Internal structures in Mal'tsev categories

Nadja Egner

Université Catholique de Louvain

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Outline

Let G be a group and $R \subseteq G \times G$ be a **reflexive** relation on G which is compatible with the group structure on G, i.e., R is a subgroup of $G \times G$.

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$$(y, x) = (x, x)(x^{-1}, y^{-1})(y, y) \in R.$$

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Thus, R is an **equivalence** relation.

Definition

Let $\mathscr{C}$ be a finitely complete category. An **internal equivalence relation** on an object X of $\mathscr{C}$ is a subobject

$$R \overset{r=(r_1,r_2)}{\rightarrowtail} X \times X$$

such that:

$$\begin{tikzcd} & R \arrow[d, tail, "r"] \\ X \arrow[ur, dashed, "\exists\delta"] \arrow[r, "\Delta_X"'] & X \times X \end{tikzcd} \qquad \begin{tikzcd} & R \arrow[d, tail, "(r_1,r_2)"] \\ R \arrow[ur, dashed, "\exists\sigma"] \arrow[r, tail, "(r_2,r_1)"'] & X \times X \end{tikzcd} \qquad \begin{tikzcd} & R \arrow[d, tail, "r"] \\ R \times_X R \arrow[ur, dashed, "\exists\tau"] \arrow[r, "(r_1 p_1, r_2 p_2)"'] & X \times X \end{tikzcd}$$

$$\begin{tikzcd} R \times_X R \arrow[r, "p_2"] \arrow[d, "p_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & R \arrow[d, "r_1"] \\ R \arrow[r, "r_2"'] & X \end{tikzcd}$$

Definition (Carboni–Lambek–Pedicchio (1990))

A finitely complete category $\mathscr{C}$ is a **Mal'tsev category** if every internal reflexive relation in $\mathscr{C}$ is an equivalence relation.

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Theorem (Carboni–Pedicchio–Pirovano (1992))

For a finitely complete category $\mathscr{C}$, the following conditions are equivalent:

1. *$\mathscr{C}$ is a Mal'tsev category.*
2. *Every reflexive relation in $\mathscr{C}$ is symmetric.*
3. *Every reflexive relation in $\mathscr{C}$ is transitive.*
4. *Every relation in $\mathscr{C}$ is difunctional.*

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4. *Every relation in $\mathscr{C}$ is difunctional.*

A relation $R \subseteq X \times Y$ in Set is **difunctional** if, for all $x, z \in X$ and $y, u \in Y$,

$$xRy, zRy, zRu \Rightarrow xRu.$$

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A regular category $\mathscr{C}$ is called **(Barr-)exact** if any internal equivalence relation is the kernel pair of some morphism.

In a regular category, any morphism factorizes as a regular epimorphism followed by a monomorphism.

$$\begin{tikzcd} \mathrm{Eq}[f] \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] & X \arrow[rr, "f"] \arrow[dr, two heads, "q"'] & & Y \\ & & I \arrow[ur, dashed, tail, "i"'] & \end{tikzcd}$$

Regular categories allow for a "good" calculus of relations.

$$
\begin{tikzcd}
 & & P \arrow[dl, "p_1"'] \arrow[dr, "p_2"] \arrow[ddll, bend right, "r_1 p_1"'] \arrow[ddrr, bend left, "s_2 p_2"] \arrow[ddd, bend left, "q"] \arrow[dd, phantom, "\lrcorner" rotate=-45, very near start] & & \\
 & R \arrow[dl, "r_1"'] \arrow[dr, "r_2"] & & S \arrow[dl, "s_1"'] \arrow[dr, "s_2"] & \\
X & & Y & & Z \\
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Theorem (Carboni–Lambek–Pedicchio (1990))

For a regular category $\mathscr{C}$, the following conditions are equivalent:

1. *$\mathscr{C}$ is a Mal'tsev category.*
2. *For any object $X \in \mathscr{C}$ and any two equivalence relations R, S on X, $S \circ R$ is an equivalence relation.*
3. *For any object $X \in \mathscr{C}$ and any two equivalence relations R, S on X, it holds that $S \circ R = R \circ S$.*

Theorem (Mal'tsev (1954))

For a variety $\mathbb{V}$ of universal algebras, the following conditions are equivalent:

1. *$\mathbb{V}$ is a Mal'tsev category.*
2. *Its algebraic theory contains a ternary term $p(x, y, z)$ such that*

$$p(x, x, y) = y = p(y, x, x).$$

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Examples (Mal'tsev varieties)

- Grp: $p(x, y, z) := xy^{-1}z$

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Examples (Mal'tsev varieties)

- Grp: $p(x,y,z) := xy^{-1}z$
- Heyt: $p(x,y,z) := ((x \Rightarrow y) \Rightarrow z) \wedge ((z \Rightarrow y) \Rightarrow x)$,
$q(x,y,x) := (y \Rightarrow (x \wedge z)) \wedge (x \vee z)$

Proposition

Let $\mathscr{C}$ be a Mal'tsev category and X be an object of $\mathscr{C}$. Then $\mathscr{C}/X$ and $X/\mathscr{C}$ are Mal'tsev categories.

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Proposition

Let $\mathscr{C}$ and $\mathscr{D}$ be finitely complete categories and $F : \mathscr{C} \to \mathscr{D}$ be a conservative functor which preserves pullbacks. If $\mathscr{D}$ is a Mal'tsev category, so is $\mathscr{C}$.

