

Internal structures in Mal'tsev categories

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Category Theory Seminar at the University of Ottawa
25/08/2023

Outline

- 1 Mal'tsev categories
- 2 Internal structures in Mal'tsev categories
- 3 Goursat and weakly Mal'tsev categories
- 4 Internal 2-groupoids

Let G be a group and $R \subseteq G \times G$ be a **reflexive** relation on G which is compatible with the group structure on G , i.e., R is a subgroup of $G \times G$.

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Thus, R is an **equivalence** relation.

Definition

Let \mathcal{C} be a finitely complete category. An **internal equivalence relation** on an object X of \mathcal{C} is a subobject

$$R \xrightarrow{r=(r_1, r_2)} X \times X$$

such that:

$$\begin{array}{ccc} & R & \\ \nearrow \exists \delta & \downarrow r & \\ X & \xrightarrow{\Delta_X} & X \times X \end{array}$$

$$\begin{array}{ccc} & R & \\ \nearrow \exists \sigma & \downarrow (r_1, r_2) & \\ R & \xrightarrow{(r_2, r_1)} & X \times X \end{array}$$

$$\begin{array}{ccc} & R & \\ \nearrow \exists \tau & \downarrow r & \\ R \times_X R & \xrightarrow{(r_1 p_1, r_2 p_2)} & X \times X \end{array}$$

$$\begin{array}{ccc} R \times_X R & \xrightarrow{p_2} & R \\ p_1 \downarrow & \lrcorner & \downarrow r_1 \\ R & \xrightarrow{r_2} & X \end{array}$$

Definition (Carboni–Lambek–Pedicchio (1990))

A finitely complete category \mathcal{C} is a **Mal'tsev category** if every internal reflexive relation in \mathcal{C} is an equivalence relation.

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Theorem (Carboni–Pedicchio–Pirovano (1992))

For a finitely complete category \mathcal{C} , the following conditions are equivalent:

- 1 \mathcal{C} is a Mal'tsev category.
- 2 Every reflexive relation in \mathcal{C} is symmetric.
- 3 Every reflexive relation in \mathcal{C} is transitive.
- 4 Every relation in \mathcal{C} is difunctional.

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A relation $R \subseteq X \times Y$ in **Set** is **difunctional** if, for all $x, z \in X$ and $y, u \in Y$,

$$xRy, zRy, zRu \Rightarrow xRu.$$

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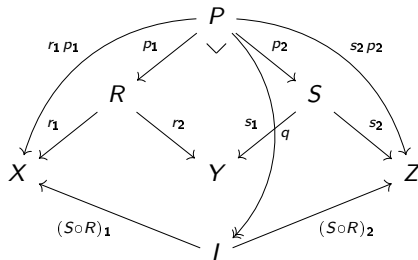
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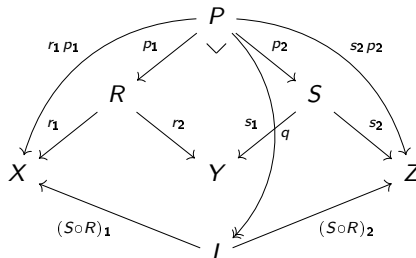
In a regular category, any morphism factorizes as a regular epimorphism followed by a monomorphism.

$$\begin{array}{ccccc} \text{Eq}[f] & \xrightarrow[p_2]{p_1} & X & \xrightarrow{f} & Y \\ & & \searrow q & & \nearrow i \\ & & I & & \end{array}$$

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Theorem (Carboni–Lambek–Pedicchio (1990))

For a regular category \mathcal{C} , the following conditions are equivalent:

- 1 \mathcal{C} is a Mal'tsev category.
- 2 For any object $X \in \mathcal{C}$ and any two equivalence relations R, S on X , $S \circ R$ is an equivalence relation.
- 3 For any object $X \in \mathcal{C}$ and any two equivalence relations R, S on X , it holds that $S \circ R = R \circ S$.

Theorem (Mal'tsev (1954))

For a variety \mathbb{V} of universal algebras, the following conditions are equivalent:

- 1** \mathbb{V} is a Mal'tsev category.
- 2** Its algebraic theory contains a ternary term $p(x, y, z)$ such that
$$p(x, x, y) = y = p(y, x, x).$$

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Examples (Mal'tsev varieties)

- Grp: $p(x, y, z) := xy^{-1}z$
- Heyt: $p(x, y, z) := ((x \Rightarrow y) \Rightarrow z) \wedge ((z \Rightarrow y) \Rightarrow x),$
 $q(x, y, x) := (y \Rightarrow (x \wedge z)) \wedge (x \vee z)$

Proposition

Let \mathcal{C} be a Mal'tsev category and X be an object of \mathcal{C} . Then \mathcal{C}/X and X/\mathcal{C} are Mal'tsev categories.

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Proposition

Let \mathcal{C} and \mathcal{D} be finitely complete categories and $F : \mathcal{C} \rightarrow \mathcal{D}$ be a conservative functor which preserves pullbacks. If \mathcal{D} is a Mal'tsev category, so is \mathcal{C} .

