Integrations in a Differential Category (with Antiderivatives)

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Today's Story

- Differential categories provide a categorical interpretation of the algebraic foundations of differential calculus.
  - R. Blute, R. Cockett, R.A.G. Seely, **Differential Categories**
  - R. Blute, R. Cockett, J.-S. P. Lemay, and R.A.G. Seely **Differential Categories Revisited**.

- Many differentiation related concepts can be formalized using differential categories. In particular, **derivations** have a very nice formalization in a differential category.
  - R. Blute, R.B.B. Lucyshyn-Wright, and K. O'NeilL **Derivations in codifferential categories**.
  - K. O'Neill **Smoothness in codifferential categories** (PhD Thesis)
  - J.-S. P. Lemay **Differential algebras in codifferential categories**

- Integration analogues of differential categories have also been worked on. In particular, we can discuss have antideriatives, the fundamental theorems of calculus, etc. in a differential category.
  - T. Ehrhard **An introduction to differential linear logic: proof-nets, models and antiderivatives**.
  - R. Cockett, J.-S. P. Lemay **Integral Categories and Calculus Categories** (and my Master's Thesis)
  - J.-S. P. Lemay **Convenient Antideriatives for Differential Linear Categories**

- **TODAY'S STORY**: interpretation of the integration analogue of derivations in a differential category (with antiderivatives).

Differential Categories vs Codiffrential Categories

Today we will actually be talking about **co**differential categories, the dual of differential categories.

- Differential categories were introduced to provide the categorical semantics of Differential Linear Logic, which is all about comonads and coalgebras
- However, codifferential categories capture the notions of derivations from algebra.

I do not like the term codifferential category... So instead I have been proposing the following:

- **Algebraic Differential Category** ≡ Codifferential Category;
- **Coalgebraic Differential Category** ≡ Differential Category.

Algebraic Differential Category - Definition

An **algebraic differential category** is a symmetric (strict) monoidal category $(\mathbb{X}, \otimes, I, \sigma)$ where:

- Each hom-set $\mathbb{X}(A, B)$ is a commutative monoid with binary operation $+$ and zero 0, that is, we can add parallel maps $f + g$ and there is a zero map 0, and such that composition and the tensor product preserves the additive structure;
- Equipped with an **algebra modality**, which is a monad (S, μ, η)

$$\mathsf{SS}A \xrightarrow{\mu} \mathsf{S}A \qquad\qquad A \xrightarrow{\eta} \mathsf{S}A$$

  equipped with two natural transformations m and u:

$$\mathsf{S}A \otimes \mathsf{S}A \xrightarrow{\mathsf{m}} \mathsf{S}A \qquad\qquad k \xrightarrow{\mathsf{u}} \mathsf{S}A$$

  such that for every object A, $(\mathsf{S}A, \mathsf{m}, \mathsf{u})$ is a commutative monoid and μ is a monoid morphism.

Rough Idea:

- $\mathsf{S}A \equiv$ set of differentiable functions;
- $\mu \equiv$ function composition;
- $\eta \equiv$ identity function/linear function;
- $\mathsf{m} \equiv$ function multiplication;
- $\mathsf{u} \equiv$ multiplication unit/constant function.

Algebraic Differential Category - Definition

An **algebraic differential category** is a symmetric (strict) monoidal category $(\mathbb{X}, \otimes, I, \sigma)$ where:

- Each hom-set $\mathbb{X}(A, B)$ is a commutative monoid with binary operation $+$ and zero 0, that is, we can add parallel maps $f + g$ and there is a zero map 0, and such that composition and the tensor product preserves the additive structure;
- Equipped with an **algebra modality**, which is a monad (S, μ, η)

$$SSA \xrightarrow{\mu} SA \qquad\qquad A \xrightarrow{\eta} SA$$

  equipped with two natural transformations m and u:

$$SA \otimes SA \xrightarrow{\mathsf{m}} SA \qquad\qquad k \xrightarrow{\mathsf{u}} SA$$

  such that for every object A, $(\mathsf{S}A, \mathsf{m}, \mathsf{u})$ is a commutative monoid and μ is a monoid morphism.
- Also equipped with a **deriving transformation** for an algebra modality is a natural transformation:

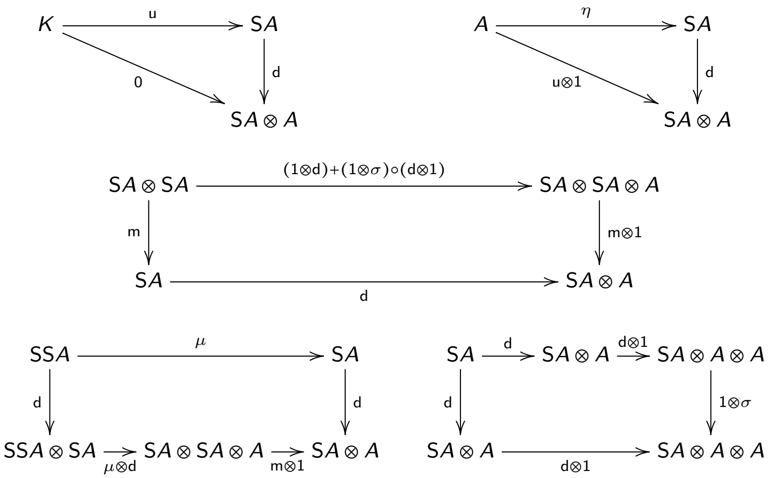
$$SA \xrightarrow{\mathsf{d}} SA \otimes A$$

  whose axioms are based on the basic identities from differential calculus.

ROUGH IDEA: $f(x) \mapsto f'(x) \otimes \mathsf{d}x$

- Constant rule: $c' = 0$
- Product rule: $(f \cdot g)'(x) = f'(x)g(x) + f(x)g'(x)$
- Linear rule: $x' = 1$
- Chain rule: $(f \circ g)'(x) = f'(g(x))g'(x)$
- Interchange rule: $\frac{d^2 f(x,y)}{dxdy} = \frac{d^2 f(x,y)}{dydx}$

Deriving Transformation - Definiton



Main Example I: Polynomials

Let k be a field and let VEC_k to be the category of all k-vector spaces and k-linear maps.

Example

A commutative monoid in VEC_k is a commutative k-algebra. Define the algebra modality Sym on VEC_k as follows: for a k-vector space V let $\mathsf{Sym}(V)$ be the free commutative k-algebra over V, also known as the free symmetric algebra on V (where $\otimes_s$ is the symmetrize tensor product):

$$\mathsf{Sym}(V) := k \oplus V \oplus (V \otimes_s V) \oplus \cdots = \bigoplus_{n \in \mathbb{N}} V \otimes_s \cdots \otimes_s V$$

If $X = \{x_1, x_2, \cdots\}$ is a basis of V, then $\mathsf{Sym}(V) \cong k[X]$. So for k^n, $\mathsf{Sym}(k^n) \cong k[x_1, \cdots, x_n]$.

Then the algebra modality structure μ and η correspond to polynomial composition, while m and u correspond to polynomial multiplication.

The deriving transformation on Sym can be described in terms of polynomials as follows:

$$\mathsf{d} : k[X] \to k[X] \otimes V$$
$$\mathsf{p}(x_1, \cdots, x_n) \mapsto \sum_{i=1}^{n} \frac{\partial \mathsf{p}}{\partial x_i}(x_1, \cdots, x_n) \otimes x_i$$

Main Example II: Smooth Functions

Let $k = \mathbb{R}$, there is another interesting algebra modality on $\mathsf{VEC}_{\mathbb{R}}$

Example

A C^∞-**ring** is commutative $\mathbb{R}$-algebra A such that for each for smooth map $f : \mathbb{R}^n \to \mathbb{R}$ there is a function $\Phi_f : A^n \to A$ and such that the Φ_f satisfy certain coherences between them.

Rough Idea: $\Phi_f(a_1, \cdots, a_n) = f(a_1, \cdots, a_n)$.

Ex. $\mathsf{C}^\infty(\mathbb{R}^n) = \{f : \mathbb{R}^n \to \mathbb{R} | f \text{ smooth}\}$ is a C^∞-ring where $\Phi_f(g_1, \cdots, g_n)(\vec{x}) = f(g_1(\vec{x}), \cdots, g_n(\vec{x}))$.

For every $\mathbb{R}$-vector space V, there is a free C^∞-ring over V, $\mathsf{S}^\infty(V)$, and this induces an algebra modality S^∞ on $\mathsf{VEC}_{\mathbb{R}}$. In particular, $\mathsf{S}^\infty(\mathbb{R}^n) = \mathsf{C}^\infty(\mathbb{R}^n)$.

In this example, μ and η correspond to composing smooth functions, while m and u correspond to multiplying smooth functions.