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- *Mal'tsev varieties internal to a finitely complete category:* $\mathrm{Grp}(\mathrm{Top})$

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- *Abelian and semi-abelian categories*
- *The dual of an elementary topos:* Set^{op}

Definition

Let $\mathscr{C}$ be a category with pullbacks. An **internal groupoid** $\mathbb{C}$ in $\mathscr{C}$ is given by a diagram

$$\begin{tikzcd} C_2 \arrow[r, "m"] & C_1 \arrow[loop above, "i"] \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "c"'] & C_0 \arrow[l, "e" description] \end{tikzcd}$$

in $\mathscr{C}$ such that:

$$\begin{tikzcd} C_2 \arrow[r, "p_2"] \arrow[d, "p_1"'] \arrow[dr, phantom, very near start, "\lrcorner"] & C_1 \arrow[d, "c"] \\ C_1 \arrow[r, "d"'] & C_0 \end{tikzcd} \qquad \begin{tikzcd} C_3 \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, very near start, "\lrcorner"] & C_1 \arrow[d, "c"] \\ C_2 \arrow[r, "dp_2"'] & C_0 \end{tikzcd} \qquad \begin{tikzcd} C_3 \arrow[r, "\pi'_2"] \arrow[d, "\pi'_1"'] \arrow[dr, phantom, very near start, "\lrcorner"] & C_2 \arrow[d, "cp_1"] \\ C_1 \arrow[r, "d"'] & C_0 \end{tikzcd}$$

(RG) $de = 1_{C_0} = ce$

(MG) $m\langle ec, 1_{C_1}\rangle = 1_{C_1} = m\langle 1_{C_1}, ed\rangle$

(C) $dm = dp_2,\ cm = cp_1,\ m\langle m\pi_1, \pi_2\rangle = m\langle \pi'_1, m\pi'_2\rangle$

(G) $di = c,\ ci = d,\ m\langle i, 1_{C_1}\rangle = ed,\ m\langle 1_{C_1}, i\rangle = ec$

Definition

Let $\mathscr{C}$ be a category with pullbacks, and $\mathbb{C}$ and $\mathbb{D}$ be two internal groupoids in $\mathscr{C}$. An **internal functor** $F : \mathbb{C} \to \mathbb{D}$ is given by morphisms $f_0 : C_0 \to D_0$ and $f_1 : C_1 \to D_1$ such that

$$\begin{tikzcd}
C_2 \arrow[r, "m"] \arrow[d, dashed] & C_1 \arrow[loop above, "i"] \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "c"'] \arrow[d, "f_1"] & C_0 \arrow[l, "e"'] \arrow[d, "f_0"] \\
D_2 \arrow[r, "m"'] & D_1 \arrow[loop below, "i"] \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "c"'] & D_0 \arrow[l, "e"']
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We denote by $\mathbf{Grpd}(\mathscr{C})$ ($\mathbf{Cat}(\mathscr{C})$, $\mathbf{MG}(\mathscr{C})$, $\mathbf{RG}(\mathscr{C})$) the category of internal groupoids (internal categories, multiplicative graphs, reflexive graphs) in $\mathscr{C}$ and internal functors between them.

Definition

A **crossed module of groups** is given by a group homomorphism $d : X \to B$ together with a group action $\triangleright : B \times X \to X$ such that for all $x, y \in X$ and $b \in B$:

(Equivariance)	$d(b \triangleright x) = bd(x)b^{-1}$
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A **morphism of crossed modules of groups** from (X, B) to (X', B') is given by two group homomorphism $f_1 : X \to X'$ and $f_0 : B \to B'$ such that for all $x \in X$ and $b \in B$:

$$f_0 d(x) = d' f_1(x)$$

$$f_1(b \triangleright x) = f_0(b) \triangleright f_1(x)$$

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$$f_1(b \triangleright x) = f_0(b) \triangleright f_1(x)$$

We denote by $\mathbf{XMod(Grp)}$ the category of crossed modules of groups and their corresponding morphisms.

Theorem (Brown–Spencer (1976))

The categories $\mathrm{Grpd}(\mathrm{Grp})$ *of internal groupoids in* Grp *and* $\mathrm{XMod}(\mathrm{Grp})$ *of crossed modules of groups are equivalent.*

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Let

$$\begin{tikzcd} C_1 \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "c"'] & C_0 \arrow[l, "e" description] \end{tikzcd}$$

be the underlying reflexive graph of an internal groupoid in Grp. Then

$$X := \mathrm{Ker}(d) \hookrightarrow C_1 \xrightarrow{c} C_0 =: B$$

and $b \rhd x := e(b)xe(b)^{-1}$ for all $x \in \mathrm{Ker}(d)$ and $b \in C_0$ define a crossed module of groups.

Theorem (Brown–Spencer (1976))

The categories $\mathrm{Grpd}(\mathrm{Grp})$ *of internal groupoids in* Grp *and* $\mathrm{XMod}(\mathrm{Grp})$ *of crossed modules of groups are equivalent.*

Let(X, B) be a crossed module of groups. Then

$$\begin{tikzcd} X \rtimes B \arrow[r, shift left=2, "p_2"] \arrow[r, shift right=2, "p_1"'] & B \arrow[l, "i_2"' description] \end{tikzcd},$$

where $p_2(x, b) := b$, $i_2(b) := (1, b)$ and $p_1(x, b) := d(x)b$ for all $x \in X$ and $b \in B$, is the underlying reflexive graph of an internal groupoid in Grp.

