoint $\exists_{\pi_i}$, and these satisfy: **Beck-Chevalley condition** and **Frobenius reciprocity**.

Introduction: doctrines

Definition (elementary doctrine)

A primary doctrine $P\colon \mathcal{C}^{op} \longrightarrow \mathbf{InfSL}$ is **elementary** if for every A in $\mathcal{C}$ there exists an object δ_A in $P(A \times A)$ such that

- the assignment
$$\exists_{\langle \mathrm{id}_A, \mathrm{id}_A \rangle}(\alpha) := P_{\pi_1}(\alpha) \wedge \delta_A$$
for an element α of $P(A)$ determines a left adjoint to $P_{\langle \mathrm{id}_A, \mathrm{id}_A \rangle}\colon P(A \times A) \to PA$;
- for every morphism e of the form $\langle \pi_1, \pi_2, \pi_2 \rangle\colon X \times A \to X \times A \times A$ in $\mathcal{C}$, the assignment
$$\exists_e(\alpha) := P_{\langle \pi_1, \pi_2 \rangle}(\alpha) \wedge P_{\langle \pi_2, \pi_2 \rangle}(\delta_A)$$
for α in $P(X \times A)$ determines a left adjoint to $P_e\colon P(X \times A \times A) \to P(X \times A)$.

Some examples

Example

Let $\mathcal{L}_{=,\exists}$ be the $(\top, \wedge, =, \exists)$-fragment of first-order Intuitionistic Logic (also called **regular logic**), and let $\mathbb{T}$ be a theory in such a fragment. Then the syntactic doctrine

$$\mathrm{LT}^{\mathbb{T}}_{=,\exists} \colon \mathcal{V}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$$

where $\mathcal{V}$ is the category of contexts and substitutions and $\mathrm{LT}^{\mathbb{T}}_{=,\exists}(\Gamma)$ is given by the Lindenbaum-Tarski algebra of well-formed formulas of $\mathcal{L}_{=,\exists}$ with free variables in Γ, is elementary and existential.

Example (weak subobjects doctrine)

Consider a category $\mathcal{D}$ with finite limits: the doctrine is given by the functor of weak subobjects (or variations)

$$\Psi_{\mathcal{D}} \colon \mathcal{D}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$$

where $\Psi_{\mathcal{D}}(A)$ is the poset reflection of the slice category $\mathcal{D}/A$. This doctrine is elementary and existential doctrine.

Example (subobjects doctrine)

Let $\mathcal{C}$ be a category with finite limits. The functor

$$\mathrm{Sub}_{\mathcal{C}} \colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$$

assigns to an object A in $\mathcal{C}$ the poset $\mathrm{Sub}_{\mathcal{C}}(A)$ of subobjects of A in $\mathcal{C}$. This is an elementary and existential doctrine if and only if the category $\mathcal{C}$ is regular.

Doctrines with Hilbert's ϵ-operators

Definition

Let $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ be an elementary, existential doctrine. An object B of $\mathcal{C}$ is equipped with Hilbert's ϵ-**operator** if, for any object A in $\mathcal{C}$ and any α in $P(A \times B)$ there exists an arrow $\epsilon_\alpha\colon A \to B$ such that

$$\exists_{\pi_1}(\alpha) = P_{\langle \mathrm{id}_A, \epsilon_\alpha \rangle}(\alpha)$$

holds in $P(A)$, where $\pi_1\colon A \times B \to A$ is the first projection.

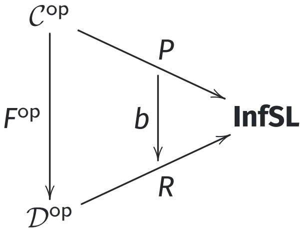
Definition

We say that an elementary, existential doctrine $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ is **equipped with** Hilbert's ϵ-**operators** if every object in $\mathcal{C}$ is equipped with ϵ-operator.

The 2-category of primary doctrines

Primary doctrines form a 2-category **PD** where:

- a 1-cell is a pair (F, b)



  such that $F\colon \mathcal{C} \longrightarrow \mathcal{D}$ is a functor preserving products, and $b\colon P \longrightarrow R \circ F^{\mathrm{op}}$ is a natural transformation.

- a 2-cell is a natural transformation $\theta\colon F \longrightarrow G$ such that for every A in $\mathcal{C}$ and every α in PA, we have

$$b_A(\alpha) \leq R_{\theta_A}(c_A(\alpha)).$$

Some subcategories of PD

We denote by:

- **ExD** the 2-full subcategory of **PD** whose objects are existential doctrines, and whose 1-cells are those 1-cells of **PD** which preserve the existential structure;
- **EED** the 2-full subcategory of **PD** whose objects are elementary and existential doctrines, and whose 1-cells preserve both the existential and the elementary structure.

