An introduction to pretorsion theories and their stable categories

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Outline

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From torsion theories to pretorsion theories

Internal preorders

The stable category

The lextensive context

Two examples of pretorsion theories in Cat

The abelian context

Definition [S.C. Dickson, 1966]

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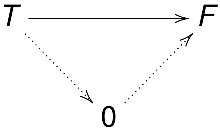
- for any $T \in \mathcal{T}$ and $F \in \mathcal{F}$ the only morphism from T to F is



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$$\begin{array}{ccc} T & \longrightarrow & F \\ & \searrow \quad \nearrow & \\ & 0 & \end{array}$$

- for any $C \in \mathbb{C}$ there is a short exact sequence

$$0 \longrightarrow T(C) \longrightarrow C \longrightarrow F(C) \longrightarrow 0$$

  with $T(C) \in \mathcal{T}$ and $F(C) \in \mathcal{F}$.

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The terminology comes from the example $(\mathcal{T}, \mathcal{F}) = (\text{Ab}_{t.}, \text{Ab}_{t.f.})$ in the category $\mathbb{C} = \text{Ab}$ of abelian groups, where

$$\mathcal{T} = \text{Ab}_{t.} \text{ is the category of torsion abelian groups}$$

and

$$\mathcal{F} = \text{Ab}_{t.f.} \text{ the category of torsion-free abelian groups.}$$

For any $A \in \text{Ab}$ one has the exact sequence

$$0 \longrightarrow T(A) \longrightarrow A \longrightarrow A/T(A) \longrightarrow 0,$$

where $T(A) = \{a \in A \mid \exists n \in \mathbb{N}_0, na = 0\}$.

The pointed case

Torsion theories have been considered in many “non-additive” pointed contexts :

- Cassidy-Hébert-Kelly, J. Austr. Math Soc. (1987)
- Bourn-Gran, J. Algebra (2006)
- Clementino-Dikranjan-Tholen, J. Algebra (2006)
- Janelidze-Tholen, Contemp. Mathem. (2007)
- Rosický-Tholen, J. Homotopy Rel. Struct. (2007)
- Clementino-Gutierres, Cah. Top. Géom. Diff. Catég. (2010)
- Tholen, Topology Appl. (2011)
- Everaert-Gran, Bull. Sciences Mathém. (2013)
- Gran-Lack, J. Algebra (2016)
- Gran-Kadjo-Vercruysse, Appl. Categ. Struct. (2016)
- Duckerts-Antoine, Adv. Math. (2017)
- Lopez Cafaggi, Cah. Top. Géom. Diff. Catég. (2022)

If $\mathbb{C}$ is a pointed category, with zero object 0, a torsion theory $(\mathcal{T}, \mathcal{F})$ in $\mathbb{C}$ can still be defined as in the abelian case :

- for any $T \in \mathcal{T}$ and $F \in \mathcal{F}$ the only morphism from T to F is

$$\begin{array}{ccc} T & \longrightarrow & F \\ & \searrow \quad \nearrow & \\ & 0 & \end{array}$$

- $\forall C \in \mathbb{C}$ there is a short exact sequence

$$0 \longrightarrow T(C) \longrightarrow C \longrightarrow F(C) \longrightarrow 0$$

with $T(C) \in \mathcal{T}$ and $F(C) \in \mathcal{F}$.

Definition (G. Janelidze, L. Márki, W. Tholen, 2002)

A finitely complete category $\mathbb{C}$ is semi-abelian if

- $\mathbb{C}$ has a 0 object
- $\mathbb{C}$ has $A + B$
- $\mathbb{C}$ is (Barr) exact
- $\mathbb{C}$ is (Bourn) protomodular : given a commutative diagram

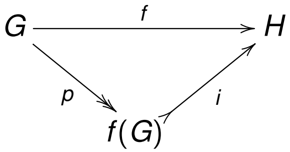
$$
\begin{array}{ccccccc}
0 & \longrightarrow & K & \xrightarrow{\;k\;} & A & \underset{f}{\overset{\longleftarrow}{\longrightarrow}} & B \\
 & & {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle w} \\
0 & \longrightarrow & K' & \xrightarrow[\;k'\;]{} & A' & \underset{f'}{\overset{\longleftarrow}{\longrightarrow}} & B'
\end{array}
$$

u, w isomorphisms $\Rightarrow$ v isomorphism.