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Theorem (G. Janelidze (2003))

Let $\mathscr{C}$ *be a semi-abelian category. Then the categories* $\mathrm{Grpd}(\mathscr{C})$ *of internal groupoids in* $\mathscr{C}$ *and* $\mathrm{XMod}(\mathscr{C})$ *of internal crossed modules in* $\mathscr{C}$ *are equivalent.*

Proposition

Any reflexive graph in Grp *admits at most one multiplicative graph structure.*

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be a multiplicative graph in Grp, where

$$C_1 \times_{C_0} C_1 = \{(g, f) \in C_1 \times C_1 \mid d(g) = c(f)\}.$$

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$$\boldsymbol{m}(\boldsymbol{g}, \boldsymbol{f}) = m(g(ed(g))^{-1}ed(g), 1f)$$

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&= \boldsymbol{g(ed(g))^{-1}f}.
\end{aligned}$$

Proposition

For a reflexive graph $\mathbb{C}$,

$$\begin{tikzcd} C_1 \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "c"'] & C_0 \arrow[l, "e"'] \end{tikzcd},$$

in Grp, *the following conditions are equivalent:*

1. $[\mathrm{Ker}(d), \mathrm{Ker}(c)] = \{1\}$
2. $\mathbb{C}$ *admits a (unique) multiplicative graph structure.*
3. $\mathbb{C}$ *admits a (unique) category structure.*
4. $\mathbb{C}$ *admits a (unique) groupoid structure.*

$(1) \Rightarrow (2)$: Let us suppose that $[\mathrm{Ker}(d), \mathrm{Ker}(c)] = \{1\}$.

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$$\begin{aligned} \boldsymbol{m}((\boldsymbol{g}', \boldsymbol{f}')(\boldsymbol{g}, \boldsymbol{f})) &= m(g'g, f'f) \\ &= (g'g)(ed(g'g))^{-1}(f'f) \end{aligned}$$

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&= g'{\color{blue}(ec(f'))^{-1}f'}{\color{red}g(ed(g))^{-1}}f \\
&= g'(ed(g'))^{-1}f'g(ed(g))^{-1}f
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&= g'g(ed(g))^{-1}(ed(g'))^{-1} f'f \\
&= g'\color{red}{g(ed(g))^{-1}}\color{blue}{(ec(f'))^{-1} f'}\color{black}{f} \\
&= g'\color{blue}{(ec(f'))^{-1} f'}\color{red}{g(ed(g))^{-1}}\color{black}{f} \\
&= g'(ed(g'))^{-1} f' g(ed(g))^{-1} f \\
&= \boldsymbol{m}(\boldsymbol{g}', \boldsymbol{f}')\boldsymbol{m}(\boldsymbol{g}, \boldsymbol{f}).
\end{aligned}
$$

(1) $\Rightarrow$ (2): Let us suppose that $[\mathrm{Ker}(d), \mathrm{Ker}(c)] = \{1\}$.
We define $m : C_1 \times_{C_0} C_1 \to C_1$ by $m(g, f) := g(ed(g))^{-1} f$. Then

$$
\begin{aligned}
\boldsymbol{m((g', f')(g, f))} &= m(g'g, f'f) \\
&= (g'g)(ed(g'g))^{-1}(f'f) \\
&= g'g(ed(g))^{-1}(ed(g'))^{-1} f'f \\
&= g' {\color{red} g(ed(g))^{-1}} {\color{blue} (ec(f'))^{-1} f'} f \\
&= g' {\color{blue} (ec(f'))^{-1} f'} {\color{red} g(ed(g))^{-1}} f \\
&= g'(ed(g'))^{-1} f' g(ed(g))^{-1} f \\
&= \boldsymbol{m(g', f')m(g, f)}.
\end{aligned}
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Furthermore,

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\boldsymbol{m(ec(f), f)} = ec(f)(edec(f))^{-1} f
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$$\begin{aligned}
\boldsymbol{m}((\boldsymbol{g}', \boldsymbol{f}')(\boldsymbol{g}, \boldsymbol{f})) &= m(g'g, f'f) \\
&= (g'g)(ed(g'g))^{-1}(f'f) \\
&= g'g(ed(g))^{-1}(ed(g'))^{-1}f'f \\
&= g'g(ed(g))^{-1}(ec(f'))^{-1}f'f \\
&= g'(ec(f'))^{-1}f'g(ed(g))^{-1}f \\
&= g'(ed(g'))^{-1}f'g(ed(g))^{-1}f \\
&= \boldsymbol{m}(\boldsymbol{g}', \boldsymbol{f}')\boldsymbol{m}(\boldsymbol{g}, \boldsymbol{f}).
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Furthermore,

$$\boldsymbol{m}(\boldsymbol{ec}(\boldsymbol{f}), \boldsymbol{f}) = ec(f)(edec(f))^{-1}f = \boldsymbol{f},$$

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Furthermore,

$$\begin{aligned}
\boldsymbol{m(ec(f), f)} &= ec(f)(edec(f))^{-1} f = \boldsymbol{f}, \\
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$(2) \Rightarrow (1)$: Let us suppose that $\mathbb{C}$ admits a multiplicative graph structure.

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To (4): We define $i : C_1 \to C_1$ by

$$i(f) = ed(f)f^{-1}ec(f).$$

Theorem (Bourn (1996))

For a finitely complete category $\mathscr{C}$, the following conditions are equivalent:

1. *$\mathscr{C}$ is a Mal'tsev category.*
2. *$\mathrm{Pt}_X(\mathscr{C})$ is a Mal'tsev category for any object X of $\mathscr{C}$.*
3. *Given a pullback of split epimorphisms*

$$\begin{tikzcd} X \times_Z Y \arrow[r, shift left, two heads, "p_2"] \arrow[d, shift right, two heads, "p_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[l, shift left, tail, "e_2"] \arrow[d, shift left, two heads, "g"] \\ X \arrow[u, shift right, tail, "e_1"'] \arrow[r, shift right, two heads, "f"'] & Z \arrow[l, shift right, tail, "r"'] \arrow[u, shift left, tail, "s"] \end{tikzcd}$$

in $\mathscr{C}$, the pullback injections e_1, e_2 are jointly extremally epimorphic.

