Measuring how much a model is not positively closed

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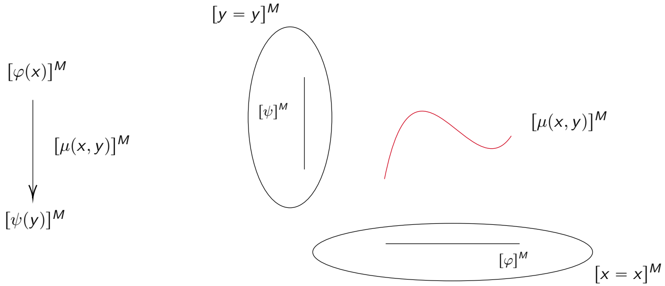
Category Theory Octoberfest 2023

Categories in logic

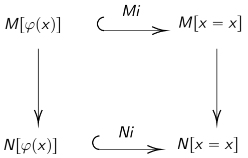
- L is a signature, M is an L-structure $\rightsquigarrow$ $Def(M)$ is the category of definable sets and definable functions.



- Idea: this is the functorial image of some "category of formulas" $\mathcal{C}$.

Categories in logic

- L is a signature, M is an L-structure $\rightsquigarrow$ $Def(M)$ is the category of definable sets and definable functions.
- Idea: this is the functorial image of some "category of formulas" $\mathcal{C}$.
- Two observations:
  - $Def(M)$ is closed under some limits/colimits (fin. products, fin. unions,...), so these should exist in $\mathcal{C}$, and M should preserve them.
  - A natural transformation is a map preserving the formulas in $\mathcal{C}$.



$\mathcal{C}$ = pos. ex. $\rightsquigarrow$ nat. tr. = homomorphisms
$\mathcal{C} = L_{\omega\omega}$ $\rightsquigarrow$ nat. tr. = elementary maps.

Definition (coherent theory)

positive existential: atomic, $\top, \wedge, \bot, \vee, \exists$
$L^g_{\omega\omega}$ (coherent): $\forall \vec{x}(\varphi(\vec{x}) \rightarrow \psi(\vec{x}))$, where φ, ψ are pos. ex. $\varphi(\vec{x}) \Rightarrow \psi(\vec{y})$

Definition (coherent category)

$\mathcal{C}$ is coherent if it has finite limits, pb.-stable eff. epi-mono factorization, pb.-stable finite unions
$F : \mathcal{C} \rightarrow \mathcal{D}$ is coherent if it preserves fin. limits, effective epis, fin. unions.

coherent theories	=	coherent categories
interpretations/ models	=	coherent functors
homomorphisms	=	natural transformations

Theorem ($\subseteq$; [Makkai-Reyes,1977])

$T \subseteq L^g_{\omega\omega} \rightsquigarrow \mathcal{C}_T$ ("the category of positive existential formulas") is a coherent category, s.t. $Mod(T) \simeq \mathbf{Coh}(\mathcal{C}_T, \mathbf{Set})$.

The syntactic category $\mathcal{C}_T$

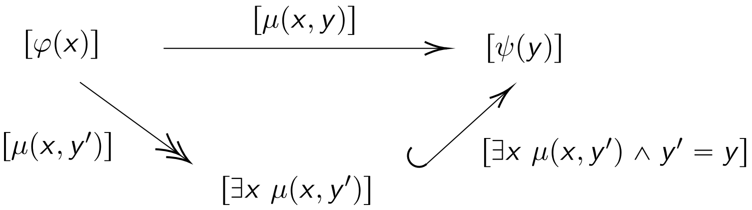
Objects: $[\varphi(\vec{x})]$: pos. ex. formulas up to (valid) renaming of free variables.

Arrows: $[\mu(\vec{x},\vec{y})] : [\varphi(\vec{x})] \to [\psi(\vec{y})]$: T-provably functional pos. ex. formulas, up to T-provable equivalence

$$\vec{x} \cap \vec{y} = \emptyset$$
$$T \vdash \mu(\vec{x},\vec{y}) \Rightarrow \varphi(\vec{x}) \wedge \psi(\vec{y})$$
$$T \vdash \varphi(\vec{x}) \Rightarrow \exists \vec{y} \mu(\vec{x},\vec{y})$$
$$T \vdash \exists \vec{x}(\mu(\vec{x},\vec{y}) \wedge \mu(\vec{x},\vec{y'})) \Rightarrow \vec{y} \approx \vec{y'}$$