Example

The category Grp is semi-abelian :

- every homomorphism f in Grp has a factorisation $f = i \circ p$



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  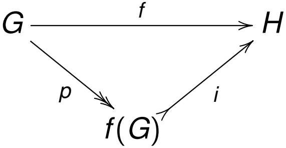

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$$\begin{array}{ccc} G & \xrightarrow{\quad f \quad} & H \\ & {\scriptstyle p}\searrow \qquad \nearrow{\scriptstyle i} & \\ & f(G) & \end{array}$$

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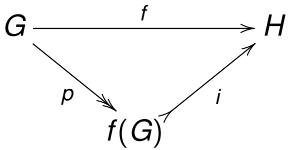

  where p is a regular epimorphism (=a coequalizer) and i is a monomorphism ;
- these factorisations are pullback stable ;
- Grp is exact (any equivalence relation is a kernel pair) ;
- the Split Short Five Lemma holds in Grp.

Examples

Grp, Rng, Alg_K, Lie_K are all semi-abelian categories.

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Any abelian category !

[$\mathbb{C}$ is abelian] $\Leftrightarrow$ [$\mathbb{C}$ and $\mathbb{C}^{op}$ are semi-abelian] !

An example of non-abelian torsion theory

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$(\mathsf{PrimHopf}_K, \mathsf{GrpHopf}_K)$ is a torsion theory in $\mathsf{Hopf}_{K,coc}$ (for K an algebraically closed field of characteristic 0).

Here the canonical short exact sequence associated with a Hopf algebra H is

$$0 \longrightarrow \mathcal{U}(L_H) \xrightarrow{i_1} H \cong \mathcal{U}(L_H) \rtimes \mathsf{K}[\mathsf{G_H}] \underset{p_2}{\overset{i_2}{\leftrightarrows}} \mathsf{K}[\mathsf{G_H}] \longrightarrow 0,$$

where $\mathcal{U}(L_H)$ is the universal enveloping algebra of the Lie algebra L_H of *primitive elements* of H

$$L_H = \{x \in H \mid \Delta(x) = 1 \otimes x + x \otimes 1\},$$

$\mathsf{K}[\mathsf{G_H}]$ is the group Hopf algebra generated by the *grouplike elements*

$$G_H = \{x \in H \mid \Delta(x) = x \otimes x, \epsilon(x) = 1\}$$

(see M. Gran, G. Kadjo, J. Vercruysse (2016)).

Torsion theories beyond the pointed case...

Any torsion theory $(\mathcal{T},\mathcal{F})$ in a pointed category $\mathbb{C}$ is such that

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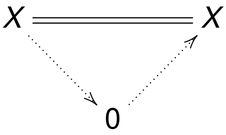
Indeed, if $X \in \mathcal{T} \cap \mathcal{F}$, then the identity 1_X factors through 0,



hence $X \cong 0$.

The idea of a pretorsion theory is to consider any two subcategories $\mathcal{T}$ and $\mathcal{F}$ and set

$$\mathcal{Z} = \mathcal{T} \cap \mathcal{F}.$$

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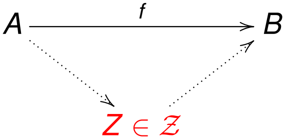
A morphism $f \colon A \to B$ is $\mathcal{Z}$-trivial if it factors through an object $Z \in \mathcal{Z}$:



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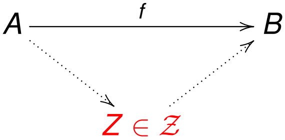
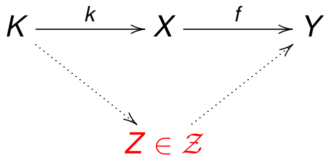


To define a pretorsion theory one needs the definition of short $\mathcal{Z}$-exact sequence.

Definition

A morphism $k\colon K \to X$ is the $\mathcal{Z}$-kernel of $f\colon X \to Y$ if

1. $K \xrightarrow{k} X \xrightarrow{f} Y$ is $\mathcal{Z}$-trivial :



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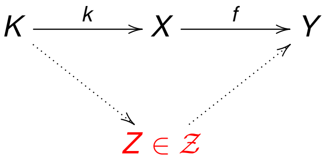
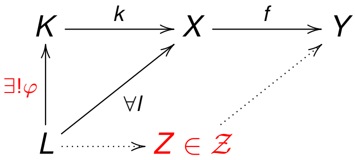


2. for any $l\colon L \to X$ such that $f \cdot l$ is $\mathcal{Z}$-trivial



there is a unique φ such that $k \cdot \varphi = l$.

