Sheaves on rings and Čech Cohomology

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Joint work with Hugo Mariano and Peter Arndt

Support: CAPES

Sheaves

Sheaves on topological spaces

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- A **presheaf** with values in *Set* is a functor $F : \mathcal{O}(X)^{op} \to Set$
- A presheaf F is a **sheaf** if for each open covering $U = \bigcup_{i \in I} U_i$, of an open set U of X, we have:

$$F(U) \xrightarrow{e} \prod_{i \in I} F(U_i) \overset{p}{\underset{q}{\rightrightarrows}} \prod_{(i,j) \in I \times I} F(U_i \cap U_j)$$

$$e(t) = \{t_{|_{U_i}} \mid i \in I\}, \quad p((t_k)_{k \in I}) = (t_{i_{|_{U_i \cap U_j}}})_{(i,j) \in I \times I}$$
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X and Y topological spaces.

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If $U = \bigcup_{i \in I} U_i$ and $f_i : U_i \to Y$ are continuous functions such that $f_{i_{|_{U_i \cap U_j}}} = f_{j_{|_{U_i \cap U_j}}}, \forall i, j \in I$ then there is a unique $f : U \to Y$ continuous such that $f_i = f_{|_{U_i}}, \forall i \in I$.

Structure sheaf

We have that $Spec(R) = \{\text{all prime ideals of } R\}$ is a topological space under the Zariski topology.

If $f \in R$, $D(f) = \{\text{prime ideals not contaning } f\}$ forms a open basis.

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$$\mathcal{O}_{SpecA}(D(f)) \longrightarrow \prod_{i \in I} \mathcal{O}_{SpecA}(D(f_i)) \rightrightarrows \prod_{(i,j) \in I \times I} \mathcal{O}_{SpecA}(D(f_i) \cap D(f_j))$$

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Since $D(f_i) \cap D(f_j) = D(f_i.f_j)$,

$$R[f^{-1}] \longrightarrow \prod_{i \in I} R[f^{-1}] \rightrightarrows \prod_{(i,j) \in I \times I} R[(f_i.f_j)^{-1}]$$

Structure Sheaf

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$$L_R(fR) \longrightarrow \prod_{i \in I} L_R(f_iR) \rightrightarrows \prod_{(i,j) \in I \times I} L_R(f_iR.f_jR)$$

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- A presheaf F is a **sheaf** if for each covering $\mathfrak{a} = \sum_{i \in I} \mathfrak{a}_i$, of an ideal $\mathfrak{a}$ of R, we have:

$$F(\mathfrak{a}) \xrightarrow{e} \prod_{i \in I} F(\mathfrak{a}_i) \overset{p}{\underset{q}{\rightrightarrows}} \prod_{(i,j) \in I \times I} F(\mathfrak{a}_i . \mathfrak{a}_j)$$

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Use quantales!

A **quantale** $(Q, \leq, \odot, 1)$ is a complete lattice $(Q, \leq)$ with a monoid $(Q, \odot, 1)$ such that

$$a \odot \left(\bigvee_{i \in I} b_i\right) = \bigvee_{i \in I}(a \odot b_i) \text{ and } \left(\bigvee_{i \in I} b_i\right) \odot a = \bigvee_{i \in I}(b_i \odot a)$$

If $\odot = \wedge$, then Q is a locale.

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Examples

- $(\mathcal{O}(X), \subseteq, \cap, X)$;
- $(\mathcal{I}(R), \subseteq, ., R)$;
- $([0, \infty], \geq, +, 0)$.

A presheaf $F : (Q, \leq, \odot, 1)^{op} \to Set$ is a **sheaf** if for each covering $b = \bigvee_{i \in I} b_i$, of an element b of Q, we have:

$$F(b) \longrightarrow \prod_{i \in I} F(b_i) \rightrightarrows \prod_{(i,j) \in I \times I} F(b_i \odot b_j)$$

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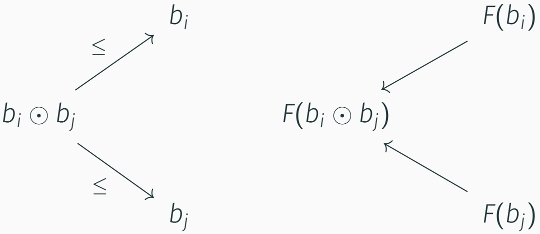

Quantale with extra properties

1. **commutative** when $(Q, \odot, 1)$ is a commutative monoid;
2. **idempotent** when $a \odot a = a$, for $a \in Q$;
3. **semicartesian** when $a \odot b \leq a, b$, for $a, b \in Q$, or, equivalently, $1 = \top$.

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- Closed right (or left) ideals of a C^*-algebra, where $\odot$ is the closure of the product of ideals.

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The inclusion $i : Sh(Q) \to PSh(Q)$ has a left adjoint $a : PSh(Q) \to Sh(Q)$ that we call sheafification.

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Warning 2: This sheafification preserves the monoidal structure in $PSh(Q)$ given by Day convolution, but it does not preserves all finite limits!