identity: $[\varphi(\vec{x}) \wedge \vec{x} \approx \vec{x'}] : [\varphi(\vec{x})] \to [\varphi(\vec{x'})]$

composition: $[\exists \vec{y}(\mu(\vec{x},\vec{y}) \wedge \nu(\vec{y},\vec{z}))] : [\varphi(\vec{x})] \xrightarrow{[\mu]} [\psi(\vec{y})] \xrightarrow{[\nu]} [\chi(\vec{z})]$



images:

$$Sub_{\mathcal{C}_T}([\psi(y)]) = \{\chi(y) : T \vdash \chi(y) \Rightarrow \psi(y)\} \big/ \sim_T$$

$\mathcal{C}_T$ is the "gluing of the pos. ex. Lindenbaum-Tarski algebras".

Theorem (⊇; [Makkai-Reyes,1977])

Every small coherent category is the syntactic category of some (many-sorted) coherent theory.

⇝ work in categorical language, with logical intuition

The lattice LM

Definition

$\mathcal{C}$ is coherent.

$Sub_{\mathcal{C}} : \mathcal{C}^{op} \to \mathbf{DLat}$

$S_{\mathcal{C}} : \mathcal{C} \xrightarrow{Sub_{\mathcal{C}}} \mathbf{DLat}^{op} \xrightarrow{Spec} \mathbf{Spec}$

$f : x \to y$

$Sub(y) \xrightarrow{f^*} Sub(x)$

$Spec(Sub(x)) \xrightarrow{(f^*)^{-1}} Spec(Sub(y))$

Definition

$$\begin{array}{ccccc} & & \mathcal{C}^{op} & \xrightarrow{\quad Sub_{\mathcal{C}} \quad} & \mathbf{DLat} \\ & & {\scriptstyle y}\big\downarrow & \cong & \nearrow {\scriptstyle Lan_y Sub_{\mathcal{C}}} \\ \mathbf{Coh}(\mathcal{C}, \mathbf{Set}) & \hookrightarrow & \mathbf{Lex}(\mathcal{C}, \mathbf{Set}) & & \end{array}$$

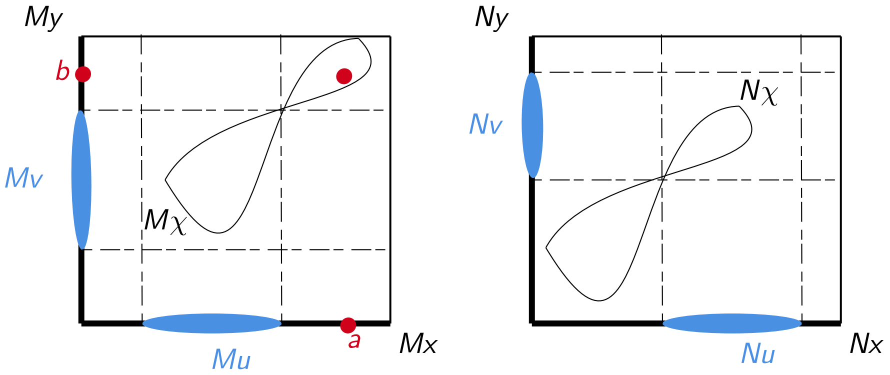
$M \mapsto LM = L_{\mathcal{C}} M$

$$LM = colim_{(x,a)\in (\int M)^{op}} Sub_{\mathcal{C}}(x)$$

$$\begin{array}{ccc} & LM = colim_{(x,a)\in (\int M)^{op}} Sub_{\mathcal{C}}(x) & \\ \nearrow & & \nwarrow \\ Sub_{\mathcal{C}}(y)^{b=Mf(a)} & \xrightarrow{f^*} & Sub_{\mathcal{C}}(x)^a \end{array}$$

- Categorically:
  LM is the set ${\{u \hookrightarrow x^a\}}/{\sim}$ where $u \hookrightarrow x^a \sim v \hookrightarrow y^b$ if there's $\chi \hookrightarrow x \times y$ s.t. $(a,b) \in M\chi$ and $\chi \cap u \times y = \chi \cap x \times v$.