Proposition

Any reflexive graph in a Mal'tsev category $\mathscr{C}$ *admits at most one multiplicative graph structure.*

Proposition

Any reflexive graph in a Mal'tsev category $\mathscr{C}$ admits at most one multiplicative graph structure.

Let $\mathbb{C}$ be a multiplicative graph in $\mathscr{C}$. We consider

$$\begin{tikzcd}
C_1 & & \\
& C_1 \times_{C_0} C_1 \arrow[ul, "m"'] \arrow[r, shift left, two heads, "p_2"] \arrow[d, shift right, two heads, "p_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & C_1 \arrow[l, shift left, tail, "e_2"] \arrow[d, shift left, two heads, "c"] \arrow[ull, bend right, "1_{C_1}"'] \\
& C_1 \arrow[u, shift right, tail, "e_1"'] \arrow[r, shift right, two heads, "d"'] \arrow[uul, bend left, "1_{C_1}"] & C_0. \arrow[l, shift right, tail, "e"'] \arrow[u, shift left, tail, "e"]
\end{tikzcd}$$

Since $e_1 = \langle 1_{C_1}, ed \rangle$, $e_2 = \langle ec, 1_{C_1} \rangle$ are jointly (strongly) epimorphic, m is uniquely determined by the equations

$$m\langle ec, 1_{C_1} \rangle = 1_{C_1} = m\langle 1_{C_1}, ed \rangle.$$

Definition (Bourn–Gran (2002))

Let R, S be two equivalence relations on an object X in a finitely complete category $\mathscr{C}$. We consider the pullback

$$\begin{tikzcd} R \times_X S \arrow[r, "p_2"] \arrow[d, "p_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & S \arrow[d, "s_1"] \\ R \arrow[r, "r_2"'] & X. \end{tikzcd}$$

A **connector** between R and S is a map $p : R \times_X S \to X$ such that

1. $xSp(x,y,z)$ and $p(x,y,z)Rz$,
2. $p(x,x,y) = y = p(y,x,x)$,
3. $p(x,y,p(y,u,v)) = p(x,u,v)$ and $p(p(x,y,u),u,v) = p(x,y,v)$.

Proposition

Let $\mathbb{C}$,

$$C_1 \underset{c}{\overset{d}{\underset{\longleftarrow}{\overset{\longrightarrow}{\longrightarrow}}}}\; C_0,$$

be a reflexive graph in a finitely complete category $\mathscr{C}$. *The connectors between* $\mathrm{Eq}[d]$ *and* $\mathrm{Eq}[c]$ *are in one-to-one correspondence with the internal groupoid structures on* $\mathbb{C}$.

Proposition

Let $\mathbb{C}$*,*

$$\begin{tikzcd} C_1 \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "c"'] & C_0 \arrow[l, "e" description] \end{tikzcd},$$

be a reflexive graph in a finitely complete category $\mathscr{C}$*. The connectors between* $\mathrm{Eq}[d]$ *and* $\mathrm{Eq}[c]$ *are in one-to-one correspondence with the internal groupoid structures on* $\mathbb{C}$*.*

Proposition (Bourn–Gran (2002))

Let $\mathscr{C}$ *be a Mal'tsev category and* R, S *two equivalence relations on an object* X *of* $\mathscr{C}$*. Then there exists at most one map* $p : R \times_X S \to X$ *such that* $p(x, x, y) = y$ *and* $p(x, y, y) = x$ *and this map is a connector.*

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Proposition (Bourn–Gran (2002))

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In this case, one says that R and S **centralize** each other.

Proposition

For a reflexive graph $\mathbb{C}$,

$$C_1 \underset{c}{\overset{d}{\underset{\longleftarrow}{\overset{\longrightarrow}{\longrightarrow}}}}_{e} C_0,$$

in a Mal'tsev category $\mathscr{C}$, *the following conditions are equivalent:*

1. $\mathrm{Eq}[d]$ *and* $\mathrm{Eq}[c]$ *centralize each other.*
2. $\mathbb{C}$ *admits a (unique) multiplicative graph structure.*
3. $\mathbb{C}$ *admits a (unique) category structure.*
4. $\mathbb{C}$ *admits a (unique) groupoid structure.*

Given two subgroups H, I of a group G, we can abstract the condition $[H, I] = \{1\}$ to the pointed Mal'tsev setting.

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Definition

Let $\mathscr{C}$ be a pointed Mal'tsev category and $f : X \to Y$ and $g : X' \to Y$ be morphisms in $\mathscr{C}$. We say that f and g **commute** if there exists a (necessarily unique) morphism $\varphi : X \times X' \to Y$ such that

$$\begin{tikzcd} X \arrow[r, "(1_X,0)"] \arrow[dr, "f"'] & X \times X' \arrow[d, dotted, "\varphi"] & X' \arrow[l, "(0,1_{X'})"'] \arrow[dl, "g"] \\ & Y & \end{tikzcd}$$

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Definition

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commutes.

Then $[H, I] = \{1\}$ if and only if the inclusions $h : H \to G$ and $i : I \to G$ commute.

