Measuring how much a model is not positively closed

Kristóf Kanalas

A formula is positive existential if it is built up from atomic formulas, $\top$, $\wedge$, $\bot$, $\vee$ and $\exists$. A coherent theory is a set of formulas of the form $\forall \vec{x}(\varphi(\vec{x}) \to \psi(\vec{x}))$ where φ and ψ are positive existential.

A model M of a coherent theory is positively closed if every homomorphism out of it reflects the validity of positive existential formulas (i.e. given $f : M \to N$, if $N \models \varphi(f\vec{a})$ then $M \models \varphi(\vec{a})$ for any positive existential φ and tuple $\vec{a}$ from M).

A coherent category is a category with finite limits, pullback-stable finite unions and pullback-stable effective epi-mono factorization, a functor is coherent if it preserves this structure. A main fact of categorical logic says that coherent theories, models and homomorphisms are the same as small coherent categories, coherent functors to **Set** and natural transformations between them.

I will associate a distributive lattice LM to any model M of a coherent theory T, and prove a possibly proper subset of the following claims:

- LM has an explicit description with formulas. Its elements are closed, positive existential formulas with parameters from M, up to the following equivalence: $\varphi(\vec{a}) \sim \psi(\vec{b})$ iff there's $\chi(\vec{x}, \vec{y})$ (positive existential), such that $M \models \chi(\vec{a}, \vec{b})$ and $T \vdash \chi(\vec{x}, \vec{y}) \wedge \varphi(\vec{x}) \;\leftrightarrow\; \chi(\vec{x}, \vec{y}) \wedge \psi(\vec{y})$.

- On the category side LM is defined by restricting the left Kan-extension

$$\begin{array}{ccc} \mathcal{C}^{op} & \xrightarrow{\;Sub_{\mathcal{C}}\;} & \mathbf{DLat} \\ {\scriptstyle y}\downarrow & \cong & \nearrow {\scriptstyle Lan_y Sub_{\mathcal{C}}} \\ \mathbf{Lex}(\mathcal{C}, \mathbf{Set}) & & \end{array}$$

  to coherent functors, where $\mathcal{C}$ is the coherent category corresponding to T.

- $\{1\} \subseteq LM$ is a prime filter.

- M is positively closed iff LM is the two-element Boolean-algebra.

- If there is an n-step chain of proper surjections out of M then the Krull-dimension of LM is at least n.
- If for any tuple of sorts $x_1, \dots x_k$ in the signature, the corresponding positive type-space $S_{x_1,\dots x_k}(T)$ has Krull-dimension at most n then LM has Krull-dimension at most n.
- I will compute LM in the two simplest possible cases: when T is a propositional theory (and hence $\mathcal{C}$ is a distributive lattice and M is a prime filter), and when T is the theory of non-empty sets over the empty signature.

Some of these results are discussed in [Kan22, Section 6].

References

[Kan22] Kristóf Kanalas. *Positive model theory of interpretations.* 2022. URL: https://arxiv.org/abs/2211.14600.