A geometric morphism is a map $f : Q' \to Q$ such that

1. f preserves arbitrary sups and 1;
2. f weakly preserves the multiplication, i.e.,
   $f(p) \odot f(q) \leq f(p \odot' q), \forall p, q \in Q$.

A **strong geometric morphism of quantales** is a geometric morphism of quantales where f preserves the multiplication.

Examples

1. The inclusion $Idem(Q) \to Q$.
2. Every surjective homomorphism $g : R \to S$ of commutative and unital rings induces a strong geometric morphism where $f : \mathcal{I}(R) \to \mathcal{I}(S)$ given by $f(J) = g(J)$.

Proposition

If $f : Q' \to Q$ is a geometric morphism and $F : Q^{op} \to Set$ is a sheaf, then $F \circ f : Q' \to Set$ is a sheaf.

Take $u = \bigvee_{i \in I} u_i$ a cover in Q'.

Then $f(u) = f(\bigvee_{i \in I} u_i) = \bigvee_{i \in I} f(u_i)$ is a cover in Q.

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Creating sheaves

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Since $f(u) \odot f(v) \leq f(u \odot' v), \forall u, v \in Q$

$$\begin{array}{ccccc} F \circ f(u) & \longrightarrow & \prod_i F \circ f(u_i) & \rightrightarrows & \prod_{i,j} F \circ f(u_i \odot' u_j) \\ & & & \searrow\!\!\searrow & \downarrow \\ & & & & \prod_{i,j} F(f(u_i) \odot f(u_j)) \end{array}$$

Example

Let X be a topological space that admits partition of unity subordinate to a cover, and $C(X)$ the ring of real-valued continuous functions on X. Define

$$\begin{aligned} \tau\colon \mathcal{I}(C(X)) &\to \mathcal{O}(X) \\ I &\mapsto \bigcup_{g \in I} g^{-1}(\mathbb{R} - \{0\}) \end{aligned}$$

$$\begin{aligned} \theta\colon \mathcal{O}(X) &\to \mathcal{I}(C(X)) \\ U &\mapsto \langle \{ f : f\restriction_{X-U} \equiv 0 \} \rangle \end{aligned}$$

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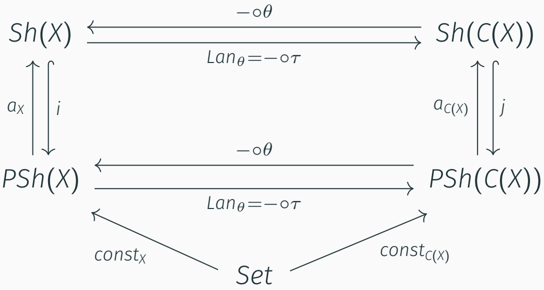
The functor τ is left adjoint to θ, τ is a strong geometric morphism and θ if a geometric morphism. Moreover, the constant sheaf K_X on $\mathcal{O}(X)$ induces a sheaf $K_X \circ \tau$ on $\mathcal{I}(C(X))$

Example

Since the following diagram commutes (up to natural isomorphism)



So $\underline{K}^a_X \circ \tau \cong \underline{K}^a_{C(X)}$.

Čech cohomology

Monoidal Čech Cohomology

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$$C^q(\mathcal{U}, F) = \prod_{i_0, \dots, i_q} F(U_{i_0} \odot \dots \odot U_{i_q}), \forall q \geq 0,$$

and its coboundary morphisms $d^q : C^q(\mathcal{U}, F) \to C^{q+1}(\mathcal{U}, F)$ are

$$(d^q \alpha) = \sum_{k=0}^{q+1} (-1)^k \alpha(\delta_k)\Big|_{(U_{i_0} \odot \dots \odot U_{i_q})}$$

where δ_k is used to indicate that we are removing i_k, and the restriction is to guarantee that $d^q \alpha \in C^{q+1}(\mathcal{U}, F)$.

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where δ_k is used to indicate that we are removing i_k, and the restriction is to guarantee that $d^q\alpha \in C^{q+1}(\mathcal{U}, F)$. The **monoidal Čech cohomology group** of $\mathcal{U}$ with coefficients in F is

$$\check{H}^q(\mathcal{U}, F) = \frac{Ker d^q}{Im d^{q-1}}.$$

Relating Čech cohomology groups

Theorem

Consider a strong geometric morphism $f : Q' \to Q$. Then $\check{\mathrm{H}}^q(\mathcal{U}', F \circ f) = \check{\mathrm{H}}^q(f(\mathcal{U}'), F)$.