Definition

The sequence $K \xrightarrow{k} X \xrightarrow{f} Y$ is short $\mathcal{Z}$-exact if

$$k = \mathcal{Z}\text{-ker}(f)$$

and

$$f = \mathcal{Z}\text{-coker}(k).$$

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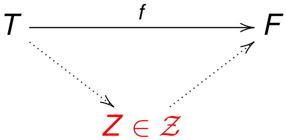
Remark

When $\mathcal{Z} = \{0\}$ one gets back the notion of short exact sequence.

Definition

A pair $(\mathcal{T}, \mathcal{F})$ of full (replete) subcategories of a category $\mathbb{C}$ is a pretorsion theory if

1. for any $T \in \mathcal{T}$ and $F \in \mathcal{F}$, any morphism from T to F is $\mathcal{Z}$-trivial :



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$$\begin{array}{ccc} T & \xrightarrow{\quad f \quad} & F \\ & \searrow \qquad \nearrow & \\ & Z \in \mathcal{Z} & \end{array}$$

2. for any object $C \in \mathbb{C}$ there is a short $\mathcal{Z}$-exact sequence in $\mathbb{C}$

$$T(C) \longrightarrow C \longrightarrow F(C),$$

with $T(C) \in \mathcal{T}$ and $F(C) \in \mathcal{F}$.

Proposition

Given any pretorsion theory $(\mathcal{T}, \mathcal{F})$ in $\mathbb{C}$, then $\mathcal{F}$ is reflective in $\mathbb{C}$

$$\mathcal{F} \underset{U}{\overset{F}{\rightleftarrows}} \mathbb{C}, \quad F \dashv U,\ U \text{ an inclusion}$$

while $\mathcal{T}$ is coreflective in $\mathbb{C}$

$$\mathcal{T} \underset{G}{\overset{V}{\rightleftarrows}} \mathbb{C}. \quad V \dashv G,\ V \text{ an inclusion}$$

Proof :

To show that $\mathcal{F}$ is reflective in $\mathbb{C}$, for any $C \in \mathbb{C}$ consider the following diagram

$$\begin{array}{ccccc} T(C) & \xrightarrow{t_C} & C & \xrightarrow{\eta_C} & F(C) \\ & & & \searrow{\scriptstyle \forall f} & \\ & & & & F_1 \end{array}$$

where the upper line is the canonical short $\mathcal{Z}$-exact sequence of the pretorsion theory, and $f\colon C \to F_1$ is any morphism with $F_1 \in \mathcal{F}$.

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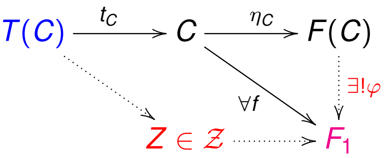

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$[f \cdot t_C$ is a $\mathcal{Z}$-trivial morphism$] \Rightarrow [\exists! \varphi$ such that $\varphi \cdot \eta_C = f]$. □

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$$X \in \mathcal{T} \Leftrightarrow \forall F \in \mathcal{F}, \ \mathrm{hom}(X, F) = \mathrm{Triv}_{\mathcal{Z}}(X, F)$$

and

$$Y \in \mathcal{F} \Leftrightarrow \forall T \in \mathcal{T}, \ \mathrm{hom}(T, Y) = \mathrm{Triv}_{\mathcal{Z}}(T, Y),$$

where $\mathrm{Triv}_{\mathcal{Z}}(X, Y)$ denotes the $\mathcal{Z}$-trivial morphisms from X to Y.

Properties

Let $(\mathcal{T}, \mathcal{F})$ be a pretorsion theory in any category $\mathbb{C}$. Then

- $\mathcal{T}, \mathcal{F}$, and $\mathcal{Z}$ are closed in $\mathbb{C}$ under retracts, and under $\mathcal{Z}$-extensions : given a short $\mathcal{Z}$-exact sequence

$$S_1 \longrightarrow X \longrightarrow S_2$$

  X belongs to $\mathcal{T}$ (to $\mathcal{F}$, or to $\mathcal{Z}$, resp.) whenever both S_1 and S_2 belong to $\mathcal{T}$ (to $\mathcal{F}$, or to $\mathcal{Z}$, resp.)

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2. for any $A \in \mathbb{C}$ the unit

$$\eta_A\colon A \to UF(A)$$

   is a $\mathcal{Z}$-cokernel.