Given an equivalence relation R on an object X in a pointed Mal'tsev category $\mathscr{C}$, we can associate to it its *normalization*

$$\begin{tikzcd} N_R := \mathrm{Ker}(r_1) \arrow[r, "\ker(r_1)"] \arrow[rr, bend right, "n_R"'] & R \arrow[r, "r_2"] & X. \end{tikzcd}$$

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Given two equivalence relations R, S on X in $\mathscr{C}$ centralizing each other, n_R and n_S commute.

The converse is not necessarily true.

Proposition

Let $\mathscr{C}$ be a Mal'tsev category. Then $\mathrm{Grpd}(\mathscr{C})$ is a full subcategory of $\mathrm{RG}(\mathscr{C})$.

$$
\begin{tikzcd}
C_2 \arrow[r, "m"] \arrow[d, dashed] & C_1 \arrow[loop above, "i"] \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "c"'] \arrow[d, "f_1"] & C_0 \arrow[l, "e"'] \arrow[d, "f_0"] \\
D_2 \arrow[r, "m"'] & D_1 \arrow[loop below, "i"'] \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "c"'] & D_0 \arrow[l, "e"']
\end{tikzcd}
$$

Definition

Let $\mathscr{C}$ be a regular category. Then $\mathscr{C}$ is called n-**permutable** if, for any two equivalence relations R, S on an object X in $\mathscr{C}$, it holds that

$$\underbrace{R \circ S \circ R \circ S \circ \cdots}_{n \text{ factors}} = \underbrace{S \circ R \circ S \circ R \circ \cdots}_{n \text{ factors}}.$$

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- A 2-permutable category is exactly a regular Mal'tsev category.
- A 3-permutable category is also called a **Goursat** category.

Theorem (Bourn (1996))

For a finitely complete category $\mathscr{C}$, the following conditions are equivalent:

1. *$\mathscr{C}$ is a Mal'tsev category.*
2. *$\mathrm{Grpd}(\mathscr{C})$ is closed in $\mathrm{RG}(\mathscr{C})$ under subobjects.*

$$\begin{tikzcd}
C_1 \times_{C_0} C_1 \arrow[r, dashed, "m"] \arrow[d, tail] & C_1 \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "c"'] \arrow[d, tail, "f_1"'] & C_0 \arrow[l, "e" description] \arrow[d, tail, "f_0"] \\
G_1 \times_{G_0} G_1 \arrow[r, "m"'] & G_1 \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "c"'] & G_0 \arrow[l, "e" description]
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Proposition (Gran–Rodelo–Tchoffo Nguefeu (2017))

For a regular category $\mathscr{C}$, the following conditions are equivalent:

1. *$\mathscr{C}$ is a Goursat category.*
2. *$\mathrm{Grpd}(\mathscr{C})$ is closed in $\mathrm{RG}(\mathscr{C})$ under quotients.*

Definition (Martins-Ferreira (2008))

A category $\mathscr{C}$ is called a **weakly Mal'tsev category** if it has pullbacks of split epimorphisms along split epimorphisms and the pullback injections e_1, e_2 of such a pullback

$$\begin{tikzcd} X \times_Z Y \arrow[r, shift left, two heads, "p_2"] \arrow[d, shift right, two heads, "p_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[l, shift left, tail, "e_2"] \arrow[d, shift left, two heads, "g"] \\ X \arrow[u, shift right, tail, "e_1"'] \arrow[r, shift right, two heads, "f"'] & Z \arrow[l, shift right, tail, "r"'] \arrow[u, shift left, tail, "s"] \end{tikzcd}$$

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are jointly epimorphic.

Proposition

Any reflexive graph in a weakly Mal'tsev category $\mathscr{C}$ admits at most one multiplicative graph structure.

Let $\mathscr{C}$ be a finitely complete category. A **strong relation** between objects X and Y of $\mathscr{C}$ is given by a strong monomorphism $r : R \rightarrowtail X \times Y$.

$$\begin{tikzcd} A \arrow[r, two heads, "e"] \arrow[d, "a"'] & B \arrow[d, "b"] \arrow[dl, dashed, "\exists! d"] \\ R \arrow[r, tail, "r"'] & X \times Y \end{tikzcd}$$

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R \arrow[r, tail, "r"'] & X \times Y
\end{tikzcd}$$

Theorem (Z. Janelidze–Martins-Ferreira (2012))

For a finitely complete category $\mathscr{C}$, the following conditions are equivalent:

1. *$\mathscr{C}$ is a weakly Mal'tsev category.*
2. *Any reflexive strong relation in $\mathscr{C}$ is an equivalence relation.*
3. *Any reflexive strong relation in $\mathscr{C}$ is symmetric.*
4. *Any reflexive strong relation in $\mathscr{C}$ is transitive.*
5. *Any strong relation in $\mathscr{C}$ is difunctional.*

Theorem (E.–Jacqmin–Martins-Ferreira, 2023)