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- *Mal'tsev varieties internal to a finitely complete category:*
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 $\text{Grp}(\text{Top})$
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- *The dual of an elementary topos: Set^{op}*

Definition

Let \mathcal{C} be a category with pullbacks. An **internal groupoid** \mathbb{C} in \mathcal{C} is given by a diagram

$$C_2 \xrightarrow{m} C_1 \begin{array}{c} \overset{i}{\curvearrowright} \\ \xleftarrow[e]{c} C_0 \end{array} \xrightleftharpoons[d]{e} C_0$$

in \mathcal{C} such that:

$$\begin{array}{ccc} \begin{array}{ccc} C_2 & \xrightarrow{p_2} & C_1 \\ p_1 \downarrow & \lrcorner & \downarrow c \\ C_1 & \xrightarrow{d} & C_0 \end{array} & \begin{array}{ccc} C_3 & \xrightarrow{\pi_2} & C_1 \\ \pi_1 \downarrow & \lrcorner & \downarrow c \\ C_2 & \xrightarrow{dp_2} & C_0 \end{array} & \begin{array}{ccc} C_3 & \xrightarrow{\pi'_2} & C_2 \\ \pi'_1 \downarrow & \lrcorner & \downarrow cp_1 \\ C_1 & \xrightarrow{d} & C_0 \end{array} \end{array}$$

$$(RG) \quad de = 1_{C_0} = ce$$

$$(MG) \quad m\langle ec, 1_{C_1} \rangle = 1_{C_1} = m\langle 1_{C_1}, ed \rangle$$

$$(C) \quad dm = dp_2, \quad cm = cp_1, \quad m\langle m\pi_1, \pi_2 \rangle = m\langle \pi'_1, m\pi'_2 \rangle$$

$$(G) \quad di = c, \quad ci = d, \quad m\langle i, 1_{C_1} \rangle = ed, \quad m\langle 1_{C_1}, i \rangle = ec$$

Definition

Let \mathcal{C} be a category with pullbacks, and \mathbb{C} and \mathbb{D} be two internal groupoids in \mathcal{C} . An **internal functor** $F : \mathbb{C} \rightarrow \mathbb{D}$ is given by morphisms $f_0 : C_0 \rightarrow D_0$ and $f_1 : C_1 \rightarrow D_1$ such that

$$\begin{array}{ccccc}
 & & i & & \\
 & & \downarrow & & \\
 C_2 & \xrightarrow{m} & C_1 & \begin{array}{c} \xrightarrow{d} \\ \xleftarrow{e} \\ \xrightarrow{c} \end{array} & C_0 \\
 \downarrow & & \downarrow f_1 & & \downarrow f_0 \\
 D_2 & \xrightarrow{m} & D_1 & \begin{array}{c} \xrightarrow{d} \\ \xleftarrow{e} \\ \xrightarrow{c} \end{array} & D_0 \\
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reasonably commutes.

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reasonably commutes.

We denote by $\mathbf{Grpd}(\mathcal{C})$ ($\mathbf{Cat}(\mathcal{C})$, $\mathbf{MG}(\mathcal{C})$, $\mathbf{RG}(\mathcal{C})$) the category of internal groupoids (internal categories, multiplicative graphs, reflexive graphs) in \mathcal{C} and internal functors between them.

Definition

A **crossed module of groups** is given by a group homomorphism $d : X \rightarrow B$ together with a group action $\triangleright : B \times X \rightarrow X$ such that for all $x, y \in X$ and $b \in B$:

$$\text{(Equivariance)} \quad d(b \triangleright x) = bd(x)b^{-1}$$

$$\text{(Peiffer identity)} \quad d(x) \triangleright y = xyx^{-1}$$

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A **morphism of crossed modules of groups** from (X, B) to (X', B') is given by two group homomorphism $f_1 : X \rightarrow X'$ and $f_0 : B \rightarrow B'$ such that for all $x \in X$ and $b \in B$:

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We denote by $\mathbf{XMod}(\mathbf{Grp})$ the category of crossed modules of groups and their corresponding morphisms.

Theorem (Brown–Spencer (1976))

The categories $\text{Grpd}(\text{Grp})$ of internal groupoids in Grp and $\text{XMod}(\text{Grp})$ of crossed modules of groups are equivalent.

Theorem (Brown–Spencer (1976))

The categories $\text{Grpd}(\text{Grp})$ of internal groupoids in Grp and $\text{XMod}(\text{Grp})$ of crossed modules of groups are equivalent.

Let

$$C_1 \begin{array}{c} \xrightarrow{d} \\ \xleftarrow{e} \\ \xrightarrow{c} \end{array} C_0$$

be the underlying reflexive graph of an internal groupoid in Grp .
Then

$$X := \text{Ker}(d) \hookrightarrow C_1 \xrightarrow{c} C_0 =: B$$

and $b \triangleright x := e(b)xe(b)^{-1}$ for all $x \in \text{Ker}(d)$ and $b \in C_0$ define a crossed module of groups.

Theorem (Brown–Spencer (1976))

The categories $\text{Grpd}(\text{Grp})$ of internal groupoids in Grp and $\text{XMod}(\text{Grp})$ of crossed modules of groups are equivalent.

Let (X, B) be a crossed module of groups. Then

$$X \rtimes B \begin{array}{c} \xrightarrow{p_2} \\ \xleftarrow{i_2} \\ \xrightarrow{p_1} \end{array} B,$$

where $p_2(x, b) := b$, $i_2(b) := (1, b)$ and $p_1(x, b) := d(x)b$ for all $x \in X$ and $b \in B$, is the underlying reflexive graph of an internal groupoid in Grp .

Theorem (Brown–Spencer (1976))

The categories $\text{Grpd}(\text{Grp})$ of internal groupoids in Grp and $\text{XMod}(\text{Grp})$ of crossed modules of groups are equivalent.

Theorem (G. Janelidze (2003))

Let \mathcal{C} be a semi-abelian category. Then the categories $\text{Grpd}(\mathcal{C})$ of internal groupoids in \mathcal{C} and $\text{XMod}(\mathcal{C})$ of internal crossed modules in \mathcal{C} are equivalent.