The deriving transformation on S^∞ corresponds to differentiating smooth functions. In particular, for $V = \mathbb{R}^n$:

$$\mathsf{d} : \mathsf{C}^\infty(\mathbb{R}^n) \to \mathsf{C}^\infty(\mathbb{R}^n) \otimes \mathbb{R}^n$$
$$f \longmapsto \sum_i \frac{\partial f}{\partial x_i} \otimes x_i$$

Classical Case: Derivations

In classical algebra, **derivations** are axiomatized by the **Leibniz rule**

$$(fg)'(x) = f'(x)g(x) + f(x)g'(x)$$

Let k be a commutative ring, A a commutative k-algebra, and M an A-module.

- A **derivation** is a k-linear map $\mathsf{D} : A \to M$ such that:

$$\mathsf{D}(ab) = a\mathsf{D}(b) + b\mathsf{D}(a)$$

- When $M = A$, we call A with a derivation $\mathsf{D} : A \to A$ a differential algebra.

In an algebraic differential category, **derivations** are instead axiomatized by the **chain rule**

$$(f \circ g)'(x) = f'(g(x))g'(x)$$

from which we can prove that they also satisfy **Leibniz rule**!

But first we need to talk about algebras and modules in an algebraic differential category,

Algebras of an Algebra Modality and their Modules

For an algebra modality S, every S-algebra $(A, \nu : \mathsf{S}A \to A)$ has a canonical commutative monoid structure where the multiplication $\mathsf{m}^\nu : A \otimes A \to A$ and unit $\mathsf{u}^\nu : k \to A$ are defined as follows:

$$A \otimes A \xrightarrow{\eta \otimes \eta} \mathsf{S}A \otimes \mathsf{S}A \xrightarrow{\mathsf{m}} \mathsf{S}A \xrightarrow{\nu} A$$

$$k \xrightarrow{\mathsf{u}} \mathsf{S}A \xrightarrow{\nu} A$$

Note that applying these constructions to $(\mathsf{S}A, \mu)$ gets us back $\mathsf{m}^\mu = \mathsf{m}$ and $\mathsf{u}^\mu = \mathsf{u}$.

For an S-algebra (A, ν), an (A, ν)-module is simply a module over the commuative monoid $(A, \mathsf{m}^\nu, \mathsf{u}^\nu)$, so a pair $(M, \alpha : A \otimes M \to M)$.

Example

The Sym-algebras are precisely the commutative k-algebras. For a commutative k-algebra A, the map $\nu : \mathsf{Sym}A \to A$ corresponds to evaluating polynomials at A:

$$p(\vec{x}) \in k[X] \longmapsto p(\vec{a}) \in A$$

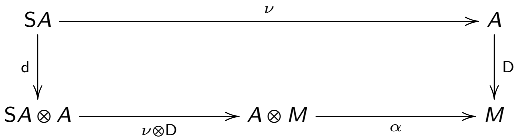
Example

The S^∞-algebras are precisely the C^∞-rings. For a C^∞-ring$A$, the map $\nu : \mathsf{S}^\infty(A) \to A$ corresponds to evaluating smooth functions at A:

$$\Phi_f(a_1, \cdots, a_n) = f(a_1, \cdots, a_n)$$

Derivations

For an S-algebra (A, ν) and a (A, ν)-module (M, α), an S-**derivation** is a map $\mathsf{D} : A \to M$ such that the following diagram commutes:

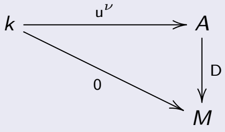


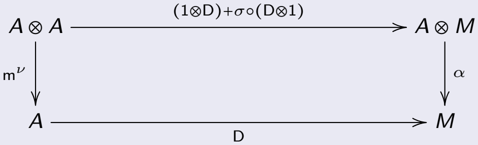
ROUGH IDEA: $\mathsf{D}(f(a)) = f'(a)\mathsf{D}(a)$

R. Blute, R.B.B. Lucyshyn-Wright, and K. O'NeilL **Derivations in codifferential categories**.

Proposition

The following diagrams commutes:





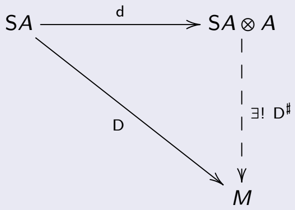
So S-derivations are indeed derivations.

Universal Derivations

Note that for the free S-algebra $(\mathsf{S}A, \mu)$, $(\mathsf{S}A \otimes A, \mathrm{m} \otimes 1)$ is a $(\mathsf{S}A, \mu)$-module.