Proposition [A. Facchini, C. Finocchiaro, M. Gran (2021)]

Let $\mathbb{C}$ be a category, $\mathcal{Z}$ a full subcategory closed under retracts in $\mathbb{C}$. Then the following are equivalent for a full subcategory $\mathcal{F}$ of $\mathbb{C}$:

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2. a) $\mathcal{F}$ is $\mathcal{Z}$-normal epireflective in $\mathbb{C}$

$$\mathcal{F} \underset{U}{\overset{F}{\underset{\hookrightarrow}{\overset{\leftarrow}{\perp}}}} \mathbb{C},$$

b) $\forall A \in \mathbb{C}$, the unit $\eta_A \colon A \to UF(A)$ has a $\mathcal{Z}$-kernel

$$t_A \colon T(A) \to A,$$

c) $\forall A \in \mathbb{C}$, $t_{T(A)} \colon T(T(A)) \to T(A)$ is an isomorphism.

An example

Recall that a preordered group $(G, \leq, +)$ is a group G endowed with a preorder relation $\leq$ on G that is "compatible" with the group operation $+$:

$$[a \leq c, \text{ and } b \leq d] \Rightarrow [a + b \leq c + d].$$

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Any preordered group $(G, \leq, +)$ has a positive cone $P_G = \{g \in G \mid 0 \leq g\}$. This is a submonoid $P_G \rightarrowtail G$ of G stable under conjugation.

Alternative presentation of PreOrdGrp :

- objects : (G, P_G) with P_G a submonoid of a group G stable under conjugation :

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- morphisms : $(G, P_G) \to (H, P_H)$ is a pair $(f, \bar{f})$

$$\begin{array}{ccc} P_G & \overset{\bar{f}}{\dashrightarrow} & P_H \\ \downarrow & & \downarrow \\ G & \underset{f}{\dashrightarrow} & H \end{array}$$

where $f \colon G \to H$ is a group homomorphism and $\bar{f} \colon P_G \to P_H$ its restriction.

Theorem [M.M. Clementino, N. Martins-Ferreira, A. Montoli (2019)]

The category PreOrdGrp is normal.

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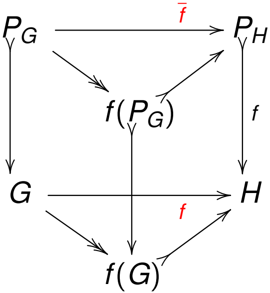
This means that any morphism $(f, \bar{f})$ factorises as a normal epimorphism (= a cokernel) followed by a monomorphism



and these factorisations are pullback stable.

The category PreOrdGrp contains the full subcategory ParOrdGrp of partially ordered groups. These are the preordered groups (G, P_G) such that P_G is a reduced monoid :

$$\forall x, y \in P_G, \qquad [x + y = 0] \Rightarrow [x = 0 = y].$$

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There is also the full subcategory ProtoPreOrdGrp of "protomodular objects" in the category PreOrdGrp : these are the preordered groups (H, P_H) with the property that P_H is a group.

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Proposition [M. Gran, A. Michel, 2021]

The pair (ProtoPreOrdGrp, ParOrdGrp) is a pretorsion theory in PreOrdGrp.

Idea of the proof :

Observe that

$$\mathcal{Z} = \mathsf{ProtoPreOrdGrp} \cap \mathsf{ParOrdGrp} = \{\, 0 \rightarrowtail G \mid G \in \mathsf{Grp}\},$$

since a reduced monoid that is also a group is trivial. This implies that any morphism from a protomodular object to a partially ordered group is trivial.

Idea of the proof :

Observe that

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since a reduced monoid that is also a group is trivial. This implies that any morphism from a protomodular object to a partially ordered group is trivial.

Next, given a preordered group $P_G \rightarrowtail G$, one defines

$$N_G = \{n \in G \mid n \in P_G \text{ and } -n \in P_G\},$$

which is a normal subgroup of G.

The canonical $\mathcal{Z}$-exact sequence associated with (G, P_G) is given by

$$
\begin{array}{ccccc}
N_G & \rightarrowtail & P_G & \overset{\overline{\eta}_G}{\twoheadrightarrow} & P_G/N_G \\
\downarrow & & \downarrow & & \downarrow \\
G & = & G & \underset{\eta_G}{\twoheadrightarrow} & G/N_G
\end{array}
$$

where $(G, N_G) \in$ ProtoPreOrdGrp and $(G/N_G, P_G/N_G) \in$ ParOrdGrp.