Proposition

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Let

$$C_1 \times_{C_0} C_1 \xrightarrow{m} C_1 \begin{array}{c} \xrightarrow{d} \\ \xleftarrow{e} \\ \xrightarrow{c} \end{array} C_0$$

be a multiplicative graph in \mathbf{Grp} , where

$$C_1 \times_{C_0} C_1 = \{(g, f) \in C_1 \times C_1 \mid d(g) = c(f)\}.$$

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$$\mathbf{m}(\mathbf{g}, \mathbf{f}) = m(g(ed(g))^{-1}ed(g), 1f)$$

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Proposition

For a reflexive graph \mathbb{C} ,

$$\mathbb{C}_1 \begin{array}{c} \xrightarrow{d} \\ \xleftarrow{e} \\ \xrightarrow{c} \end{array} \mathbb{C}_0,$$

in \mathbf{Grp} , the following conditions are equivalent:

- 1** $[\mathrm{Ker}(d), \mathrm{Ker}(c)] = \{1\}$
- 2** \mathbb{C} admits a (unique) multiplicative graph structure.
- 3** \mathbb{C} admits a (unique) category structure.
- 4** \mathbb{C} admits a (unique) groupoid structure.

(1) \Rightarrow (2): Let us suppose that $[\text{Ker}(d), \text{Ker}(c)] = \{1\}$.

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 &= g'(ec(f'))^{-1}f'g(ed(g))^{-1}f \\
 &= g'(ed(g'))^{-1}f'g(ed(g))^{-1}f \\
 &= m(g', f')m(g, f).
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 \end{aligned}$$

Furthermore,

$$m(ec(f), f) = ec(f)(edec(f))^{-1}f$$

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 &= g'(ed(g'))^{-1}f'g(ed(g))^{-1}f \\
 &= m(g', f')m(g, f).
 \end{aligned}$$

Furthermore,

$$m(ec(f), f) = ec(f)(edec(f))^{-1}f = f,$$

(1) \Rightarrow (2): Let us suppose that $[\text{Ker}(d), \text{Ker}(c)] = \{1\}$.

We define $m : C_1 \times_{C_0} C_1 \rightarrow C_1$ by $m(g, f) := g(ed(g))^{-1}f$. Then

$$\begin{aligned}
 m((g', f')(g, f)) &= m(g'g, f'f) \\
 &= (g'g)(ed(g'g))^{-1}(f'f) \\
 &= g'g(ed(g))^{-1}(ed(g'))^{-1}f'f \\
 &= g'g(ed(g))^{-1}(ec(f'))^{-1}f'f \\
 &= g'(ec(f'))^{-1}f'g(ed(g))^{-1}f \\
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Furthermore,

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 m(ec(f), f) &= ec(f)(edec(f))^{-1}f = f, \\
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(2) \Rightarrow (1): Let us suppose that \mathbb{C} admits a multiplicative graph structure.

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$$\mathbf{fg} = m(f, 1)m(1, g)$$

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$$\mathbf{fg} = m(f, 1)m(1, g) = m(f, g) = m(1, g)m(f, 1) = \mathbf{gf}.$$

To (4): We define $i : C_1 \rightarrow C_1$ by

$$i(f) = ed(f)f^{-1}ec(f).$$

Theorem (Bourn (1996))

For a finitely complete category \mathcal{C} , the following conditions are equivalent:

- 1** \mathcal{C} is a Mal'tsev category.
- 2** $\text{Pt}_X(\mathcal{C})$ is a Mal'tsev category for any object X of \mathcal{C} .
- 3** Given a pullback of split epimorphisms

$$\begin{array}{ccc}
 X \times_Z Y & \begin{array}{c} \xleftarrow{p_2} \\ \xrightarrow{e_2} \end{array} & Y \\
 \begin{array}{c} \downarrow p_1 \\ \uparrow e_1 \end{array} \lrcorner & & \begin{array}{c} \downarrow s \\ \uparrow g \end{array} \\
 X & \begin{array}{c} \xleftarrow{r} \\ \xrightarrow{f} \end{array} & Z
 \end{array}$$

in \mathcal{C} , the pullback injections e_1, e_2 are jointly extremally epimorphic.

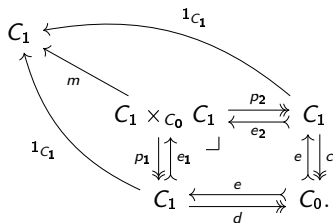
Proposition

Any reflexive graph in a Mal'tsev category \mathcal{C} admits at most one multiplicative graph structure.

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Let \mathbb{C} be a multiplicative graph in \mathcal{C} . We consider



Since $e_1 = \langle 1_{C_1}, ed \rangle$, $e_2 = \langle ec, 1_{C_1} \rangle$ are jointly (strongly) epimorphic, m is uniquely determined by the equations

$$m\langle ec, 1_{C_1} \rangle = 1_{C_1} = m\langle 1_{C_1}, ed \rangle.$$

Definition (Bourn–Gran (2002))

Let R, S be two equivalence relations on an object X in a finitely complete category \mathcal{C} . We consider the pullback

$$\begin{array}{ccc} R \times_X S & \xrightarrow{p_2} & S \\ p_1 \downarrow & \lrcorner & \downarrow s_1 \\ R & \xrightarrow{r_2} & X. \end{array}$$

A **connector** between R and S is a map $p : R \times_X S \rightarrow X$ such that

- 1 $xSp(x, y, z)$ and $p(x, y, z)Rz$,
- 2 $p(x, x, y) = y = p(y, x, x)$,
- 3 $p(x, y, p(y, u, v)) = p(x, u, v)$ and $p(p(x, y, u), u, v) = p(x, y, v)$.

Proposition

Let \mathbb{C} ,

$$C_1 \begin{array}{c} \xrightarrow{d} \\ \xleftarrow{e} \\ \xrightarrow{c} \end{array} C_0,$$

be a reflexive graph in a finitely complete category \mathcal{C} . The connectors between $\text{Eq}[d]$ and $\text{Eq}[c]$ are in one-to-one correspondence with the internal groupoid structures on \mathbb{C} .