Proposition

The deriving transformation $\mathrm{d} : \mathsf{S}A \to \mathsf{S}A \otimes A$ *is an S-derivation and furthermore it is universal:*



where $\mathrm{D}^\sharp : \mathsf{S}A \otimes A \to A$ *is an SA-module morphism.*

If certain coequalizers exist, we can construct universal derivations for an S-algebra (A, ν).

Derivations Examples

Example

For a commutative k-algebra A and an A-module M, Sym-derivations $\mathrm{D} : A \to M$ correspond precisely to classical derivations.

$\Rightarrow$: We already saw that every Sym-derivation is a derivation.

$\Leftarrow$: Why is this true? Thanks to the Leibniz rule, derivations do satisfy a chain rule with respect to polynomials:

$$\mathrm{D}\left(p(a_1, \cdots, a_n)\right) = \sum_{i=1}^{n} \frac{\partial \mathrm{p}}{\partial x_i}(a_1, \cdots, a_n)\mathrm{D}(a)$$

Example

For a C^∞-ring A and an A-module M, a S^∞-derivation corresponds to a C^∞-derivation, which is a derivation for the Fermat theory of smooth functions in the sense of:

Dubuc and Kock, **On 1-form classifiers**.

So a C^∞-derivation is a map $\mathrm{D} : A \to M$ such that:

$$\mathrm{D}\left(\Phi_f(a_1, \cdots, a_n)\right) = \sum_{i=1}^{n} \Phi_{\frac{\partial f}{\partial x_i}}(a_1, \cdots, a_n)\mathrm{D}(a)$$

You can see that a C^∞-derivation is a derivation if you take $f(x_1, x_2) = x_1 x_2$.

Classical Case: Integrations

In classical algebra, **integrations** are axiomatized by the **Rota-Baxter rule**, which is an integral only version of the **integration by parts rule**:

$$f(x)g(x) = \int_0^x f(t)g'(t)\mathrm{d}t + \int_0^t f'(t)g(t)\mathrm{d}t$$

Then setting $f(x) = \int_0^x F(u)\mathrm{d}u$ and $g(x) = \int_0^x G(u)\mathrm{d}u$, gives us that:

$$\left(\int_0^x F(u)\mathrm{d}u\right)\left(\int_0^x G(u)\mathrm{d}u\right) = \int_0^x \left(\int_0^t F(u)\mathrm{d}u\right) G(t)\mathrm{d}t + \int_0^t F(t)\left(\int_0^x G(u)\mathrm{d}u\right)\mathrm{d}t$$

Let k be a commutative ring, A a commutative k-algebra, and M an A-module.

- An **integration** is a k-linear map $\mathsf{P} : M \to A$ such that:

$$\mathsf{P}(m)\mathsf{P}(n) = \mathsf{P}\left(\mathsf{P}(m)n\right) + \mathsf{P}\left(\mathsf{P}(n)m\right)$$

- When $M = A$, we call A with an integration $\mathsf{P} : A \to A$ a Rota-Baxter algebra.

In an algebraic differential category, **integrations** are instead axiomatized by the **integration by substitution rule**

$$\int_0^x f(g(t))g'(t)\mathrm{d}t = \int_{g(0)}^{g(a)} f(x)\mathrm{d}x$$

from which we can prove that they also satisfy the **Rota-Baxter rule**.

Moving onto Integration

- We can already define what an integration would be in a differential category...

- But why talk about integrations in a differential category and not in an integral category?

- Briefly an integral category, we instead have an **integral transformation** $\mathsf{s} : SA \otimes A \to SA$.

  R. Cockett, J.-S. P. Lemay **Integral Categories and Calculus Categories** (and my Master's Thesis)

- So why don't we ask an integration to interact with s?

- **ANSWER:** s does not have a composition axiom... there is no **integral only** version of the integration by substitution rule that holds in the multivariable case (more on this later).

- So an integration in a differential category will satisfy the analogue of the integration by substitution rule which includes the integration and the deriving transformation.

But before that, let's discuss antiderivatives in a differential category.

Antiderivatives in an Algebraic Differential Category

Define the natural transformation $\mathsf{L} : SA \to SA$ as follows:

$$SA \xrightarrow{\mathsf{d}} SA \otimes A \xrightarrow{1 \otimes \eta} SA \otimes SA \xrightarrow{\mathsf{m}} SA$$

An algebraic differential category is said to have **antiderivatives** if the natural transformation $\mathsf{K} : SA \to SA$ defined as :

$$\mathsf{K} := \mathsf{L} + \mathsf{S}(0)$$

is a natural isomorphism.