- Logically:
  LM is the set of closed pos. ex. formulas with parameters from M up to: $\varphi(\vec{a}) \sim \psi(\vec{b})$ if there's pos. ex. $\chi(\vec{x},\vec{y})$, s.t. $M \models \chi(\vec{a},\vec{b})$ and $T \vdash \chi \wedge \varphi \Leftrightarrow \chi \wedge \psi$



$a \in Mu,\ b \in Mv \quad \rightsquigarrow \quad u \hookrightarrow x^a \sim v \hookrightarrow y^b \ (\chi = u \times y \ \cap \ x \times v)$

$a \in Mu,\ b \notin Mv \quad \rightsquigarrow \quad u \hookrightarrow x^a \nsim v \hookrightarrow y^b$

$a \notin Mu,\ b \in Mv \quad \rightsquigarrow \quad u \hookrightarrow x^a \nsim v \hookrightarrow y^b$

$a \notin Mu,\ b \notin Mv \quad \rightsquigarrow$ who knows?

Definition ([Barr,1986])

$\mathcal{C}$, $F, G : \mathcal{C} \to \textbf{Set}$ lex. A natural transformation $\eta : F \Rightarrow G$ is elementary if for each mono $u \hookrightarrow x$ in $\mathcal{C}$, the naturality square is a pullback.

Theorem (e.g. [Poizat-Yeshkeyev,2018])

$\mathcal{C}$, $M : \mathcal{C} \to \textbf{Set}$ coherent. TFAE:

1. *For any $N : \mathcal{C} \to \textbf{Set}$ coherent, every nat. tr. $\eta : M \Rightarrow N$ is elem.*
2. *For any $w \hookrightarrow z$ mono and $c \in Mz \setminus Mw$ there's $\chi \hookrightarrow z$ with $c \in M\chi$ and $w \cap \chi = \emptyset$.*

In this case M is said to be positively closed.

Proposition

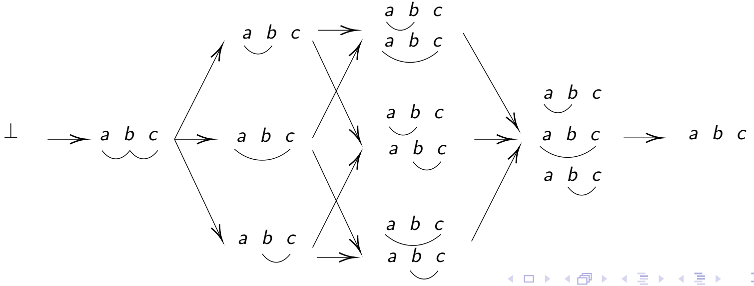
M is positively closed iff $LM = 2$

$\Rightarrow$: $(a, b) \in M(x \times y) \setminus M(u \times y \ \cup \ x \times v) \rightsquigarrow$ there's χ.
$\Leftarrow$: $c \in Mz \setminus Mw \rightsquigarrow w \hookrightarrow z^c \not\sim 1 \hookrightarrow 1^* \rightsquigarrow w \hookrightarrow z^c \sim \emptyset \hookrightarrow 1^* \rightsquigarrow$ there's χ.

Examples

- $\mathcal{C} = K$ is a dist. lattice, $M = p : K \to 2$ is a prime filter:
  $L_K p = {}^{K}\!\diagup_{\sim_p}$ where $u \sim_p v$ iff there's $w \in p$ with $u \cap w = v \cap w$.
  (Remark: $Lp = 2$ iff p is maximal)
- $\Sigma = \{\{X\}; \varnothing; \varnothing; \varnothing\}$ $T = \{\exists x : x = x\} \subseteq \Sigma_{\omega\omega}$
  models (M) are non-empty sets, homomorphisms are functions.
  LM: finite lists of finitely supported partitions of M, where in each list we only keep the finest partitions.

When $M = \{a, b, c\}$:



Prime filteres on LM

$Lan_y Sub_{\mathcal{C}} \dashv \mathbf{DLat}(Sub_{\mathcal{C}}(-), \bullet)$, i.e. for $F : \mathcal{C} \to \mathbf{Set}$ lex and K dist. lat.:

$$\begin{array}{rcl} \mathbf{DLat}(LF, K) & \cong & Nat(F, \mathbf{DLat}(Sub_{\mathcal{C}}(-), K)) \\ Spec(LF) & \cong & Nat(F, S_{\mathcal{C}}) \end{array}$$

explicitly:

$$\begin{array}{lcl} p \subseteq LF & \rightsquigarrow & \alpha_{p,x} : F(x) \ni a \mapsto \{u \hookrightarrow x : [u \hookrightarrow x^a] \in p\} \in S_{\mathcal{C}}(x) \\ \alpha : F \Rightarrow S_{\mathcal{C}} & \rightsquigarrow & p_\alpha = \{[u \hookrightarrow x^a] : u \in \alpha_x(a)\} \end{array}$$