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Proposition (Bourn–Gran (2002))

Let \mathcal{C} be a Mal'tsev category and R, S two equivalence relations on an object X of \mathcal{C} . Then there exists at most one map $p : R \times_X S \rightarrow X$ such that $p(x, x, y) = y$ and $p(x, y, y) = x$ and this map is a connector.

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In this case, one says that R and S **centralize** each other.

Proposition

For a reflexive graph \mathbb{C} ,

$$\mathbb{C}_1 \begin{array}{c} \xrightarrow{d} \\ \xleftarrow{e} \\ \xrightarrow{c} \end{array} \mathbb{C}_0,$$

in a Mal'tsev category \mathcal{C} , the following conditions are equivalent:

- 1** $\text{Eq}[d]$ and $\text{Eq}[c]$ centralize each other.
- 2** \mathbb{C} admits a (unique) multiplicative graph structure.
- 3** \mathbb{C} admits a (unique) category structure.
- 4** \mathbb{C} admits a (unique) groupoid structure.

Given two subgroups H, I of a group G , we can abstract the condition $[H, I] = \{1\}$ to the pointed Mal'tsev setting.

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Definition

Let \mathcal{C} be a pointed Mal'tsev category and $f : X \rightarrow Y$ and $g : X' \rightarrow Y$ be morphisms in \mathcal{C} . We say that f and g **commute** if there exists a (necessarily unique) morphism $\varphi : X \times X' \rightarrow Y$ such that

$$\begin{array}{ccccc}
 X & \xrightarrow{(1_X, 0)} & X \times X' & \xleftarrow{(0, 1_{X'})} & X' \\
 & \searrow f & \downarrow \varphi & \swarrow g & \\
 & & Y & &
 \end{array}$$

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 & & Y & &
 \end{array}$$

commutes.

Then $[H, I] = \{1\}$ if and only if the inclusions $h : H \rightarrow G$ and $i : I \rightarrow G$ commute.

Given an equivalence relation R on an object X in a pointed Mal'tsev category \mathcal{C} , we can associate to it its *normalization*

$$N_R := \text{Ker}(r_1) \xrightarrow{\ker(r_1)} R \xrightarrow{r_2} X.$$

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Given two equivalence relations R, S on X in \mathcal{C} centralizing each other, n_R and n_S commute.

The converse is **not** necessarily true.

Proposition

Let \mathcal{C} be a Mal'tsev category. Then $\text{Grpd}(\mathcal{C})$ is a full subcategory of $\text{RG}(\mathcal{C})$.

$$\begin{array}{ccccc}
 & & i & & \\
 & & \downarrow & & \\
 C_2 & \xrightarrow{m} & C_1 & \begin{array}{c} \xrightarrow{d} \\ \xleftarrow{e} \\ \xrightarrow{c} \end{array} & C_0 \\
 & & \downarrow f_1 & & \downarrow f_0 \\
 D_2 & \xrightarrow{m} & D_1 & \begin{array}{c} \xrightarrow{d} \\ \xleftarrow{e} \\ \xrightarrow{c} \end{array} & D_0 \\
 & & \uparrow i & &
 \end{array}$$

Definition

Let \mathcal{C} be a regular category. Then \mathcal{C} is called **n -permutable** if, for any two equivalence relations R, S on an object X in \mathcal{C} , it holds that

$$\underbrace{R \circ S \circ R \circ S \circ \cdots}_{n \text{ factors}} = \underbrace{S \circ R \circ S \circ R \circ \cdots}_{n \text{ factors}}.$$

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- A 2-permutable category is exactly a regular Mal'tsev category.
- A 3-permutable category is also called a **Goursat** category.

Theorem (Bourn (1996))

For a finitely complete category \mathcal{C} , the following conditions are equivalent:

- 1 \mathcal{C} is a Mal'tsev category.
- 2 $\text{Grpd}(\mathcal{C})$ is closed in $\text{RG}(\mathcal{C})$ under subobjects.

$$\begin{array}{ccccc}
 C_1 \times_{C_0} C_1 & \overset{m}{\dashrightarrow} & C_1 & \overset{d}{\underset{c}{\rightleftarrows[e]}} & C_0 \\
 \downarrow & & \downarrow f_1 & & \downarrow f_0 \\
 G_1 \times_{G_0} G_1 & \xrightarrow{m} & G_1 & \overset{d}{\underset{c}{\rightleftarrows[e]}} & G_0
 \end{array}$$

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 \downarrow & & \downarrow f_1 & & \downarrow f_0 \\
 G_1 \times_{G_0} G_1 & \xrightarrow{\quad m \quad} & G_1 & \begin{array}{c} \xrightarrow{d} \\ \xleftarrow{e} \\ \xrightarrow{c} \end{array} & G_0
 \end{array}$$

Proposition (Gran–Rodelo–Tchoffo Nguéfeu (2017))

For a regular category \mathcal{C} , the following conditions are equivalent:

- 1 \mathcal{C} is a Goursat category.
- 2 $\text{Grpd}(\mathcal{C})$ is closed in $\text{RG}(\mathcal{C})$ under quotients.

Definition (Martins-Ferreira (2008))

A category \mathcal{C} is called a **weakly Mal'tsev category** if it has pullbacks of split epimorphisms along split epimorphisms and the pullback injections e_1, e_2 of such a pullback

$$\begin{array}{ccccc}
 X \times_Z Y & & Y & \xleftarrow[p_2]{\rightrightarrows} & Y \\
 \downarrow p_1 \uparrow e_1 & \lrcorner & & & \uparrow s \downarrow g \\
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Proposition

Any reflexive graph in a weakly Mal'tsev category \mathcal{C} admits at most one multiplicative graph structure.