T. Ehrhard **An introduction to differential linear logic: proof-nets, models and antiderivatives**.

R. Cockett, J.-S. P. Lemay **Integral Categories and Calculus Categories** (and my Master's Thesis)

J.-S. P. Lemay **Convenient Antideriatives for Differential Linear Categories**

The **antiderivative integral transformation** is the natural transformation $\mathsf{s} : SA \otimes A \to SA$ defined as follows:

$$SA \otimes A \xrightarrow{1 \otimes \eta} SA \otimes SA \xrightarrow{\mathsf{m}} SA \xrightarrow{\mathsf{K}^{-1}} SA$$

ROUGH IDEA: $f(x) \otimes x \mapsto \int f(x) \mathsf{d}x$

This antiderivative integral transformations satisfies the Rota-Baxter rule and also the Fundamental Theorems of Calculus! More on this later...

Main Example I: Polynomials

Example

If $X = \{x_1, x_2, \cdots\}$ is a basis of V, then $\mathsf{K} : k[X] \to k[X]$ is given by multiplying monomials by their degree and sending constants to themselves:

$$\mathsf{K}\left(x_1^{a_1} \cdots x_n^{a_n}\right) = (a_1 + \cdots + a_n) x_1^{a_1} \cdots x_n^{a_n} \qquad \mathsf{K}(c) = c$$

If the field k is of characteristic zero, then K is an isomorphism with inverse:

$$\mathsf{K}^{-1}\left(x_1^{a_1} \cdots x_n^{a_n}\right) = \frac{1}{a_1 + \cdots + a_n} x_1^{a_1} \cdots x_n^{a_n} \qquad \mathsf{K}^{-1}(c) = c$$

The antiderivative integral transformation $\mathsf{s} : k[X] \otimes V \to k[X]$ is given by:

$$\mathsf{s}\left(x_1^{a_1} \cdots x_n^{a_n} \otimes x_i\right) = \frac{1}{a_1 + \cdots + a_n + 1} x_1^{a_1} \cdots x_i^{a_i + 1} \cdots x_n^{a_n}$$

Main Example II: Smooth Functions

Example

For S^∞, $\mathsf{K} : \mathsf{S}^\infty V \to \mathsf{S}^\infty V$ is a natural isomorphism. In particular for $V = \mathbb{R}^n$, $\mathsf{K} : \mathsf{C}^\infty(\mathbb{R}^n) \to \mathsf{C}^\infty(\mathbb{R}^n)$ is an isomorphism where:

$$\mathsf{K}(f)(\vec{v}) = \nabla(f)(\vec{v}) \cdot \vec{v} + f(\vec{0}) \qquad \mathsf{K}^{-1}(f)(\vec{v}) = \int_0^1 \int_0^1 \nabla(f)(st\vec{v}) \cdot \vec{v} \mathsf{d}s\mathsf{d}t + f(\vec{0})$$

where $\nabla(f)$ is the gradient of f and $\cdot$ is the dot product.

The antiderivative integral transformation $\mathsf{s} : \mathsf{C}^\infty(\mathbb{R}^n) \otimes \mathbb{R}^n \to \mathsf{C}^\infty(\mathbb{R}^n)$ is given by:

$$\mathsf{s}(f \otimes x_i)(\vec{v}) = \int_0^1 f(t\vec{v}) v_i \mathsf{d}t$$

this is the line integral. And if you take f to be a polynomial, then you get the integral formula from the previous slide.

G. Cruttwell, J.-S. P. Lemay, R.B.B. Lucyshyn-Wright **Integral and differential structure on the free C^∞-ring modality**.

Integrations

So now we want to define S-integrations whose axiom in some analogue of the integration by substitution rule, and such that the antiderivative integral transformation $\mathsf{s} : SA \otimes A \to SA$ is also an S-integration.

Integrations – First Attempt

For an S-algebra (A, ν) and a (A, ν)-module (M, α), an S-**integration** is a map $\mathrm{P} : M \to A$ such that **???**. One's first attempt might be to take the integration by substitution rule:

$$\int_0^x f(g(t))g'(t)\mathrm{d}t = \int_{g(0)}^{g(a)} f(x)\mathrm{d}x$$

then set $f(x) = \int_0^x F(u)\mathrm{d}u$ and $g(x) = \int_0^x G(u)\mathrm{d}u$, giving us:

$$\int_0^x F\left(\int_0^t G(u)\mathrm{d}u\right) G(t)\mathrm{d}t = \int_0^{\int_0^x G(u)\mathrm{d}u} F(t)\mathrm{d}t$$