Outline

The category $\mathrm{PreOrd}(\mathbb{C})$ of internal preorders in an exact category $\mathbb{C}$

- objects : reflexive and transitive relations

$$\rho \underset{r_2}{\overset{r_1}{\rightrightarrows}} A$$

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$$\begin{array}{ccc} \rho & \xrightarrow{\ \overline{f}\ } & \sigma \\ r_1 \Big\downarrow\Big\downarrow r_2 & & s_1 \Big\downarrow\Big\downarrow s_2 \\ A & \xrightarrow[\ f\ ]{} & B \end{array}$$

  such that $s_1 \cdot \overline{f} = f \cdot r_1$ and $s_2 \cdot \overline{f} = f \cdot r_2$.

A pretorsion theory in PreOrd($\mathbb{C}$)

An internal preorder (A, ρ) is a partial order if $\rho \cap \rho^o = \Delta_A$ (anti-symmetry), where Δ_A is the **discrete relation** on A :

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A preorder (A, ρ) is an equivalence relation if $\rho^o = \rho$ (symmetry).
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Proposition [A. Facchini, C. Finocchiaro, M. Gran, 2021]

When $\mathbb{C}$ is exact, the pair $(\mathsf{Eq}(\mathbb{C}), \mathsf{ParOrd}(\mathbb{C}))$ is a pretorsion theory in $\mathsf{PreOrd}(\mathbb{C})$.

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\begin{array}{ccccc}
\sim_\rho & \overset{i}{\rightarrowtail} & \rho & \overset{\overline{\pi}}{\twoheadrightarrow} & \pi(\rho) \\
\downarrow\downarrow & & r_1 \downarrow\downarrow r_2 & & \downarrow\downarrow \\
A & = & A & \underset{\pi}{\twoheadrightarrow} & \frac{A}{\sim_\rho}
\end{array}
$$

Any morphism

$$\begin{array}{ccc} \rho & \xrightarrow{\bar{f}} & \sigma \\ r_1 \Big\downarrow \Big\downarrow r_2 & & s_1 \Big\downarrow \Big\downarrow s_2 \\ A & \xrightarrow[f]{} & B \end{array}$$

from an equivalence relation (A, ρ) to a partial order (B, σ) is $\mathcal{Z}$-trivial :

$$\begin{aligned} f(\rho) &= f(\rho \cap \rho^o) \\ &\leq f(\rho) \cap f(\rho^o) \\ &\leq \sigma \cap \sigma^o \\ &= \Delta_B, \end{aligned}$$

hence $(f, \bar{f})$ factors through the discrete relation $(B, \Delta_B) \in \mathcal{Z}$. □

Outline

From torsion theories to pretorsion theories

Internal preorders

The stable category

The lextensive context

Two examples of pretorsion theories in Cat

The stable category

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The stable category is a **quotient** of PreOrd having the property that this quotient sends

- any trivial preorder to the zero object,
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With F. Borceux and F. Campanini we have looked at this construction from a categorical perspective. We proposed a new definition of the stable category of $\mathsf{PreOrd}(\mathbb{C})$, where $\mathbb{C}$ is a pretopos.

A pretopos is an exact category $\mathbb{C}$ with finite sums that is also

- extensive : in any commutative diagram

$$\begin{array}{ccccc} X' & \longrightarrow & A & \longleftarrow & Y' \\ \downarrow & & \downarrow & & \downarrow \\ X & \xrightarrow[s_1]{} & X \coprod Y & \xleftarrow[s_2]{} & Y, \end{array}$$

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Examples

Set, G-Set, *HComp* (compact Hausdorff spaces), any topos, etc.

Complemented subobjects

In a pretopos $\mathbb{C}$ a subobject $A \overset{\alpha}{\rightarrowtail} B$ of B is **complemented** if there is another subobject $A^c \overset{\alpha^c}{\rightarrowtail} B$ with the property that

$$A \cap A^c = 0 \text{ and } A \cup A^c = B.$$

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The “idea” of the stable category is to identify two morphisms in PreOrd($\mathbb{C}$) if they coincide on a (complemented) subobject and are both $\mathcal{Z}$-trivial on its complement.

To define the stable category, we first build the category $\mathsf{PaPreOrd}(\mathbb{C})$ of partial morphisms in $\mathsf{PreOrd}(\mathbb{C})$.