Existential free elements

Definition (existential splitting)

Let $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ be an existential doctrine. An object α of the fibre $P(A)$ is said to be an **existential splitting** if for every projection $\pi_A\colon A \times B \to A$ and for every element β of the fibre $P(A \times B)$, whenever $\alpha = \exists_{\pi_A}(\beta)$ holds then there exists an arrow $h\colon A \to B$ such that $\alpha = P_{\langle \mathrm{id}_A, h\rangle}(\beta)$.

Definition (existential free elements)

Let $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ be an existential doctrine. An object α of the fibre $P(A)$ is said to be **existential free** if for every morphism $f\colon B \to A$, $P_f(\alpha)$ is an existential splitting.

M.E. Maietti and D. Trotta (2023), *A characterization of generalized existential completions*, Ann. Pure Appl. Log.

Doctrines with enough existential free objects

Definition

Given an existential doctrine $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$, we say that an element α of the fibre $P(A)$ is **covered** by an element $\beta \in P(A \times B)$ if $\alpha = \exists_{\pi_A}(\beta)$.

Definition

We say that an existential doctrine $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ has **enough existential free objects** if for every object A of $\mathcal{C}$, any element $\alpha \in P(A)$ is covered by some existential free element $\beta \in P(A \times B)$ for some object B of $\mathcal{C}$, namely β is an existential free element and

$$\alpha = \exists_{\pi_A}(\beta).$$

Existential cover

Definition

Let $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ be an existential doctrine equipped with a full subdoctrine $P'\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$. We say that P' is an **existential cover** of P if for any object A, every element α' of $P'(A)$ is existential splitting for P (and hence existential free) and every element α of $P(A)$ is covered by an element of P'.

Lemma

Let $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ be an existential doctrine, and let $P'\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ be a full subdoctrine of P. If P′ is an existential cover of P, then the existential free elements of P coincides exactly with the elements of P′. Hence if an existential cover exists, it is unique.

M.E. Maietti and D. Trotta (2023), *A characterization of regular and exact completions of pure existential completions*, arXiv

Theorem (characterization of existential completions)

Let $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ *be an existential doctrine. Then the following are equivalent:*

1. *P has the universal property of being the existential completion $(P')^{\exists}$ of a primary doctrine P';*
2. *P satisfies the following points:*
   - 2.1 *P satisfies the rule of choice RC, i.e. every top element $\top_A$ is existential free;*
   - 2.2 *for every existential free object α and β of $P(A)$, then $\alpha \wedge \beta$ is an existential free.*
   - 2.3 *P has enough existential free elements;*
3. *P has a (unique) existential cover.*

Corollary

Every elementary existential doctrine is equipped with Hilbert's ϵ-operators if and only if it provides an existential cover of itself.

Regular completion

Definition

Let $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ be an elementary, existential doctrine. The **regular completion** of P is the category $\mathrm{Reg}(P)$ given by the following data:

- objects are pairs (A, α) where A is an object of $\mathcal{C}$ and $\alpha \in P(A)$;
- an arrow from (A, α) to (B, β) is given by an element ϕ of $P(A \times B)$ such that:
  1. $\phi \leq P_{\pi_1}(\alpha) \wedge P_{\pi_2}(\beta)$;
  2. $\alpha \leq \exists_{\pi_1}(\phi)$;
  3. $P_{\langle \pi_1, \pi_2 \rangle}(\phi) \wedge P_{\langle \pi_1, \pi_3 \rangle}(\phi) \leq P_{\langle \pi_2, \pi_3 \rangle}(\delta_B)$.

Theorem

The category $\mathrm{Reg}(P)$ *is regular, and the assignment* $P \mapsto \mathrm{Reg}(P)$ *extends to a 2-functor*

$$\mathrm{Reg}(-)\colon \mathbf{EED} \to \mathbf{Reg}$$

which is left biadjoint to the inclusion $\mathbf{Reg} \to \mathbf{EED}$.

Examples

Example

The regular completion $\mathrm{Reg}(\Psi_{\mathcal{D}})$ of the weak subobjects doctrine $\Psi_{\mathcal{D}}\colon \mathcal{D}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ coincides with the regular completion $(\mathcal{D})_{\mathrm{reg/lex}}$ of the lex category $\mathcal{D}$.