A variety $\mathbb{V}$ of universal algebras is a weakly Mal'tsev category if and only if there exist integers $k, m, N \geqslant 0$, binary terms $f_1, g_1, \ldots, f_k, g_k$, ternary terms $p_1, \ldots, p_m$, $(2(k+2m+1))$-ary terms $s_1, \ldots, s_N$, $(2(k+m+2))$-ary terms $\sigma_1, \ldots, \sigma_{N+1}$ and, for all $i \in \{1, \ldots, N+1\}$, $(k+m+1)$-ary terms $\eta_1^{(i)}, \eta_2^{(i)}, \epsilon_1^{(i)}, \epsilon_2^{(i)}$ such that, for all $i \in \{1, \ldots, k\}$, $j \in \{1, \ldots, N+1\}$ and $\alpha \in \{1, 2\}$,

$$f_i(x,x) = g_i(x,x) \tag{1}$$

$$\begin{aligned} &\eta_\alpha^{(j)}(y, f_1(x,y), \ldots, f_k(x,y), p_1(x,x,y), \ldots, p_m(x,x,y)) \\ =&\epsilon_\alpha^{(j)}(y, f_1(x,y), \ldots, f_k(x,y), p_1(x,x,y), \ldots, p_m(x,x,y)), \end{aligned} \tag{2a}$$

$$\begin{aligned} &\eta_\alpha^{(j)}(x, g_1(x,y), \ldots, g_k(x,y), p_1(x,y,y), \ldots, p_m(x,y,y)) \\ =&\epsilon_\alpha^{(j)}(x, g_1(x,y), \ldots, g_k(x,y), p_1(x,y,y), \ldots, p_m(x,y,y)), \end{aligned} \tag{2b}$$

Theorem

for all $i \in \{1, \dots, N\}$,

$$\begin{aligned}&\sigma_i(u, \vec{v}, \vec{w}, u', \vec{v}', \vec{w}', \epsilon_1^{(i)}(u, \vec{v}, \vec{w}), \epsilon_2^{(i)}(u', \vec{v}', \vec{w}'))\\ =&s_i(u, \vec{v}, \vec{w}, \vec{w}, u', \vec{v}', \vec{w}', \vec{w}'),\end{aligned} \tag{3a}$$

$$\begin{aligned}&s_i(u, \vec{v}, \vec{w}, \vec{w}', u', \vec{v}', \vec{w}', \vec{w})\\ =&\sigma_{i+1}(u, \vec{v}, \vec{w}, u', \vec{v}', \vec{w}', \eta_1^{(i+1)}(u, \vec{v}, \vec{w}), \eta_2^{(i+1)}(u', \vec{v}', \vec{w}')),\end{aligned} \tag{3b}$$

$$u = \sigma_1(u, \vec{v}, \vec{w}, u', \vec{v}', \vec{w}', \eta_1^{(1)}(u, \vec{v}, \vec{w}), \eta_2^{(1)}(u', \vec{v}', \vec{w}')), \tag{4a}$$

$$u' = \sigma_{N+1}(u, \vec{v}, \vec{w}, u', \vec{v}', \vec{w}', \epsilon_1^{(N+1)}(u, \vec{v}, \vec{w}), \epsilon_2^{(N+1)}(u', \vec{v}', \vec{w}')), \tag{4b}$$

where $\vec{v} = (v_1, \dots, v_k)$, $\vec{w} = (w_1, \dots, w_m)$ *and analogosuly for* $\vec{v}'$ *and* $\vec{w}'$.

Examples (Martins-Ferreira (2012, 2015))

- *Mal'tsev categories*

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Proposition (Martins-Ferreira, 2008)

Let $\mathscr{C}$ be a weakly Mal'tsev category. Then any multiplicative graph in $\mathscr{C}$ is an internal category.

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Proposition (Martins-Ferreira, 2008)

Let $\mathscr{C}$ be a weakly Mal'tsev category. Then any multiplicative graph in $\mathscr{C}$ is an internal category.

Not any internal category in a weakly Mal'tsev category yields an internal groupoid:

$$C_0 := (\mathbb{N}, +),$$
$$C_1 := \{(a, b) \mid a, b \in \mathbb{N} \text{ with } a \leqslant b\}$$

yield a category in $\mathrm{CancCommMon}$ that doesn't allow for inverses.

Proposition (Martins-Ferreira–Van der Linden (2014))

For a weakly Mal'tsev category $\mathscr{C}$, the following conditions are equivalent:

1. *Any internal category is an internal groupoid.*
2. *Any preorder is an equivalence relation.*

Proposition (Martins-Ferreira–Van der Linden (2014))

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Note: weakly Mal'tsev + Goursat $\neq$ Mal'tsev

Theorem (Gran (1999))

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In this case, an internal functor

$$\begin{tikzcd}
C_2 \arrow[r, "m"] \arrow[d, dashed] & C_1 \arrow[loop above, "i"] \arrow[r, shift left=3, "d"] \arrow[r, shift right=3, "c"'] \arrow[d, "f_1"] & C_0 \arrow[l, "e"'] \arrow[d, "f_0"] \\
D_2 \arrow[r, "m"'] & D_1 \arrow[loop below, "i"] \arrow[r, shift left=3, "d"] \arrow[r, shift right=3, "c"'] & D_0 \arrow[l, "e"']
\end{tikzcd}$$

in $\mathrm{Grpd}(\mathrm{C})$ is a regular epimorphism if and only f_0, f_1 are regular epimorphisms in $\mathscr{C}$.