This translates to saying the following diagram commutes:

$$\begin{array}{ccccccc}
SM \otimes M & \xrightarrow{\mathrm{s}} & SM & & \xrightarrow{\mathrm{S(P)}} & & SA \\
\downarrow{\scriptstyle \mathrm{S(P)}\otimes 1} & & & & & & \downarrow{\scriptstyle \nu} \\
SA \otimes M & \xrightarrow[\nu\otimes 1]{} & A \otimes M & \xrightarrow[\alpha]{} & M & \xrightarrow[\mathrm{P}]{} & A
\end{array}$$

Problems: We don't want to assume antiderivatives for our definition, and even if we did, $\mathrm{s} : SA \otimes A \to SA$ does not satisfy this! (it only does when $A = I$ essentially)

Integrations – Correct Attempt

Instead, we will use this formula:

$$f\left(\int_0^x G(u)\mathrm{d}u\right) = \int_0^x \frac{\mathrm{d}f}{\mathrm{u}}\left(\int_0^t G(u)\mathrm{d}u\right) G(t)\mathrm{d}t + f(0)$$

which translates too:

For an S-algebra (A, ν) and a (A, ν)-module (M, α), an S-**integration** is a map $\mathrm{P} : M \to A$ such that the following equality holds:

$$SM \xrightarrow{\mathrm{S(P)}} SA \xrightarrow{\nu} A$$

$$=$$

$$SM \xrightarrow{\mathrm{d}} SM \otimes M \xrightarrow{\mathrm{S(P)}\otimes 1} SA \otimes M \xrightarrow{\nu \otimes 1} A \otimes M$$

$$\xrightarrow[\alpha]{} M \xrightarrow[\mathrm{P}]{} A$$

$$+$$

$$SM \xrightarrow{\mathrm{S}(0)} SA \xrightarrow{\nu} A$$

The extra summand $\nu \circ \mathrm{S}(0)$ is carried around for the constant parts of SM.

Integrations – Rota-Baxter Rule

Proposition

An S-integration $\mathrm{P} : M \to A$ *satisfies the Rota-Baxter identity:*

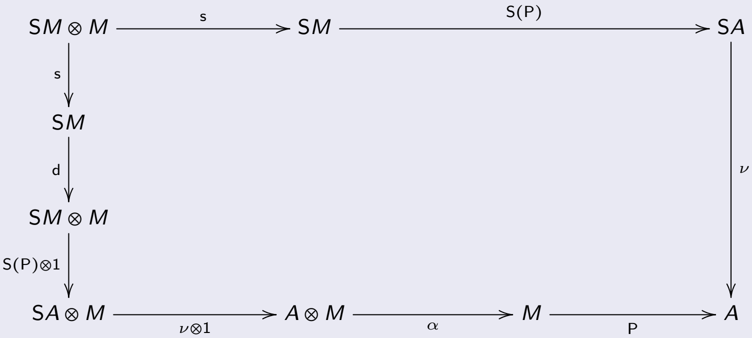
$$\begin{array}{ccc} M \otimes M & \xrightarrow{(\mathrm{P}\otimes 1)+\sigma\circ(1\otimes \mathrm{P})} & A \otimes M \\ & & \Big\downarrow{\scriptstyle \alpha} \\ \Big\downarrow{\scriptstyle \mathrm{P}\otimes\mathrm{P}} & & M \\ & & \Big\downarrow{\scriptstyle \mathrm{P}} \\ A \otimes A & \xrightarrow[\mathrm{m}^{\nu}]{} & A \end{array}$$

So S-integrations are indeed integrations.

Integrations – Antiderivatives

Proposition

In an algebraic differential category with antiderivatives, $\mathsf{P} : M \to A$ *is an S-integration if and only if the following diagram commutes:*



Proposition

The antiderivative integral transformation $\mathsf{s} : SA \otimes A \to SA$ *is an S-integration.*

Integrations Examples

Example

For a commutative k-algebra A and an A-module M, Sym-integrations $\mathrm{P} : M \to A$ correspond precisely to classical integrations.

$\Rightarrow$: We already saw that every Sym-integration is an integration.