Consider a covering $\{u'_i\}_{i \in I} = \mathcal{U}'$ in Q'. Then

$$\begin{aligned} C^q(\mathcal{U}', F \circ f) &= \prod_{i_0 < \ldots < i_q} F \circ f(u'_{i_0} \odot' \ldots \odot' u'_{i_q}) \\ &= \prod_{i_0 < \ldots < i_q} F(f(u'_{i_0}) \odot \ldots \odot f(u'_{i_q})) \\ &= C^q(f(\mathcal{U}'), F). \end{aligned}$$

So the following diagram commutes

$$\begin{array}{ccccccccc} \dots & \longrightarrow & C^{q-1}(\mathcal{U}', F \circ f) & \xrightarrow{d^{q-1}} & C^{q}(\mathcal{U}', F \circ f) & \xrightarrow{d^{q}} & C^{q+1}(\mathcal{U}', F \circ f) & \longrightarrow & \dots \\ & & \downarrow{id} & & \downarrow{id} & & \downarrow{id} & & \\ \dots & \longrightarrow & C^{q-1}(f(\mathcal{U}'), F) & \xrightarrow[d^{q-1}]{} & C^{q}(f(\mathcal{U}'), F) & \xrightarrow[d^{q}]{} & C^{q+1}(f(\mathcal{U}'), F) & \longrightarrow & \dots \end{array}$$

Then $\check{\mathrm{H}}^q(\mathcal{U}', F \circ f) = \check{\mathrm{H}}^q(f(\mathcal{U}'), F)$.

The **Čech cohomology group of an element** $u \in Q$ with coefficient in a sheaf F is the directed (co)limit

$$\check{\mathrm{H}}^q(u, F) := \varinjlim_{\mathcal{U} \in \mathrm{Cov}(u)} \check{\mathrm{H}}^q(\mathcal{U}, F).$$

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Theorem

Consider a strong geometric morphism $Q \underset{f_*}{\overset{f}{\leftrightarrows}} Q'$ such that f_* preserves unity and arbitrary joins. Then $\check{\mathrm{H}}^q(1', F \circ f) \cong \check{\mathrm{H}}^q(1, F)$.

Proof

We have

$$\check{\mathrm{H}}^q(1', F \circ f) := \varinjlim_{\mathcal{U}' \in \mathrm{Cov}(1')} \check{\mathrm{H}}^q(\mathcal{U}', F \circ f)$$
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Proof

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Since that $f(f_*(\mathcal{U}))$ is a refinement of $\mathcal{U}$, for every covering $\mathcal{U}$ of 1 there is a covering $\mathcal{U}'$ of $1'$ such that $f(\mathcal{U}') \subseteq \mathcal{U}$. Then

$$\check{\mathrm{H}}^q(1, F) := \varinjlim_{\mathcal{U} \in \mathrm{Cov}(1)} \check{\mathrm{H}}^q(\mathcal{U}, F)$$
$$\cong \varinjlim_{\mathcal{U}' \in \mathrm{Cov}(1')} \check{\mathrm{H}}^q(f(\mathcal{U}'), F)$$

Application

Since $\tau : \mathcal{I}(C(X)) \to \mathcal{O}(X)$ is a strong geometric morphism and $\theta : \mathcal{O}(X) \to \mathcal{I}(C(X))$ is a geometric morphism, then

$$\check{\mathrm{H}}^q(C(X), \underline{K}^a_{C(X)}) \cong \check{\mathrm{H}}^q(C(X), \underline{K}^a_X \circ \tau) \cong \check{\mathrm{H}}^q(X, \underline{K}^a_X)$$

Reminder

- If X is a smooth manifold of dimension n then there is an isomorphism $H^q_{dR}(X) \cong \check{H}^q(X, \mathbb{R})$, for all $q \leq n$;
- The dimension of $H^0_{dR}(X)$ corresponds to the number of connected components of X;
- X is connected if and only if $C(X)$ only had trivial idempotent elements.

Conjecture: The Čech cohomology group in degree zero of a ring R with coefficients in a constant sheaf is related to the number of idempotent elements of R.

Related work

More about $Sh(Q)$

In $Sh(L)$, the subobject classifier is $\Omega(u) = \{q \in L \,:\, q \leq u\}$ such that for all $v \leq u$

$$\begin{aligned} \Omega(u) &\to \Omega(v) \\ q &\mapsto q \odot v \end{aligned}$$

More about $Sh(Q)$

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$$\begin{aligned} \Omega(u) &\to \Omega(v) \\ q &\mapsto q \odot v \end{aligned}$$

Suppose $\bigvee_{i \in I} q_i^- = (\bigvee_{i \in I} q_i)^-$, for each $\{q_i : i \in I\} \subseteq Q$.

$$q^- := \bigvee \{p \in Idem(Q) : p \leq q \odot p\}$$

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$$\Omega^-(u) = \{q \in Q \,:\, q \odot u^- = q\}$$

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If Q is ”good enough”

$$q^+ := \bigwedge \{p \in Q : q \leq q \odot p\}$$

Define

$$\Omega^+(u) := \{q \in Q : q^+ \odot u = q\}$$

$$\begin{aligned} \Omega^+(u) &\to \Omega^+(v) \\ q &\mapsto q^+ \odot v \end{aligned}$$

Conclusions

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- We extended Čech cohomology, showing the potential of the theory to relate algebraic and geometric properties;

- We have mentioned logical aspects of $Sh(Q)$;

- We developed an extension of sheaves for Grothendieck pretopologies that encompasses our sheaves on quantales.

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Thank you!