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- objects : internal preorders (A, ρ) in $\mathbb{C}$;
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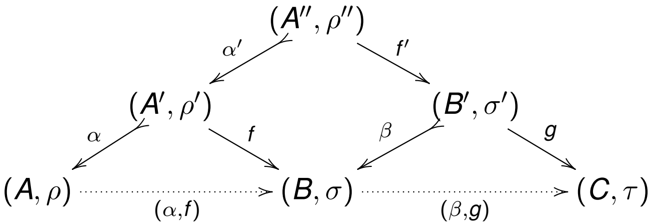
$$\begin{array}{ccc} & (A', \rho') & \\ \swarrow^{\alpha} & & \searrow^{f} \\ (A, \rho) & \underset{(\alpha, f)}{\cdots\cdots\cdots\cdots\rightarrow} & (B, \sigma), \end{array}$$

where $f\colon (A', \rho') \to (B, \sigma)$ is a morphism in PreOrd($\mathbb{C}$) and $(A', \rho') \overset{\alpha}{\rightarrowtail} (A, \rho)$ is a complemented subobject in PreOrd($\mathbb{C}$).

- <u>composition</u> : given two composable morphisms, one considers the pullback



and set

$$(\beta, g) \circ (\alpha, f) = (\alpha\alpha', gf').$$

There is a functor

$$I\colon \mathbf{PreOrd}(\mathbb{C}) \to \mathbf{PaPreOrd}(\mathbb{C})$$

sending a morphism

$$f\colon (A,\rho) \to (B,\sigma)$$

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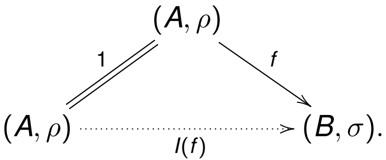

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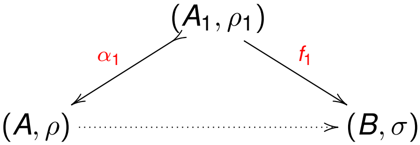
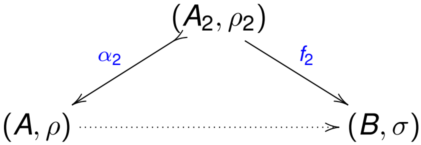
$$\begin{array}{ccccc} & & (A,\rho) & & \\ & {\scriptstyle 1}\swarrow\!\!\!\!= & & \searrow{\scriptstyle f} & \\ (A,\rho) & & \xrightarrow[I(f)]{\cdots\cdots\cdots} & & (B,\sigma). \end{array}$$

The stable category $\mathsf{Stab}(\mathbb{C})$ of $\mathsf{PreOrd}(\mathbb{C})$ is a quotient of $\mathsf{PaPreOrd}(\mathbb{C})$.

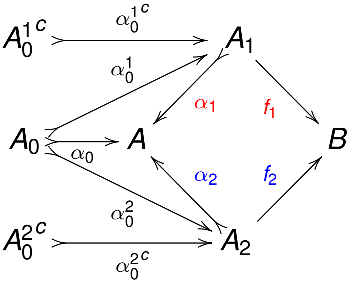
Two parallel morphisms (α_1, f_1) and (α_2, f_2) in $\mathsf{PaPreOrd}(\mathbb{C})$



and



are equivalent for $\sim$ if there is a **congruence diagram** between them :



The equivalence relation $\sim$ is "compatible" with the composition, and is a congruence on $\mathsf{PaPreOrd}(\mathbb{C})$. One has the quotient

$$\mathsf{PaPreOrd}(\mathbb{C}) \xrightarrow{\pi} \frac{\mathsf{PaPreOrd}(\mathbb{C})}{\sim} := \mathsf{Stab}(\mathbb{C}).$$

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We then get the functor

$$\Sigma : \mathsf{PreOrd}(\mathbb{C}) \xrightarrow{I} \mathsf{PaPreOrd}(\mathbb{C}) \xrightarrow{\pi} \mathsf{Stab}(\mathbb{C}).$$

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- the stable category $\mathrm{Stab}(\mathbb{C})$ is pointed ;

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- the functor $\mathsf{PreOrd}(\mathbb{C}) \xrightarrow{\Sigma} \mathsf{Stab}(\mathbb{C})$ preserves finite coproducts and monomorphisms.