$\Leftarrow$: Why is this true? Well the Sym-integration axiom is essentially just a higher-order Rota-Baxter rule:

$$p\left(\mathrm{P}(m_1)\cdots\mathrm{P}(m_n)\right) = \sum_{i=1}^{n} \mathrm{P}\left(\frac{\partial \mathrm{p}}{\partial x_i}(\mathrm{P}(m_1), \cdots, \mathrm{P}(m_1))m_i\right) + p(0, \cdots, 0)$$

Example

For a C^∞-ring A and an A-module M, a S^∞-integration is a map $\mathrm{P} : M \to A$ such that:

$$\Phi_f\left(\mathrm{P}(m_1), \cdots, \mathrm{P}(m_n)\right) = \sum_{i=1}^{n} \mathrm{P}\left(\Phi_{\frac{\mathrm{d}f}{\mathrm{d}x_i}}\left(\mathrm{P}(m_1), \cdots, \mathrm{P}(m_n)\right) \cdot m_i\right) + \Phi_f(0, \cdots, 0)$$

Interaction between Derivations and Integrations

A derivation and an integration should be compatible via analogues of the **Fundamental Theorems of Calculus!**

Second Fundamental Theorem of Calculus

The Second Fundamental Theorem of Calculus states that:

$$\int_0^x \frac{\mathrm{d}f}{\mathrm{d}x}(t)\mathrm{d}(t) = f(x) - f(0)$$

In the multivariable case, this is called the fundamental theorem of line integration.

To interpret this in our algebraic differential category setting, we need our evaluation map. This is captured by an idempotent S-algebra morphism $\mathsf{e} : (A, \nu) \to (A, \nu)$.

An S-derivation $\mathsf{D} : A \to M$ and an S-integration $\mathsf{P} : M \to A$ satisfy **FTC2 at** e if the following equalities hold:

$$\mathsf{D} \circ \mathsf{e} = 0 \qquad \mathsf{e} \circ \mathsf{P} = 0 \qquad \mathsf{P} \circ \mathsf{D} + \mathsf{e} = 1$$

For $(\mathsf{S}A, \mu)$, $\mathsf{S}(0)$ is an idempotent S-algebra morphism and corresponds to evaluating at zero is captured by the map .

Proposition

The deriving transformation $\mathsf{d} : \mathsf{S}A \to \mathsf{S}A \otimes A$ *and the antiderivative integral transformation* $\mathsf{s} : \mathsf{S}A \otimes A \to \mathsf{S}A$ *together satisfy FTC2 at* $\mathsf{S}(0)$*.*

$$\mathsf{s} \circ \mathsf{d} + \mathsf{S}(0) = 1$$

First Fundamental Theorem of Calculus

The First Fundamental Theorem of Calculus states that:

$$\frac{\mathrm{d} \int_0^u f(t) \mathrm{d}(t)}{\mathrm{d}u}(x) = f(x)$$

An S-derivation $\mathsf{D} : A \to M$ and an S-integration $\mathsf{P} : M \to A$ satisfy **FTC1** if the following equalities hold:

$$\mathsf{D} \circ \mathsf{P} = 1$$

The First Fundamental Theorem of Calculus holds for the one variable case, this does not hold for the multivariable case! Thus n general, the deriving transformation $\mathsf{d} : SA \to SA \otimes A$ and the antiderivative integral transformation $\mathsf{s} : SA \otimes A \to SA$ do **NOT** satisfy FTC1.

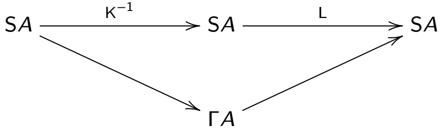
But when $A = I$ they do! This is important since for a categorical model of Differential Linear Logic, having antiderivatives if completely determined by if SI has an S-integration which satisfies FTC1 and FTC2 with the deriving transformation.

J.-S. P. Lemay **Convenient Antideriatives for Differential Linear Categories**

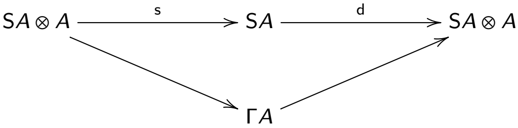
WANT: An (SA, μ)-module ΓA which has an $\mathsf{D} : SA \to \Gamma A$ and an S-integration $\mathsf{P} : \Gamma A \to SA$ which satisfy both FTC1 and FTC2.