Let $\mathscr{C}$ be an exact Mal'tsev category. We have an adjunction

$$\begin{tikzcd} \mathscr{C} \arrow[rr, bend right, "\mathsf{D}"'] \arrow[rr, phantom, "\perp"] & & \mathrm{Grpd}(\mathscr{C}), \arrow[ll, bend right, "\pi_0"'] \end{tikzcd}$$

where:

$$\mathsf{D}(X) = (X = X)$$

$$\pi_0(C_1 \underset{c}{\overset{d}{\rightrightarrows}} C_0) = \mathrm{Coeq}(d, c)$$

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Discrete groupoids are closed under subobjects and quotients in groupoids.

$$\begin{tikzcd} X \arrow[r, two heads, "e_1"] \arrow[d, shift right=6, "1_X"'] \arrow[d, shift left=6, "1_X"] & C_1 \arrow[d, shift right=6, "d"'] \arrow[d, shift left=6, "c"] \\ X \arrow[r, two heads, "e_0"'] \arrow[u, "1_X" description] & C_0 \arrow[u, "e" description] \end{tikzcd}$$

Let $\mathscr{C}$ be an exact category and $\mathscr{X}$ be a replete full reflective sub-category of $\mathscr{C}$.

$$\begin{tikzcd} \mathscr{C} \arrow[r, bend left, "\mathsf{I}"] \arrow[r, phantom, "\perp"] & \mathscr{X} \arrow[l, bend left, "\mathsf{H}"] \end{tikzcd}$$

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Remark

The Birkhoff subcategories of a variety $\mathbb{V}$ of universal algebras are exactly the subvarieties of $\mathbb{V}$.

Definition (Janelidze–Kelly (1994))

Let $\mathscr{X}$ be a Birkhoff subcategory of an exact Mal'tsev category $\mathscr{C}$ and $f : A \to B$ be a regular epimorphism in $\mathscr{C}$.

1. We call f a **trivial extension** if the naturality square

$$\begin{tikzcd} A \arrow[r, two heads, "\eta_A"] \arrow[d, two heads, "f"'] & \mathrm{HI}A \arrow[d, two heads, "\mathrm{HI}(f)"] \\ B \arrow[r, two heads, "\eta_B"'] & \mathrm{HI}B \end{tikzcd}$$

is a pullback.

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   is a pullback.

2. We call f a **central extension** if p_1 in

$$\begin{tikzcd} A \times_B A \arrow[r, "p_2"] \arrow[d, "p_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "f"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

   is a trivial extension.

Example

A surjective group homomorphism $f : A \to B$ *is central with respect to the subcategory* Ab *of abelian groups of the category* Grp *of groups if and only if* $\mathrm{Ker}(f) \subseteq Z(A)$.

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Example (Gran (2001))

An exact Mal'tsev category $\mathscr{C}$ yields a Birkhoff subcategory of $\mathrm{Grpd}(\mathscr{C})$ and a regular epimorphism $(f_0, f_1) : \mathbb{C} \to \mathbb{D}$ in $\mathrm{Grpd}(\mathscr{C})$ is central with respect to $\mathscr{C}$ if and only if it is a discrete fibration.

$$\begin{tikzcd}
C_1 \arrow[r, "f_1"] \arrow[d, "c"'] \arrow[dr, phantom, "\lrcorner", very near start] & D_1 \arrow[d, "c"] \\
C_0 \arrow[r, "f_0"'] & D_0
\end{tikzcd}$$

Let $\mathscr{C}$ be an exact Mal'tsev category. Then the category

$$\mathrm{Grp}^2(\mathscr{C}) := \mathrm{Grpd}(\mathrm{Grpd}(\mathscr{C}))$$

of **internal double groupoids** in $\mathscr{C}$ is exact Mal'tsev as well.

$$\begin{tikzcd}[column sep=huge, row sep=huge]
C_1^1 \arrow[r, shift left=4, "d^1"] \arrow[r, shift right=4, "c^1"'] \arrow[d, shift right=4, "d_1"'] \arrow[d, shift left=4, "c_1"] & C_0^1 \arrow[l, "e^1" description] \arrow[d, shift right=4, "d_0"'] \arrow[d, shift left=4, "c_0"] \\
C_1^0 \arrow[u, "e_1" description] \arrow[r, shift left=4, "d^0"] \arrow[r, shift right=4, "c^0"'] & C_0^0 \arrow[l, "e^0" description] \arrow[u, "e_0" description]
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$$\begin{tikzcd}
C_1^1 \arrow[r, shift left=2, "d^1"] \arrow[r, shift right=2, "c^1"'] \arrow[d, shift right=3, "d_1"'] \arrow[d, shift left=3, "c_1"] & C_0^1 \arrow[l, "e^1" description] \arrow[d, shift right=3, "d_0"'] \arrow[d, shift left=3, "c_0"] \\
C_1^0 \arrow[r, shift left=2, "d^0"] \arrow[r, shift right=2, "c^0"'] \arrow[u, "e_1" description] & C_0^0 \arrow[l, "e^0" description] \arrow[u, "e_0" description]
\end{tikzcd}$$

We consider, in $\mathrm{Grpd}^2(\mathscr{C})$, the full replete subcategory

$$2\text{–}\mathrm{Grpd}(\mathscr{C})$$

of **internal** 2-**groupoids** in $\mathscr{C}$, which contains all the double groupoids with $C_1^0 = C_0^0$ and $d^0 = 1_{C_0^0} = c^0$.

Proposition (E.–Jacqmin–Gran (2023))

Let $\mathscr{C}$ be an exact Mal'tsev category with finite colimits. Then $2\text{–}\mathrm{Grpd}(\mathscr{C})$ is a Birkhoff subcategory of $\mathrm{Grpd}^2(\mathscr{C})$.