Let \mathcal{C} be a finitely complete category. A **strong relation** between objects X and Y of \mathcal{C} is given by a strong monomorphism $r : R \rightarrowtail X \times Y$.

$$\begin{array}{ccc}
 A & \xrightarrow{e} \twoheadrightarrow & B \\
 a \downarrow & \swarrow \exists! d & \downarrow b \\
 R & \xrightarrow{r} \rightarrowtail & X \times Y
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Theorem (Z. Janelidze–Martins-Ferreira (2012))

For a finitely complete category \mathcal{C} , the following conditions are equivalent:

- 1 \mathcal{C} is a weakly Mal'tsev category.
- 2 Any reflexive strong relation in \mathcal{C} is an equivalence relation.
- 3 Any reflexive strong relation in \mathcal{C} is symmetric.
- 4 Any reflexive strong relation in \mathcal{C} is transitive.
- 5 Any strong relation in \mathcal{C} is difunctional.

Theorem (E.–Jacqmin–Martins-Ferreira, 2023)

A variety \mathbb{V} of universal algebras is a weakly Mal'tsev category if and only if there exist integers $k, m, N \geq 0$, binary terms $f_1, g_1, \dots, f_k, g_k$, ternary terms p_1, \dots, p_m , $(2(k + 2m + 1))$ -ary terms s_1, \dots, s_N , $(2(k + m + 2))$ -ary terms $\sigma_1, \dots, \sigma_{N+1}$ and, for all $i \in \{1, \dots, N + 1\}$, $(k + m + 1)$ -ary terms $\eta_1^{(i)}, \eta_2^{(i)}, \epsilon_1^{(i)}, \epsilon_2^{(i)}$ such that, for all $i \in \{1, \dots, k\}$, $j \in \{1, \dots, N + 1\}$ and $\alpha \in \{1, 2\}$,

$$f_i(x, x) = g_i(x, x) \quad (1)$$

$$\begin{aligned} & \eta_\alpha^{(j)}(y, f_1(x, y), \dots, f_k(x, y), p_1(x, x, y), \dots, p_m(x, x, y)) \\ &= \epsilon_\alpha^{(j)}(y, f_1(x, y), \dots, f_k(x, y), p_1(x, x, y), \dots, p_m(x, x, y)), \end{aligned} \quad (2a)$$

$$\begin{aligned} & \eta_\alpha^{(j)}(x, g_1(x, y), \dots, g_k(x, y), p_1(x, y, y), \dots, p_m(x, y, y)) \\ &= \epsilon_\alpha^{(j)}(x, g_1(x, y), \dots, g_k(x, y), p_1(x, y, y), \dots, p_m(x, y, y)), \end{aligned} \quad (2b)$$

Theorem

for all $i \in \{1, \dots, N\}$,

$$\begin{aligned} & \sigma_i(u, \vec{v}, \vec{w}, u', \vec{v}', \vec{w}', \epsilon_1^{(i)}(u, \vec{v}, \vec{w}), \epsilon_2^{(i)}(u', \vec{v}', \vec{w}')) \\ &= s_i(u, \vec{v}, \vec{w}, \vec{w}', u', \vec{v}', \vec{w}', \vec{w}'), \end{aligned} \quad (3a)$$

$$\begin{aligned} & s_i(u, \vec{v}, \vec{w}, \vec{w}', u', \vec{v}', \vec{w}', \vec{w}) \\ &= \sigma_{i+1}(u, \vec{v}, \vec{w}, u', \vec{v}', \vec{w}', \eta_1^{(i+1)}(u, \vec{v}, \vec{w}), \eta_2^{(i+1)}(u', \vec{v}', \vec{w}')), \end{aligned} \quad (3b)$$

$$u = \sigma_1(u, \vec{v}, \vec{w}, u', \vec{v}', \vec{w}', \eta_1^{(1)}(u, \vec{v}, \vec{w}), \eta_2^{(1)}(u', \vec{v}', \vec{w}')), \quad (4a)$$

$$u' = \sigma_{N+1}(u, \vec{v}, \vec{w}, u', \vec{v}', \vec{w}', \epsilon_1^{(N+1)}(u, \vec{v}, \vec{w}), \epsilon_2^{(N+1)}(u', \vec{v}', \vec{w}')), \quad (4b)$$

where $\vec{v} = (v_1, \dots, v_k)$, $\vec{w} = (w_1, \dots, w_m)$ and analogously for \vec{v}' and \vec{w}' .

Examples (Martins-Ferreira (2012, 2015))

- *Mal'tsev categories*

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Let \mathcal{C} be a weakly Mal'tsev category. Then any multiplicative graph in \mathcal{C} is an internal category.

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Proposition (Martins-Ferreira, 2008)

Let \mathcal{C} be a weakly Mal'tsev category. Then any multiplicative graph in \mathcal{C} is an internal category.

Not any internal category in a weakly Mal'tsev category yields an internal groupoid:

$$C_0 := (\mathbb{N}, +),$$

$$C_1 := \{(a, b) \mid a, b \in \mathbb{N} \text{ with } a \leq b\}$$

yield a category in $\mathbf{CancCommMon}$ that doesn't allow for inverses.

Proposition (Martins-Ferreira–Van der Linden (2014))

For a weakly Mal'tsev category \mathcal{C} , the following conditions are equivalent:

- 1 Any internal category is an internal groupoid.*
- 2 Any preorder is an equivalence relation.*

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For a variety \mathbb{V} , the following conditions are equivalent:

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Proposition (Martins-Ferreira–Rodelo–Van der Linden (2014))

Let \mathcal{C} be an n -permutable category. Then any internal category is an internal groupoid.

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Proposition (Martins-Ferreira–Rodelo–Van der Linden (2014))

Let \mathcal{C} be an n -permutable category. Then any internal category is an internal groupoid.