Split Antiderivatives

An algebraic differential category is said to have **split antiderivatives** if it has antiderivatives and the idempotent $\mathsf{L} \circ \mathsf{K}^{-1}$ splits:



which is equivalent to asking that the idempotent $\mathsf{d} \circ \mathsf{s}$ splits:



From Split Antiderivatives to FTC 1 & 2

Proposition

If we have split antiderivatives, then:

- ΓA *is an* (SA, μ)*-module via* $SA \otimes \Gamma A \to SA \otimes SA \xrightarrow{\mathsf{m}} SA \to \Gamma A$;
- $\mathsf{D} : SA \xrightarrow{\mathsf{L}} SA \to \Gamma A$ *is an* S*-derivation;*
- $\mathsf{P} : \Gamma A \to SA \xrightarrow{\mathsf{K}^{-1}} SA$ *is an* S*-integration;*
- D *and* P *satisfy FTC1;*
- D *and* P *satisfy FTC2 at* $\mathsf{S}(0)$.
- *The following diagrams commute:*

$$\begin{array}{ccccc} A \otimes A & \xrightarrow{1 \otimes \eta} & A \otimes SA & \xrightarrow{\sigma} & SA \otimes A \\ \downarrow{\scriptstyle \eta \otimes 1} & & & & \downarrow{\scriptstyle \mathsf{D}^\sharp} \\ SA \otimes A & & \xrightarrow[\mathsf{D}^\sharp]{} & & \Gamma A \end{array}$$

$$\begin{array}{ccccc} SA \otimes \Gamma A & \xrightarrow{1 \otimes \mathsf{P}} & SA \otimes SA & \xrightarrow{1 \otimes \mathsf{L}} & SA \otimes SA \\ \downarrow{\scriptstyle \alpha} & & & & \downarrow{\scriptstyle \mathsf{m}} \\ \Gamma A & \xrightarrow[\mathsf{P}]{} & SA & \xrightarrow[\mathsf{L}]{} & SA \end{array}$$

From FTC 1 & 2 to Split Antiderivatives

Proposition

If for every A, $(\mathsf{S}A, \mu)$ has a $(\mathsf{S}A, \mu)$-module $(\Gamma A, \alpha)$ with an S-derivation $\mathsf{D} : \mathsf{S}A \to \Gamma A$ and S-integration $\mathsf{P} : \Gamma A \to \mathsf{S}A$ which satisfy FTC1 and FTC2, and the following diagrams commute:

$$\begin{array}{ccccc} A \otimes A & \xrightarrow{1\otimes\eta} & A \otimes \mathsf{S}A & \xrightarrow{\sigma} & \mathsf{S}A \otimes A \\ \downarrow{\scriptstyle \eta\otimes 1} & & & & \downarrow{\scriptstyle \mathsf{D}^\sharp} \\ \mathsf{S}A \otimes A & & \xrightarrow{\mathsf{D}^\sharp} & & \Gamma A \end{array} \qquad \begin{array}{ccccc} \mathsf{S}A \otimes \Gamma A & \xrightarrow{1\otimes\mathsf{P}} & \mathsf{S}A \otimes \mathsf{S}A & \xrightarrow{1\otimes\mathsf{L}} & \mathsf{S}A \otimes \mathsf{S}A \\ \downarrow{\scriptstyle \alpha} & & & & \downarrow{\scriptstyle \mathsf{m}} \\ \Gamma A & \xrightarrow{\mathsf{P}} & \mathsf{S}A & \xrightarrow{\mathsf{L}} & \mathsf{S}A \end{array}$$

Then we have split antiderivatives, and in particular the antiderivative integral transformation $\mathsf{s} : \mathsf{S}A \otimes A \to \mathsf{S}A$ factors through ΓA as follows:

$$\mathsf{S}A \otimes A \xrightarrow{\mathsf{D}^\sharp} \Gamma A \xrightarrow{\mathsf{P}} \mathsf{S}A$$

Differential Algebras and Rota-Baxter Algebras

Every S-algebra (A, ν) is a (A, ν)-module over itself as (A, m^ν).

An S-algebra (A, ν) with an S-derivations $A \to A$ is called an S-differential algebra.

J.-S. P. Lemay **Differential algebras in codifferential categories**

- For S-differential algebras, we can apply D multiple times, and so get a higher order Leibniz rule and also a Faa di Bruno formula.
- If we have countable coproducts, we can construct free S-differential algebra over an object.
- If we have countable products, we can construct cofree S-differential algebra over a S-algebra.

An S-algebra (A, ν) with an S-integrations $A \to A$ is called S-Rota-Baxter algebra.

- Question: Unclear to me if the free Rota-Baxter algebra construction over an algebra generalizes to give a free S-Rota-Baxter algebra over an S-algebra. (I need to show how the shuffle algebra is an S-algebra or something like this...)

HOPE YOU ENJOYED

THANK YOU FOR LISTENING!

HAPPY HALLOWEEN

MERCI!