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\begin{tikzcd}
C_1^1 \arrow[rr, shift left=2, "d^1"] \arrow[rr, shift right=2, "c^1"'] \arrow[d, shift right=2, "d_1"'] \arrow[d, shift left=2, "c_1"] \arrow[ddr, ->>] & & C_0^1 \arrow[ll, "e^1" description] \arrow[d, shift right=2, "d_0"'] \arrow[d, shift left=2, "c_0"] \arrow[ddr, ->>] & \\
C_1^0 \arrow[rr, shift left=2, "d^0"] \arrow[rr, shift right=2, "c^0"'] \arrow[u, "e_1" description] \arrow[ddr, ->>, "\mathrm{coeq}(d^0,c^0)d^0"'] & & C_0^0 \arrow[ll, "e^0" description] \arrow[u, "e_0" description] \arrow[ddr, "\mathrm{coeq}(d^0,c^0)"] & \\
& C_1^1 +_{C_1^0} \mathrm{Coeq}(d^0,c^0) \arrow[rr, shift left=2, "\delta^1"] \arrow[rr, shift right=2, "\gamma^1"'] \arrow[d, shift right=2, "\delta_1"'] \arrow[d, shift left=2, "\gamma_1"] & & C_0^1 +_{C_0^0} \mathrm{Coeq}(d^0,c^0) \arrow[ll, "\epsilon^1" description] \arrow[d, shift right=2, "\delta_0"'] \arrow[d, shift left=2, "\gamma_0"] \\
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\begin{tikzcd}[column sep=large, row sep=large]
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$$
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$$
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C_1^0 \arrow[r, shift left=2, "d^0"] \arrow[r, shift right=2, "c^0"'] \arrow[u, "e_1" description] \arrow[ddr, two heads, "\mathrm{coeq}(d^0,c^0)d^0" description] & C_0^0 \arrow[l, "e^0" description] \arrow[u, "e_0" description] \arrow[ddr, two heads, "\mathrm{coeq}(d^0,c^0)" description] & \\
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C_1^0 \arrow[u, "e_1" description] \arrow[r, shift left=2, "d^0"] \arrow[r, shift right=2, "c^0"'] \arrow[ddr, two heads, "\mathrm{coeq}(d^0,c^0)d^0" description] & C_0^0 \arrow[u, "e_0" description] \arrow[l, "e^0" description] \arrow[ddr, two heads, "\mathrm{coeq}(d^0,c^0)" description] & \\
& C_1^1 +_{C_1^0} \mathrm{Coeq}(d^0,c^0) \arrow[r, shift left=2, "\delta^1"] \arrow[r, shift right=2, "\gamma^1"'] \arrow[d, shift right=3, "\delta_1"'] \arrow[d, shift left=3, "\gamma_1"] & C_0^1 +_{C_0^0} \mathrm{Coeq}(d^0,c^0) \arrow[l, "\epsilon^1" description] \arrow[d, shift right=3, "\delta_0"'] \arrow[d, shift left=3, "\gamma_0"] \\
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C_1^0 \arrow[u, "e_1"] \arrow[rr, shift left=2, "d^0"] \arrow[rr, shift right=2, "c^0"'] \arrow[dddrr, two heads, "\mathrm{coeq}(d^0, c^0)d^0"'] & & C_0^0 \arrow[ll, "e^0"'] \arrow[u, "e_0"] \arrow[dddrr, two heads, "\mathrm{coeq}(d^0, c^0)"] & & \\
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C_1^1 \arrow[r, shift left=2, "d^1"] \arrow[r, shift right=2, "c^1"'] \arrow[d, shift right=2, "d_1"'] \arrow[d, shift left=2, "c_1"] \arrow[ddr, two heads] & C_0^1 \arrow[l, "e^1"' description] \arrow[d, shift right=2, "d_0"'] \arrow[d, shift left=2, "c_0"] \arrow[ddr, two heads] & \\
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C_1^1 \arrow[r, shift left=2, "d^1"] \arrow[r, shift right=2, "c^1"'] \arrow[d, shift right=2, "d_1"'] \arrow[d, shift left=2, "c_1"] \arrow[ddr, two heads] & C_0^1 \arrow[l, "e^1" description] \arrow[d, shift right=2, "d_0"'] \arrow[d, shift left=2, "c_0"] \arrow[ddrr, two heads] & & \\
C_1^0 \arrow[u, "e_1" description] \arrow[r, shift left=2, "d^0"] \arrow[r, shift right=2, "c^0"'] \arrow[ddr, two heads, "\mathrm{coeq}(d^0,c^0)d^0"'] & C_0^0 \arrow[l, "e^0" description] \arrow[u, "e_0" description] \arrow[ddrr, two heads, "\mathrm{coeq}(d^0,c^0)"] & & \\
& C_1^1 +_{C_1^0} \mathrm{Coeq}(d^0,c^0) \arrow[rr, shift left=2, "\delta^1"] \arrow[rr, shift right=2, "\gamma^1"'] \arrow[d, shift right=2, "\delta_1"'] \arrow[d, shift left=2, "\gamma_1"] & & C_0^1 +_{C_0^0} \mathrm{Coeq}(d^0,c^0) \arrow[ll, "\epsilon^1" description] \arrow[d, shift right=2, "\delta_0"'] \arrow[d, shift left=2, "\gamma_0"] \\
& \mathrm{Coeq}(d^0,c^0) \arrow[u, "\epsilon_1" description] \arrow[rr, equal] & & \mathrm{Coeq}(d^0,c^0) \arrow[u, "\epsilon_0" description]
\end{tikzcd}$$

Proposition

Let $\mathscr{C}$ be a regular Mal'tsev category with coequalizers. Then $\mathrm{Gprd}(\mathscr{C})$ is a Birkhoff subcategory of $\mathrm{RG}(\mathscr{C})$.

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Corollary

Let $\mathbb{V}$ be a Mal'tsev variety of universal algebras. Then $\mathrm{Grpd}(\mathbb{V})$, $2\!-\!\mathrm{Grpd}(\mathbb{V})$, $\mathrm{Grpd}^2(\mathbb{V})$ are varieties.

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