Proposition [F. Borceux, F. Campanini, M.G., 2022]

The functor $\mathsf{PreOrd}(\mathbb{C}) \xrightarrow{\Sigma} \mathsf{Stab}(\mathbb{C})$ is a torsion theory functor :

- the pretorsion theory $(\mathsf{Eq}(\mathbb{C}), \mathsf{ParOrd}(\mathbb{C}))$ "becomes" a torsion theory in the pointed category $\mathsf{Stab}(\mathbb{C})$;

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- the canonical short $\mathcal{Z}$-exact sequence of the pretorsion theory

$$\begin{array}{ccccc} \sim_\rho & \overset{i}{\rightarrowtail} & \rho & \overset{\overline{\pi}}{\twoheadrightarrow} & \pi(\rho) \\ \downarrow\downarrow & & r_1 \downarrow\downarrow r_2 & & \downarrow\downarrow \\ A & = & A & \underset{\pi}{\twoheadrightarrow} & \frac{A}{\sim_\rho} \end{array}$$

“becomes” the canonical short exact sequence of the torsion theory in $\mathsf{Stab}(\mathbb{C})$.

Universal property

The functor $\mathsf{PreOrd}(\mathbb{C}) \xrightarrow{\Sigma} \mathsf{Stab}(\mathbb{C})$ is universal among all finite coproduct preserving torsion theory functors $G\colon \mathsf{PreOrd}(\mathbb{C}) \to \mathbb{X}$, where $\mathbb{X}$ is equipped with a torsion theory :

$$\begin{array}{ccc} \mathsf{PreOrd}(\mathbb{C}) & \xrightarrow{\quad\Sigma\quad} & \mathsf{Stab}(\mathbb{C}) \\ & {\scriptstyle \forall G}\searrow \quad \swarrow{\scriptstyle \exists! \overline{G}} & \\ & \mathbb{X}. & \end{array}$$

The unique $\overline{G}$ such that $\overline{G} \circ \Sigma = G$ is a torsion theory functor that preserves finite coproducts.

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The stable category $\mathrm{Stab}(\mathbb{C})$ provides the "universal torsion theory" associated with the pretorsion theory $(\mathrm{Eq}(\mathbb{C}), \mathrm{ParOrd}(\mathbb{C}))$.

Outline

The lextensive context

It is interesting to analyse the structural reasons why the construction of the stable category works well for $\mathsf{PreOrd}(\mathbb{C})$.

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Examples

Set, any pretopos, CRng^{op}, Top, PreOrd, $\mathrm{PreOrd}(\mathbb{C})$ (for $\mathbb{C}$ a pretopos), Cat.

Let $(\mathcal{T}, \mathcal{F})$ be a pretorsion theory in a lextensive category $\mathbb{C}$ satisfying the following properties :

- $\mathcal{Z} = \mathcal{T} \cap \mathcal{F}$ is closed in $\mathbb{C}$ under complemented subobjects ;
- $\mathcal{Z} = \mathcal{T} \cap \mathcal{F}$ is closed in $\mathbb{C}$ under binary coproducts.

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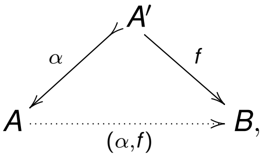
Under these assumptions one can define the category $\mathsf{Par}(\mathbb{C})$ of partial morphisms in $\mathbb{C}$, exactly as we did in the case of the internal preorders.

The category Par($\mathbb{C}$)

Let $\mathbb{C}$ be a lextensive category. Define :

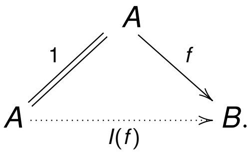
- objects : same as in $\mathbb{C}$;
- morphisms : a pair (α, f) as in



where $f\colon A' \to B$ is a morphism in $\mathbb{C}$ and $A' \xrightarrow{\alpha} A$ is a complemented subobject.

The stable category

- There is a functor $\mathbb{C} \xrightarrow{I} \mathrm{Par}(\mathbb{C})$, sending a morphism $f\colon A \to B$ to



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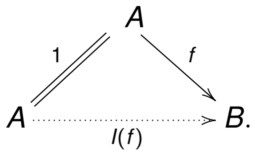
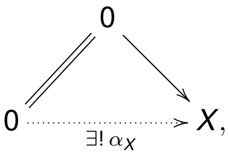
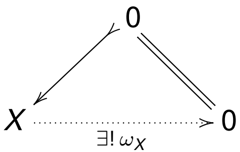


- $\mathrm{Par}(\mathbb{C})$ is pointed : the initial object 0 of $\mathbb{C}$ becomes a zero object in $\mathrm{Par}(\mathbb{C})$:





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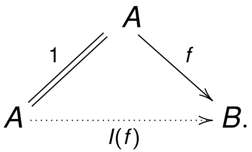

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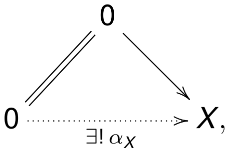

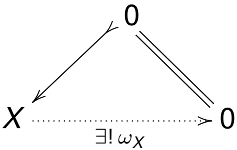

- The quotient category $\frac{\text{Par}(\mathbb{C})}{\sim} = \text{Stab}(\mathbb{C})$ of Par($\mathbb{C}$) by the equivalence relation $\sim$ on the morphisms in Par($\mathbb{C}$) gives the **stable category**.

The composite

$$\Sigma\colon \mathbb{C} \longrightarrow \mathsf{Par}(\mathbb{C}) \longrightarrow \frac{\mathsf{Par}(\mathbb{C})}{\sim} = \mathsf{Stab}(\mathbb{C})$$

is the universal functor sending the pretorsion theory $(\mathcal{T}, \mathcal{F})$ to a torsion theory :

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Universal property

Let $(\mathcal{T}, \mathcal{F})$ be a pretorsion theory in a lextensive category $\mathbb{C}$, with $\mathcal{T}$ closed in $\mathbb{C}$ under complemented subobjects. If $\mathcal{Z}$-kernels and $\mathcal{Z}$-cokernels exist, then the functor $\Sigma\colon \mathbb{C} \to \mathsf{Stab}(\mathbb{C})$ is universal among all finite coproduct preserving torsion theory functors $G\colon \mathbb{C} \to \mathbb{X}$, where $\mathbb{X}$ is equipped with a torsion theory :

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Outline

The example of “symmetric” and “antisymmetric” categories (J. Xarez, 2022)

Let us consider the category Cat of (small) categories and functors.

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We write SymCat for the full subcategory of Cat whose objects are *symmetric categories*, i.e. those categories having the property :

$$\text{for any } X, Y, \text{ if } \mathrm{hom}(X, Y) \neq \emptyset, \text{ then } \mathrm{hom}(Y, X) \neq \emptyset.$$

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Let AntiSymCat denote the full subcategory of antisymmetric categories :

$$\text{if } \hom(X, Y) \neq \emptyset \text{ and } \hom(Y, X) \neq \emptyset, \quad \text{then } X = Y.$$

Theorem (J. Xarez (2022))

The pair (SymCat, AntiSymCat) *is a pretorsion theory in* Cat*.*

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In this example of pretorsion theory, the trivial objects in

$$\mathcal{Z} = \mathsf{SymCat} \cap \mathsf{AntiSymCat}$$

are the classes of monoids.

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We write Grpd for the full subcategory of Cat whose objects are groupoids.

We write SkelCat for the full subcategory of Cat whose objects are skeletal categories (= categories where every isomorphism is an automorphism).

In this case

$$\mathcal{Z} = \mathsf{Grpd} \cap \mathsf{SkelCat}$$

are the classes of groups.

Given a small category $\mathcal{C}$, there is always the subgroupoid $\mathsf{Iso}(\mathcal{C}) \in \mathsf{Grpd}$ of its isomorphisms :

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In order to build the skeletal category $\mathcal{S}$ associated with $\mathcal{C}$ one forms the following coequalizer in Cat

$$\coprod_{\sigma \in \mathsf{Iso}(\mathcal{C})} 1 \underset{c}{\overset{d}{\rightrightarrows}} \mathcal{C} \overset{q}{\twoheadrightarrow} \mathcal{S},$$

where 1 is the terminal category, *d* and *c* are the functors associating - with any component indexed by an isomorphism σ - its “domain” and “codomain”, respectively.

Theorem (F. Borceux, F. Campanini, M. Gran, W. Tholen, 2023)

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$$\mathsf{Iso}(\mathcal{C}) \xrightarrow{i} \mathcal{C} \xrightarrow{q} \mathcal{S}.$$

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A key ingredient in the proof comes from a property of coequalizers of morphisms in Cat with a “discrete” domain : they are faithful and reflect isomorphisms.

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Remark

When the small category $\mathcal{C}$ is a preordered set, the $\mathcal{Z}$-exact sequence above gives back the canonical $\mathcal{Z}$-exact sequence for (Eq(Set), ParOrd(Set)).

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