Example

The regular completion $\mathrm{Reg}(\mathrm{LT}^{\mathbb{T}}_{=,\exists})$ of the doctrine $\mathrm{LT}^{\mathbb{T}}_{=,\exists}\colon \mathcal{V}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ provides exactly the syntactic category denoted $\mathcal{C}^{\mathrm{reg}}_{\mathbb{T}}$ associated with the theory of the regular fragment of first-order logic in the book *Sketches of an elephant*.

Existential splittings provide regular projectives

Theorem

Let $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ *be an elementary and existential doctrine. Then:*

- *every object* (A, α) *where* α *is existential splitting is regular projective in* $\mathrm{Reg}(P)$*;*
- *if* P *has enough existential free objects, every object of* $\mathrm{Reg}(P)$ *is covered by a regular projective object.*

M.E. Maietti and D. Trotta (2023), *A characterization of regular and exact completions of pure existential completions*, arXiv

Category of predicates

Definition

Given an elementary doctrine $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow$ **InfSL** , its **category of predicates** Prd_P is defined as follows:

- an object of Prd_P is a pair (A, α) where A is an object of $\mathcal{C}$ and $\alpha \in P(A)$;
- an arrow $[f]_\sim\colon (A, \alpha) \to (B, \beta)$ is an equivalence class of arrows $f\colon A \to B$ of $\mathcal{C}$ satisfying $\alpha \leq P_f(\beta)$. The equivalence relation $f \sim g$ is given by $\top_A \leq P_{\langle f,g\rangle}(\delta_B)$.

Definition (graph functor)

Given an elementary, existential doctrine P and an elementary subdoctrine P' we can define an embedding, called **graph functor** $G_{|_{P'}}\colon \mathrm{Prd}_{P'} \to \mathrm{Reg}(P)$ by mapping (A, α) of $\mathrm{Prd}_{P'}$ into (A, α) of $\mathrm{Reg}(P)$ and an arrow $[f]\colon (A, \alpha) \to (B, \beta)$ of $\mathrm{Prd}_{P'}$ into the arrow $G_{|_{P'}}([f]) = P_{f\times \mathrm{id}_B}(\delta_B) \wedge \ (P_{\pi_1}(\alpha) \ \wedge \ P_{\pi_2}(\beta)\ )$ from (A, α) to (B, β) of $\mathrm{Reg}(P)$.

Characterization of the regular completion

Theorem

Let $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ *be an elementary and existential doctrine. Then* P *has an existential cover* P' *if and only if the functor* $G^{\mathrm{reg}}_{|_{P'}}\colon (\mathrm{Prd}_{P'})_{\mathrm{reg/lex}} \to \mathrm{Reg}(P)$ *provides an equivalence* $\mathrm{Reg}(P) \equiv (\mathrm{Prd}_{P'})_{\mathrm{reg/lex}}$.

M.E. Maietti and D. Trotta (2023), *A characterization of regular and exact completions of pure existential completions*, arXiv

Exact completion

Definition (exact completion)

Let P be an elementary, existential doctrine. We call the category $\mathrm{Ex}(P) := (\mathrm{Reg}(P))_{\mathrm{ex/reg}}$ the **exact completion** of P.

Theorem (exact completion of an elementary, existential doctrine)

Let $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ *be an elementary and existential doctrine. Then the category* $\mathrm{Ex}(P)$ *is exact, and the assignment* $P \mapsto \mathrm{Ex}(P)$ *extends to a 2-functor*

$$\mathrm{Ex}(-)\colon\ \mathbf{EED} \to \mathbf{ExCat}$$

which is left biadjoint to the inclusion of the 2-category **ExCat** *of exact categories in the 2-category* **EED** *of elementary and existential doctrines.*

M.E. Maietti and G. Rosolini (2013), *Unifying exact completions*, Appl. Categ. Structures.

Example

The exact completion $\mathrm{Ex}(\Psi_{\mathcal{D}})$ of the weak subobjects doctrine $\Psi_{\mathcal{D}}\colon \mathcal{D}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ coincides with the ex completion $(\mathcal{D})_{\mathrm{ex/lex}}$ of the lex category $\mathcal{D}$.

Example

The exact completion $\mathrm{Ex}(\mathrm{LT}^{\mathbb{T}}_{=,\exists}) = (\mathrm{Reg}(\mathrm{LT}^{\mathbb{T}}_{=,\exists}))_{\mathrm{ex/reg}}$ of the syntactic doctrine $\mathrm{LT}^{\mathbb{T}}_{=,\exists}\colon \mathcal{V}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ provides the exact category called the *effectivization* the syntactic category $\mathcal{C}^{\mathrm{reg}}_{\mathbb{T}}$, also denoted by $\mathcal{E}_{\mathbb{T}} := \mathbf{Eff}(\mathcal{C}^{\mathrm{reg}}_{\mathbb{T}})$.