Example

$M : \mathcal{C} \to \mathbf{Set}$ coherent.
$tp_M : M \Rightarrow S_{\mathcal{C}}$ takes an element to its type: $tp_{M,x}(a) = \{u \hookrightarrow x : a \in Mu\}$

Corollary

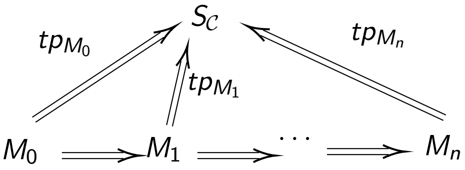
(M coherent.) $\{1\} \subseteq LM$ is a prime filter.

Corollary

(M coherent.) M is positively closed iff tp_M is the unique $M \Rightarrow S_{\mathcal{C}}$ nat. tr.

Krull-dimension

Idea:



$\rightsquigarrow \quad \{1\} = p_0 \subseteq p_1 \subseteq \cdots \subseteq p_n$

If all $M_i \Rightarrow M_{i+1}$ maps are prop. surjections (on underlying sets, not pointwise) $\rightsquigarrow$ no triangles commute (no maps are elementary) $\rightsquigarrow$ $M_0 \Rightarrow M_i \Rightarrow S_{\mathcal{C}}$ are different $\rightsquigarrow$ $p_0 \subsetneq p_1 \subsetneq \cdots \subsetneq p_n$.

Proposition

If there's an n-step chain of proper surjections out of M, then $dim(LM) \geqslant n$.

Remark

If for each $x \in \mathcal{C}$: $dim(Sub_{\mathcal{C}}(x)) \leqslant n$ then $dim(LM) \leqslant n$.

Functoriality

- $Lan_y Sub_{\mathcal{C}} : \mathbf{Lex}(\mathcal{C}, \mathbf{Set}) \to \mathbf{DLat}$ preserves all
  - colimits (it's a left adjoint),
  - strong monos (can write them as a filtered colimit of strong monos between representables, those are coming from effective epis in $\mathcal{C}$, pullback along them is injective),
  - if $\mathcal{C}$ has disjoint unions: finite products (finite products form a sound doctrine in the sense of [Adámek-Borceux-Lack-Rosický,2002]).

Boolean-valued models

Theorem ([Lurie,2018])

($\mathcal{C}$ has disjoint coproducts.) $F : \mathcal{C} \to \mathbf{Set}$ *regular.*
It can be factored as $\mathcal{C} \xrightarrow{M_F} Sh(F(2)) \xrightarrow{\Gamma} \mathbf{Set}$, *where* M_F *is coherent,* $F(2)$ *is a Boolean-alg. with the coherent top.,* Γ *is global sections.*
Conversely, if $M : \mathcal{C} \to Sh(B)$ *is coherent then* ΓM *is regular.*
There's an equivalence: $\mathbf{Reg}(\mathcal{C}, \mathbf{Set}) \simeq \mathcal{C} \downarrow Sh(\mathbf{BAlg})$.

Theorem

$\mathcal{C}$ *has disjoint coproducts,* $F : \mathcal{C} \to \mathbf{Set}$ *regular.*
$B \subseteq LF$ *is the Boolean-algebra of complemented subobjects.*

- $B \cong F(2)$
- $\downarrow : LF \to (Id(B), \subseteq)$ *(taking* x *to* $\{b : b \leqslant x\}$*) is a homomorphism.*
  So there's a "retraction": $\downarrow : B \hookrightarrow LF \xrightarrow{\downarrow} Id(B)$

References

[Makkai-Reyes,1977]: First-order categorical logic

[Barr,1986]: Representation of categories

[Adámek-Borceux-Lack-Rosický,2002]: A classification of accessible categories

[Poizat-Yeshkeyev,2018]: Positive Jonsson Theories

[Lurie,2018]: Lecture notes in categorical logic (`https://www.math.ias.edu/~lurie/278x.html`)