Note: weakly Mal'tsev + Goursat \neq Mal'tsev

Theorem (Gran (1999))

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Let \mathcal{C} be an exact Mal'tsev category. Then $\text{Grpd}(\mathcal{C})$ is an exact Mal'tsev category.

In this case, an internal functor

$$\begin{array}{ccccc}
 & & i & & \\
 & & \downarrow & & \\
 C_2 & \xrightarrow{m} & C_1 & \begin{array}{c} \xrightarrow{d} \\ \xleftarrow{e} \\ \xrightarrow{c} \end{array} & C_0 \\
 & & \downarrow f_1 & & \downarrow f_0 \\
 D_2 & \xrightarrow{m} & D_1 & \begin{array}{c} \xrightarrow{d} \\ \xleftarrow{e} \\ \xrightarrow{c} \end{array} & D_0 \\
 & & \uparrow i & &
 \end{array}$$

in $\text{Grpd}(\mathcal{C})$ is a regular epimorphism if and only if f_0, f_1 are regular epimorphisms in \mathcal{C} .

Let \mathcal{C} be an exact Mal'tsev category. We have an adjunction

$$\begin{array}{ccc} & \xleftarrow{\pi_0} & \\ \mathcal{C} & \perp & \mathbf{Grpd}(\mathcal{C}) \\ & \xrightarrow{D} & \end{array}$$

where:

$$D(X) = (X = X)$$

$$\pi_0(C_1 \begin{smallmatrix} d \\ \rightrightarrows \\ c \end{smallmatrix} C_0) = \mathrm{Coeq}(d, c)$$

Let \mathcal{C} be an exact Mal'tsev category. We have an adjunction

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where:

$$D(X) = (X = X)$$

$$\pi_0(C_1 \rightrightarrows_C C_0) = \mathrm{Coeq}(d, c)$$

Discrete groupoids are closed under subobjects and quotients in groupoids.

$$\begin{array}{ccccc} X & \xrightarrow{e_1} \twoheadrightarrow & C_1 & & \\ 1_X \downarrow \uparrow 1_X & & d \downarrow \uparrow c & & \\ & & e & & \\ X & \xrightarrow{e_0} \twoheadrightarrow & C_0 & & \end{array}$$

Let \mathcal{C} be an exact category and \mathcal{X} be a replete full reflective subcategory of \mathcal{C} .

$$\begin{array}{ccc} & \downarrow & \\ \mathcal{C} & \xrightarrow{\quad I \quad} & \mathcal{X} \\ & \uparrow & \\ & \text{H} & \end{array}$$

We assume that $I\text{H} = 1_{\mathcal{X}}$.

Let \mathcal{C} be an exact category and \mathcal{X} be a replete full reflective subcategory of \mathcal{C} .

$$\mathcal{C} \begin{array}{c} \xrightarrow{\quad} \\ \perp \\ \xleftarrow{\quad} \\ \text{H} \end{array} \mathcal{X}$$

We assume that $\text{IH} = 1_{\mathcal{X}}$.

Definition

\mathcal{X} is called a **Birkhoff subcategory** of \mathcal{C} if it is closed under subobjects and quotients in \mathcal{C} .

Let \mathcal{C} be an exact category and \mathcal{X} be a replete full reflective subcategory of \mathcal{C} .

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Remark

The Birkhoff subcategories of a variety \mathbb{V} of universal algebras are exactly the subvarieties of \mathbb{V} .

Definition (Janelidze–Kelly (1994))

Let \mathcal{X} be a Birkhoff subcategory of an exact Mal'tsev category \mathcal{C} and $f : A \rightarrow B$ be a regular epimorphism in \mathcal{C} .

- 1 We call f a **trivial extension** if the naturality square

$$\begin{array}{ccc}
 A & \xrightarrow{\eta_A} & \mathrm{H}A \\
 f \downarrow & & \downarrow \mathrm{H}f \\
 B & \xrightarrow{\eta_B} & \mathrm{H}B
 \end{array}$$

is a pullback.

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- 2 We call f a **central extension** if p_1 in

$$\begin{array}{ccc} A \times_B A & \xrightarrow{p_2} & A \\ p_1 \downarrow & \lrcorner & \downarrow f \\ A & \xrightarrow{f} & B \end{array}$$

is a trivial extension.

Example

A surjective group homomorphism $f : A \rightarrow B$ is central with respect to the subcategory \mathbf{Ab} of abelian groups of the category \mathbf{Grp} of groups if and only if $\mathrm{Ker}(f) \subseteq Z(A)$.

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Example (Gran (2001))

An exact Mal'tsev category \mathcal{C} yields a Birkhoff subcategory of $\mathbf{Grpd}(\mathcal{C})$ and a regular epimorphism $(f_0, f_1) : \mathbb{C} \rightarrow \mathbb{D}$ in $\mathbf{Grpd}(\mathcal{C})$ is central with respect to \mathcal{C} if and only if it is a discrete fibration.

$$\begin{array}{ccc}
 C_1 & \xrightarrow{f_1} & D_1 \\
 c \downarrow & \lrcorner & \downarrow c \\
 C_0 & \xrightarrow{f_0} & D_0
 \end{array}$$

Let \mathcal{C} be an exact Mal'tsev category. Then the category

$$\mathrm{Grp}^2(\mathcal{C}) := \mathrm{Grpd}(\mathrm{Grpd}(\mathcal{C}))$$

of **internal double groupoids** in \mathcal{C} is exact Mal'tsev as well.

$$\begin{array}{ccccc}
 & & \xrightarrow{d^1} & & \\
 C_1^1 & \xleftarrow{e^1} & C_0^1 & & \\
 \uparrow & \xrightarrow{c^1} & \uparrow & & \\
 d_1 \downarrow & e_1 \uparrow & c_1 \downarrow & & d_0 \downarrow & e_0 \uparrow & c_0 \downarrow \\
 C_1^0 & \xleftarrow{e^0} & C_0^0 & & \\
 & \xrightarrow{c^0} & & &
 \end{array}$$