Example

The exact completion of a subobjects doctrine $\mathrm{Sub}_{\mathcal{C}}\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ coincides with the well known construction of the exact on regular category $(\mathcal{C})_{\mathrm{ex/reg}}$.

Exact completions of doctrines as ex / lex-completions of Prd

Theorem

Let $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ *be an elementary and existential doctrine. Then P has an existential cover P′ if and only if the functor* $G^{\mathrm{ex}}_{|P'} : (\mathrm{Prd}_{P'})_{\mathrm{ex/lex}} \to \mathrm{Ex}(P)$ *provides an equivalence* $\mathrm{Ex}(P) \equiv (\mathrm{Prd}_{P'})_{\mathrm{ex/lex}}$.

M.E. Maietti and D. Trotta (2023), *A characterization of regular and exact completions of pure existential completions*, arXiv

Applications

Corollary

Let $P\colon \mathcal{C}^{op} \longrightarrow \mathbf{InfSL}$ *be an elementary and existential doctrine. Then the following are equivalent:*

- P *is equipped with Hilbert's* ϵ*-operators;*
- *the functor* $G^{reg}\colon (\mathrm{Prd}_P)_{reg/lex} \to \mathrm{Reg}(P)$ *provides an equivalence* $\mathrm{Reg}(P) \equiv (\mathrm{Prd}_P)_{reg/lex}$*;*
- *the functor* $G^{ex}\colon (\mathrm{Prd}_P)_{ex/lex} \to \mathrm{Ex}(P)$ *provides an equivalence* $\mathrm{Ex}(P) \equiv (\mathrm{Prd}_P)_{ex/lex}$*.*

Corollary

Every Joyal's arithmetic universe on a Skolem theory S *is equivalent to the exact completion* $\mathrm{Ex}(\mathsf{R}^{\exists})$ *of the existential completion* $\mathsf{R}^{\exists}\colon \mathsf{S}^{op} \longrightarrow \mathbf{InfSL}$ *of the elementary doctrine* $\mathsf{R}\colon \mathsf{S}^{op} \longrightarrow \mathbf{InfSL}$ *of* S*-primitive recursive predicates.*

Applications

Corollary

Let $\mathbb{T}_0$ *be a regular theory given by the fragment* $\mathcal{L}_{=,\exists}$ *of first-order Intuitionistic Logic and no extra-logical axioms on a generic signature. Let* H_0 *be the Horn theory given by the corresponding fragment* $\mathcal{L}_{=}$ *with no extra-logical axioms on the same signature.*
The syntactic category $\mathcal{C}^{\mathsf{reg}}_{\mathbb{T}_0}$ *of* $\mathbb{T}_0$ *is equivalent to the* reg / lex*-completion* $(\mathrm{Prd}_{\mathrm{LT}^{\mathsf{H}_0}_{=}})_{\mathsf{reg/lex}}$ *of the category of predicates of the syntactic doctrine* $\mathrm{LT}^{\mathsf{H}_0}_{=}$ *of* H_0*.*
Hence, also its effectivization $\mathcal{E}_{\mathbb{T}_0}$ *is the* ex / lex*-completion* $(\mathrm{Prd}_{\mathrm{LT}^{\mathsf{H}_0}_{=}})_{\mathsf{ex/lex}}$ *of the category of predicates of* $\mathrm{LT}^{\mathsf{H}_0}_{=}$*.*

M.E. Maietti and D. Trotta (2023), *A characterization of regular and exact completions of pure existential completions*, arXiv

Main references

- M.E. Maietti (2010), Joyal's arithmetic universe as list-arithmetic pretopos, *Theory Appl. Categ.*
- M.E. Maietti and G. Rosolini (2013), Quotient completion for the foundation of constructive mathematics, *Log. Univers.*
- M.E. Maietti and G. Rosolini (2013), Unifying exact completions, *Appl. Categ. Structures.*
- M.E. Maietti, F. Pasquali, and G. Rosolini (2017), Triposes, exact completions, and Hilbert's ϵ-operator, *Tbil. Math. J.*
- J. Frey (2015), Triposes, q-toposes and toposes, *Ann. Pure Appl. Log.*
- M.E. Maietti and D. Trotta (2023), A characterization of generalized existential completions, *Ann. Pure Appl. Log.*
- M.E. Maietti and D. Trotta (2023), A characterization of regular and exact completions of pure existential completions, *arXiv*
- D. Trotta (2020), The existential completion, *Theory App. Categ.*
- D. Trotta, M. Spadetto, and V. de Paiva, Dialectica principles via Gödel doctrines, *Theoret. Comput. Sci.*