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$$\begin{array}{ccccc}
 & & d^1 & & \\
 & & \longrightarrow & & \\
 C_1^1 & \xleftarrow{e^1} & C_0^1 & & \\
 & & c^1 & & \\
 \uparrow & & \longrightarrow & & \uparrow \\
 d_1 \downarrow & e_1 \downarrow & c_1 \downarrow & & d_0 \downarrow & e_0 \downarrow & c_0 \downarrow \\
 C_1^0 & \xleftarrow{e^0} & C_0^0 & & \\
 & & c^0 & & \\
 & & \longrightarrow & &
 \end{array}$$

We consider, in $\mathrm{Grp}^2(\mathcal{C})$, the full replete subcategory

$$2\text{-Grpd}(\mathcal{C})$$

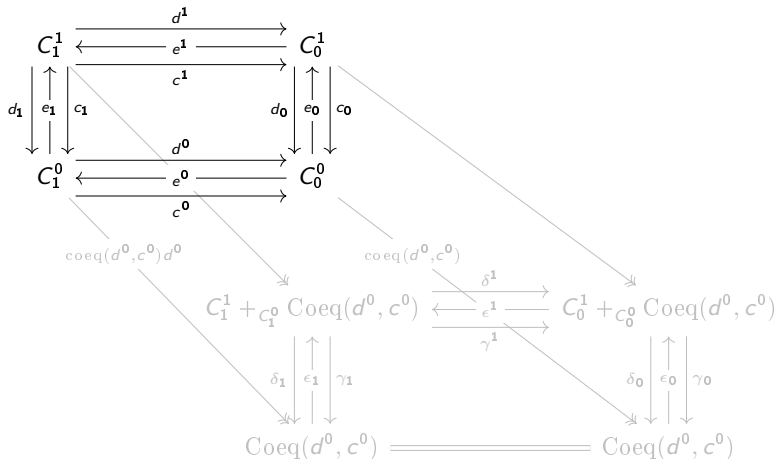
of **internal 2-groupoids** in \mathcal{C} , which contains all the double groupoids with $C_1^0 = C_0^0$ and $d^0 = 1_{C_0^0} = c^0$.

Proposition (E.–Jacqmin–Gran (2023))

Let \mathcal{C} be an exact Mal'tsev category with finite colimits. Then $2\text{-Grpd}(\mathcal{C})$ is a Birkhoff subcategory of $\text{Grpd}^2(\mathcal{C})$.

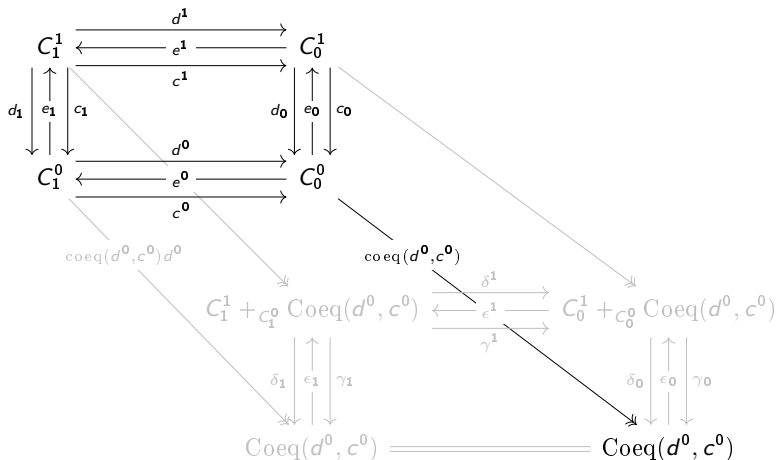
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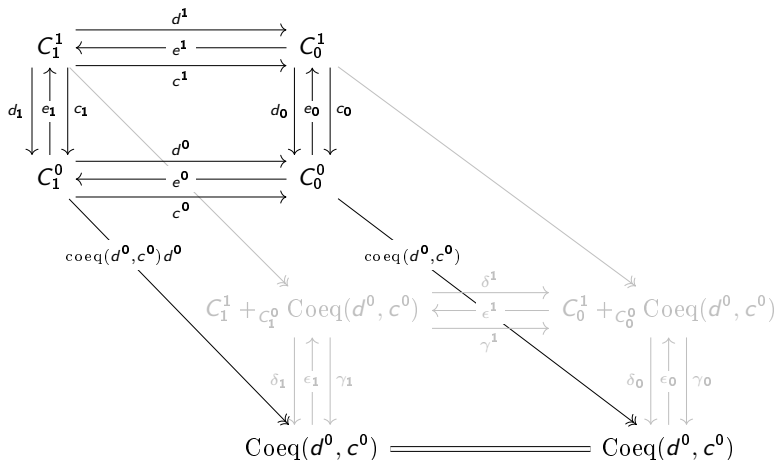
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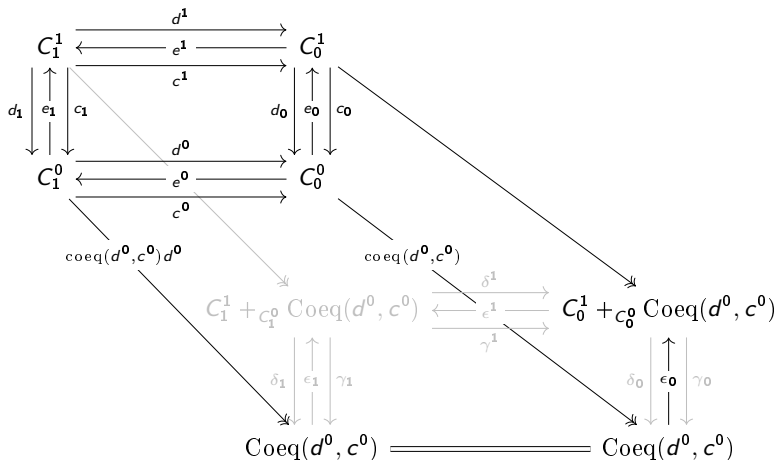
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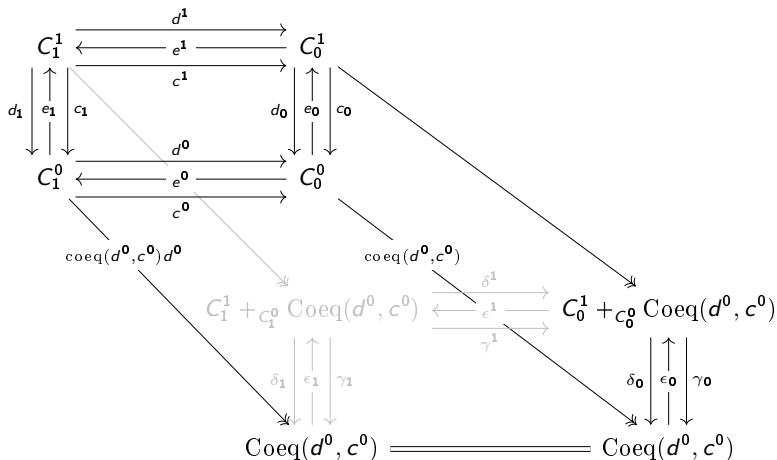
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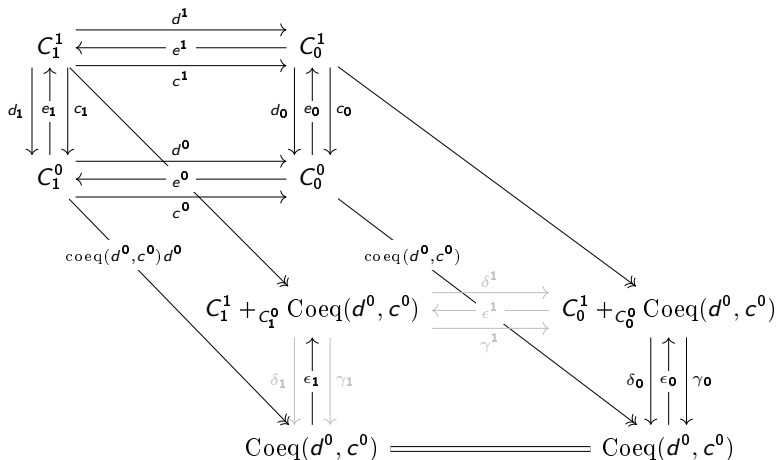
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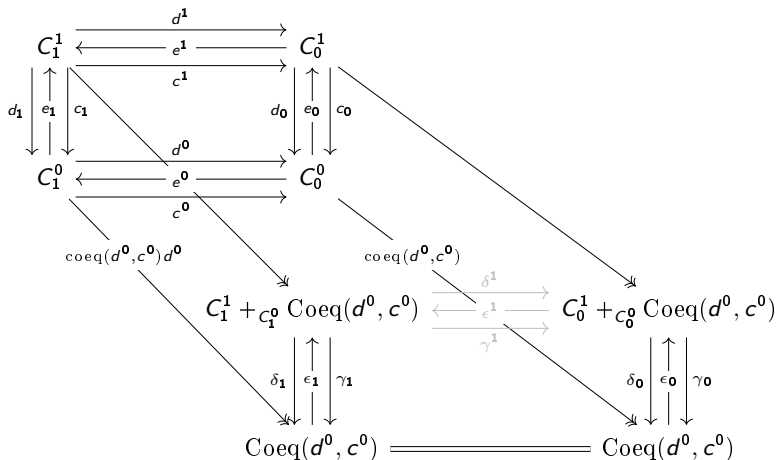
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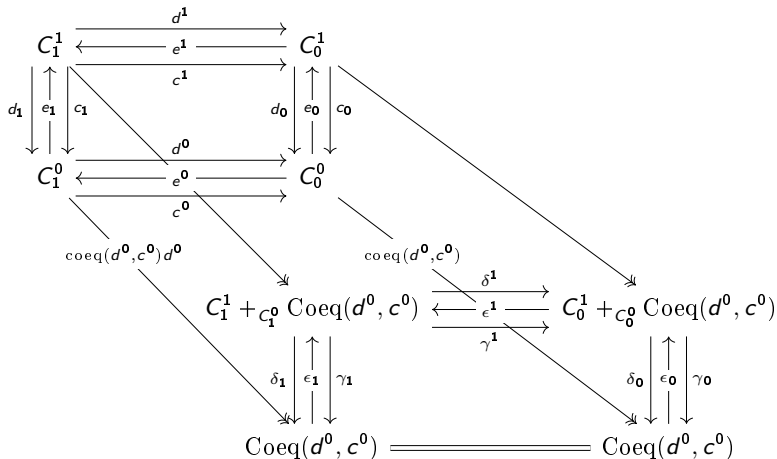
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Proposition

Let \mathcal{C} be a regular Mal'tsev category with coequalizers. Then $\mathbf{Gprd}(\mathcal{C})$ is a Birkhoff subcategory of $\mathbf{RG}(\mathcal{C})$.

Proposition

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Corollary

Let \mathbb{V} be a Mal'tsev variety of universal algebras. Then $\mathbf{Grpd}(\mathbb{V})$, $2\text{-}\mathbf{Grpd}(\mathbb{V})$, $\mathbf{Grpd}^2(\mathbb{V})$ are varieties.

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